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The Generalized Apex Magic Trick

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The Generalized Apex Magic Trick

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Abstract: *Many card tricks are based on mathematic principles. One such trick is known as the "apex number triangle". In this trick, the basic tool the magician uses to make a prediction is the "modulo 9" vector product between a row of numbers chosen by a spectator and the corresponding row in Pascal's triangle. In this paper we will show how to explore the fractal structure of the modulo versions of Pascal's triangle to produce different "magic" procedures, how to "work" them under different initial conditions, and how to prove pictorially a theorem regarding these kind of problems.*

Key-words: mathematical card tricks, Martin Gardner, Pascal's triangle.

1 Introduction

There seems to be a deep connection between mathematics and magic. Perhaps this is because some results of mathematical reasoning, methodology and procedure are somewhat surprising and counter-intuitive. One example of such a mathematical "self-working" magic trick is the so-called "apex number triangle" that was introduced by Martin Gardner [1] in his book *Mathematical Carnival* and further studied and generalized by Colm Mulcahy [2] and others [3].

Five playing cards, face-valued 1-9, are chosen by a spectator and placed face-up in a row on the table. In this trick, the suit of the cards is not important, so for all purposes, instead of using playing cards, one could simply take five slips of paper and have the spectator write down a number between 1-9 on each slip. The magician predicts a certain number and writes down his prediction. Then, the spectator is asked to add the two numbers on each pair of adjacent cards in the row. If the sum is larger than 9, the two digits of the result are added again to get a one-digit number, in other words, addition mod 9. A card with the face value of the sum is chosen from the pack and placed in a row above and in-between the two cards. As a result of this, a 4-card row of the modulo 9 sums of the pairs of cards in the row beneath is created. This procedure is repeated to create a 3-card row, then a 2-card row and finally one card - the "apex" of the triangle of cards that has just been formed. This card, of course, turns out to be the "predicted" card.

The math "behind" the trick can be unveiled using algebra, assigning variables to the set of initial numbers and performing the procedure. Once done, it is easy to see that the magician "predicts" the "apex" card by multiplying the face value of the spectator's initial five cards by 1,4,6,4,1 respectively and adding the sum modulo 9. Interestingly, the coefficients 1,4,6,4,1 are the numbers in the fifth row of Pascal's triangle modulo 9. Figure 1 shows an example of the trick played out. The spectator chose the cards: 5,6,2,7,5.

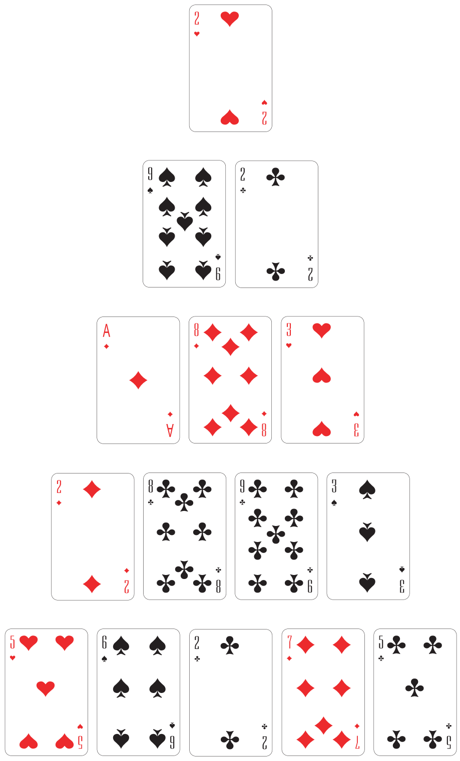

Figure 1: Example of the played out apex magic trick.

The magician predicts that the apex card will be

$$1 \times 5 + 4 \times 6 + 6 \times 2 + 4 \times 7 + 1 \times 5 = 2 \mod 9.$$

2 Generalizations

This trick can be generalized by using any initial number of cards and the corresponding row in Pascal's triangle modulo 9. This is due to the fact that the algebraic construction of the magic trick and Pascal's triangle is the same. Figure 2 shows the first 49 rows of the Pascal's triangle modulo 9. The fourth row, for example, is 1,3,3,1. These are the coefficients to be used for a 4-card trick. The magician predicts the apex card in this case by multiplying 1,3,3,1 by the numbers on the four cards that the spectator chooses, respectively. The tenth row is 1,0,0,3,0,0,3,0,0,1 and can be used for a 10-card trick. In this case, the magician predicts the apex card by multiplying these coefficients by ten numbers (or cards) that the spectator chooses. Since the coefficients have many zeroes, this is particularly easy to perform mentally. There is an infinite number of similar rows, where the magician only needs to remember four numbers in the bottom row of the triangle of cards. The next row is the twenty-eighth row: 1,0,0,0,0,0,0,0,0,3,0,0,0,0,0,0,0,0,3,0,0,0,0,0,0,0,0,1 and after that, rows 82, 244 and 730 etc. In general, any $3^n + 1$ row, where n is a positive whole number, will generate such a row. These rows can easily be spotted if we use a scheme to color the different numbers in the modulo 9 Pascal triangle, generating a fractal visual cue for similar-pattern rows.

Note that the fractal shows the general rules for the repetition of any row combination. The $3^n + 1$ rows, for example, are the 3-colored base of the top downwards-pointing Sierpinski style triangles in the fractal. In contrast, the 3^n rows are the 4-colored base of the top upwards-pointing Sierpinski style triangles in the fractal. This corresponds to rows where all the coefficients are non-zero.

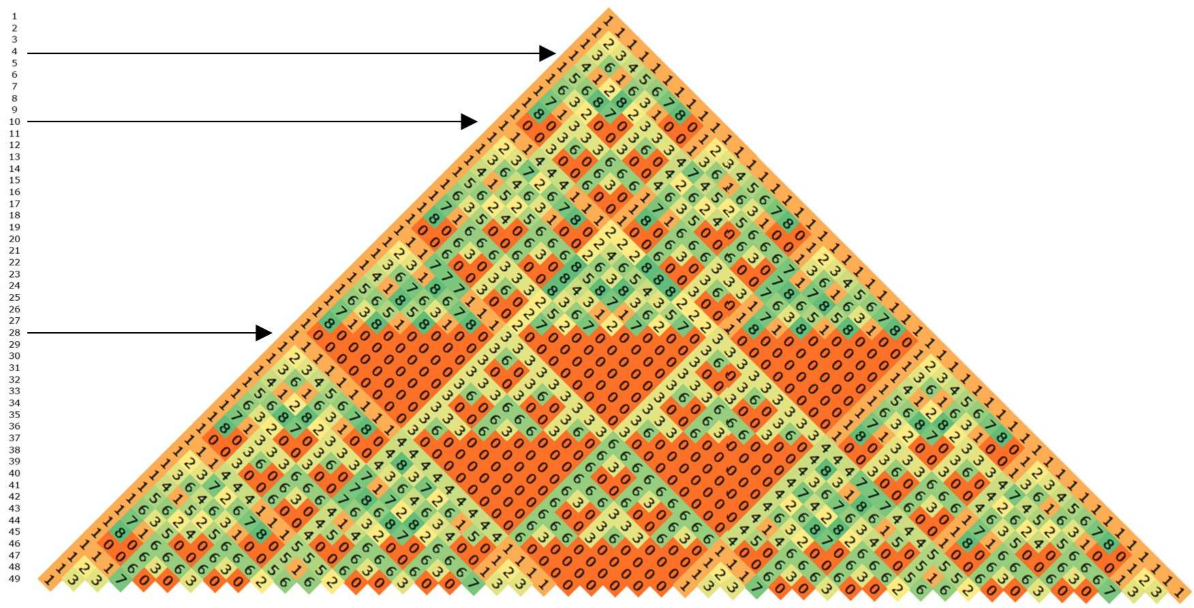

Figure 2: Color-coded mod 9 Pascal Triangle fractal.

The five-card trick can also be performed using different moduli. In a recent study, Behrends and Humble [4] consider the modulo 3 case. This is the best scenario for the magician since some rows in the modulo 3 Pascal triangle consist entirely of zeroes except the first and last numbers in the row that are 1. Behrends and Humble called these rows "Φ-simple". This means that all the magician needs to do is to add the first and last number in the row of cards that the spectator lays out and the result will be the apex number.

In particular, they rigorously prove that if a given row d is the smallest Φ-simple row, then a row $n > d$ is Φ-simple if and only if n is $(d-1)^s + 1$ where s is a positive whole number. In the modulo 3 case, the fourth row 1,0,0,1, is the smallest Φ-simple row, hence, rows 10, 28, 81 etc ... will also be Φ-simple. The fractal generation of the modulo 3 Pascal triangle can be used as a "pictorial proof" of Behrends and Humbles' theorem (Figure 3). It is obvious from the picture that the fractal grows by a factor of 3, hence, Behrend and Humbles theorem is proven, at least for this special case, and also proves that non-Φ-simple patterned rows repeat with the same factor.

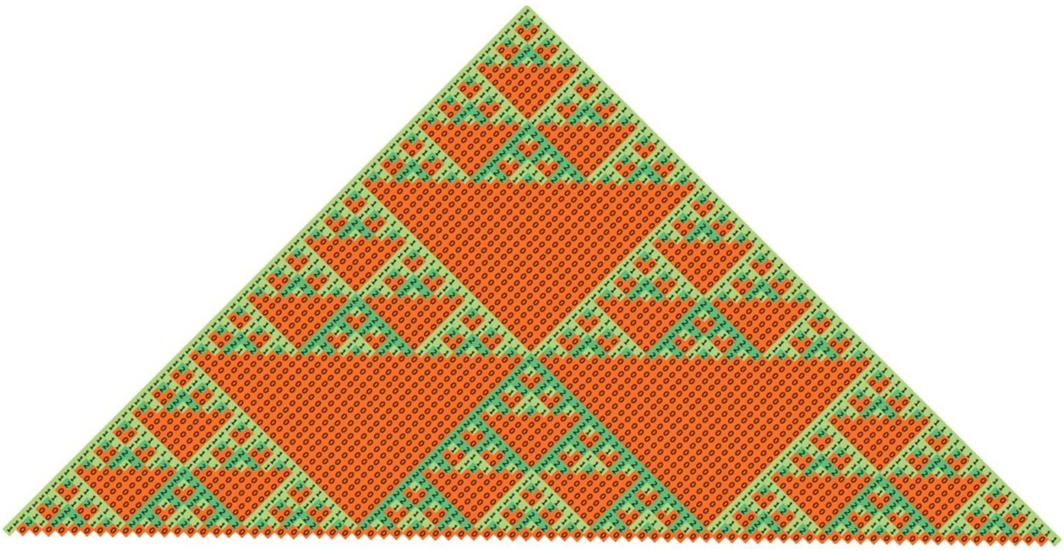

Figure 3: Color-coded mod 3 Pascal Triangle fractal.

We can explore this fractal structure to devise a "large audience" magic trick. Arrange a triangular formation of seats in an auditorium or large classroom so that there is one chair at the back of the room, two chairs in the row in front, three chairs in the next row and so on, until there are 10, 28 or 81 chairs in the last row (depending on the space available). Tell the people on the first row in the room, the row with the Φ-simple number of chairs, that they have to choose between three different gestures. They can raise one hand, raise both hands or sit with their arms folded. After each person chooses a position, you immediately predict and write down on a piece of paper one of the positions. You then ask each person in the second row to choose a gesture according to the combination of gestures of the two people in the row in front. If both have both hands raised or one of them has one hand raised and the other has folded his arms, then the person should raise both his hands. If both have one hand raised or one has both hands raised and the other has folded his arms, then the person should fold his arms. If both have arms folded or one has one hand raised and the other has both hands raised, then the person should raise one hand.

This procedure carries on until the person sitting in the last row has made his gesture, which of course, turns out to be the prediction. The analogy to the five card apex trick is immediate. All the magician does is to combine the gestures of the first and last people in the first row. Since the row is Φ-simple, this will be the apex gesture. The procedure itself is just addition modulo 3 where 0 = "both hands raised", 1="one hand raised" and 2="arms folded".

The nice thing about the fractal structure is that it allows one to perform this trick with any number of rows of chairs, albeit increasing the difficulty of the mental calculation. This is useful if one suspects that a clever audience might suggest that only the two end chairs predict the outcome. An easy choice would be to use rows of length: 7, 19, 55 etc. that correspond to the base of the downward pointing 2-triangle rows. The calculation is the same as before, except that twice the gesture of the middle chair in the first row is also added. Then, if the audience suspects as before that the end chairs cause the result, one can "prove" that this is not so.

Note that the Φ-simple rule for the modulo 3 case ($3^n + 1$) is the same rule that we found for the case of modulo 9. Although none of the rows in the modulo 9 case are Φ-simple, the rule does in fact generate all the rows with the minimal number of non-zero elements. This is easily explained by the visual similarity of the Pascal triangle fractals (Figs. 2 and 3).

The modulo 10 case presents another "neat" magic trick. Write eleven numbers in a row. Add each pair of numbers, keeping only the units digit and create a triangle as before. Although the modulo 10 Pascal triangle is not Φ-simple, as can be observed by its corresponding fractal, Figure 4, the prediction is very easy. Multiply the value of the spectator's first and last numbers by 1, fourth and eighth numbers by 5 and sixth number by 2 - and then mentally sum the results, keeping only the units digit (this is akin to addition modulo 10). This is the apex number. Note that if the fourth and eighth number have the same parity, they can be ignored because they will cancel each other out, so summation of the first, last and twice the middle card is all that is needed and the prediction is made.

3 Summary

We have shown just a few magic tricks that can be performed using the different moduli versions of Pascal's triangle. Each trick can be adapted to different initial conditions by exploring the fractal structure of the triangle. Indeed, the fractal coloring of different moduli Pascal triangles is intriguing. Observation of the emerging structure of the fractals leads to pictorial proofs regarding the position of Φ-simple, "minimal non-zero element" and similar-structure rows. These immediately provide the magician with many more puzzles and tricks that can be generated. We intend to study further generalizations, including possible uses of different shapes in the fractal other than triangles, and developing tricks using the different diagonals of the moduli versions of Pascal's triangle. On a final note, we would like to encourage educators and others to use the ideas

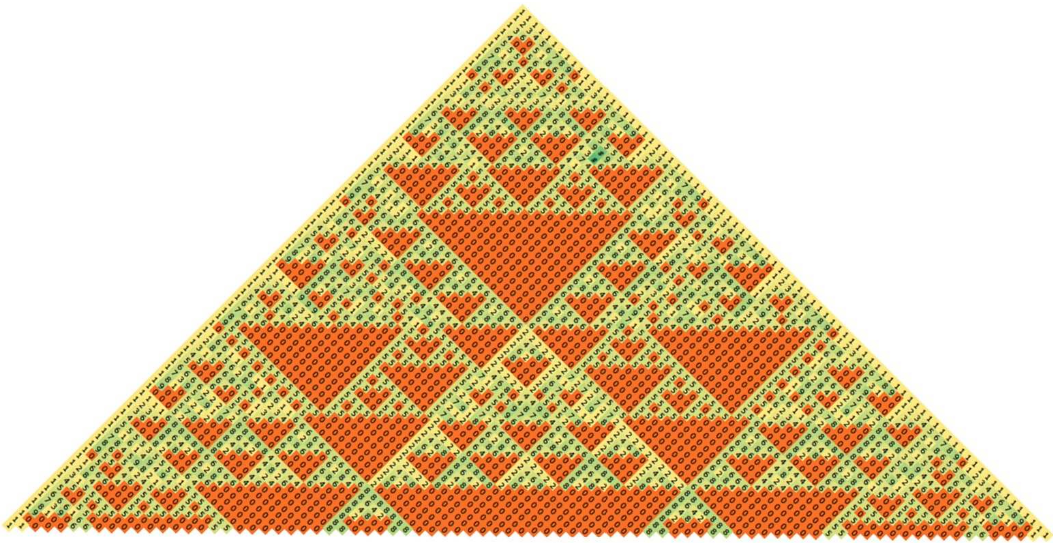

Figure 4: Color-coded mod 10 Pascal Triangle fractal.

developed in this paper for educational purposes. It is our firm belief, and indeed our own experience, that introducing mathematical topics through recreational mathematics is well received and inspiring for teachers and students and can help eradicate negative attitudes towards mathematics.

References

[1] Martin Gardner. "Pascal's Triangle", *Mathematical Carnival*, Mathematical Association of America, Washington D.C., 1989.

[2] `http://cardcolm-maa.blogspot.co.il/2012/10/all-or-nothing-trickle-treat.html`

[3] N. Gary. "The Pyramid Collection", *The Australian Mathematics Teacher Journal*, 61, pp. 9-13, 2005.

[4] E. Behrends, S. Humble. "Triangle Mysteries", *The Mathematical Intelligencer*, 35, pp. 10-15, 2013.