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The Secrets of Notakto: Winning at X-only Tic-Tac-Toe

Thane E. Plambeck, Greg Whitehead
Gathering for Gardner, Counterwave
tplambeck@gmail.com

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The Secrets of Notakto: Winning at X-only Tic-Tac-Toe

Thane E. Plambeck, Greg Whitehead
Gathering for Gardner, Counterwave
tplambeck@gmail.com

Abstract: *We analyze misere play of "impartial" tic-tac-toe—a game suggested by Bob Koca in which* **both** *players make X's on the board, and the first player to complete three-in-a-row* **loses**. *This game was recently discussed on mathoverflow.net in a thread created by Timothy Y. Chow.*

Key-words: combinatorial game theory, misere play, tic-tac-toe.

1 Introduction

Suppose tic-toe-toe is played on the usual 3×3 board, but where both players make X's on the board. The first player to complete a line of three-in-a-row *loses* the game.

Who should win? The answer for a single 3×3 board is given in a recent mathoverflow.net discussion [Chow]:

> *In the 3×3 misere game, the first player wins by playing in the center, and then wherever the second player plays, the first player plays a knight's move away from that.*

Kevin Buzzard pointed out that any other first-player move loses:

> *The reason any move other than the centre loses for [the first player to move] in the 3×3 game is that [the second player] can respond with a move diametrically opposite [the first player's] initial move. This makes the centre square unplayable, and then player two just plays the "180 degree rotation" strategy which clearly wins.*

In this note we generalize these results to give a complete analysis of multiboard impartial tic-tac-toe under the disjunctive misere-play convention.

2 Disjunctive misere play

A *disjunctive* game of 3×3 impartial tic-tac-toe is played not just with one tic-tac-toe board, but more generally with an arbitrary (finite) number of such boards forming the start position. On a player's move, he or she selects a single one of the boards, and makes an X on it (a board that already has a three-in-a-row configuration of X's is considered unavailable for further moves and out of play).

Play ends when every board has a three-in-a-row configuration. The player who completes the last three-in-a-row on the last available board is the loser.

3 The misere quotient of 3×3 impartial tic-tac-toe

We can give a succinct and complete analysis of the best misere play of an arbitrarily complicated disjunctive sum of impartial 3×3 tic-tac-toe positions by introducing a certain 18-element commutative monoid Q given by the presentation

$$Q = \langle\, a, b, c, d \mid a^2 = 1,\ b^3 = b,\ b^2c = c,\ c^3 = ac^2,\ b^2d = d,\ cd = ad,\ d^2 = c^2 \rangle. \tag{1}$$

The monoid Q has eighteen elements

$$Q = \{1, a, b, ab, b^2, ab^2, c, ac, bc, abc, c^2, ac^2, bc^2, abc^2, d, ad, bd, abd\}, \tag{2}$$

and it is called the **misere quotient** of impartial tic-tac-toe[1].

A complete discussion of the misere quotient theory (and how Q can be calculated from the rules of impartial tic-tac-toe) is outside the scope of this document. General information about misere quotients and their construction can be found in [MQ1], [MQ2], [MQ3], and [MQ4]. One way to think of Q is that it captures the misere analogue of the "nimbers" and "nim addition" that are used in normal play disjunctive impartial game analyses, but localized to the play of this particular impartial game, misere impartial 3×3 tic-tac-toe.

In the remainder of this paper, we simply take Q as given.

[1]For the cognoscenti: Q arises as the misere quotient of the hereditary closure of the sum G of two impartial misere games $G = 4 + \{2+, 0\}$. The game $\{2+, 0\}$ is the misere canonical form of the 3×3 single board start position, and "4" represents the nim-heap of size 4, which also happens to occur as a single-board position in impartial tic-tac-toe. In describing these misere canonical forms, we've used the notation of John Conway's *On Numbers and Games*, on page 141, Figure 32.

4 Outcome determination

Figure 6 (on page 53, after the References) assigns an element of Q to each of the conceivable 102 non-isomorphic positions[2] in 3×3 single-board impartial tic-tac-toe.

To determine the outcome of a multi-board position (ie, whether the position is an *N-position*—a Next player to move wins in best play, or alternatively, a *P-position*—second player to move wins), one first multiplies the corresponding elements of Q from the dictionary together. The resulting word is then reduced via the relations 1, that we started with above, necessarily eventually arriving at at one of the eighteen words (in the alphabet a, b, c, d) that make up the elements of Q.

If that word ends up being one of the four words in the set P

$$P = \{a, b^2, bc, c^2\}, \tag{3}$$

the position is P-position; otherwise, it's an N-position.

5 Example analysis

To illustrate outcome calculation for Impartial Tic-Tac-Toe, we consider the two-board start position shown in Figure 1.

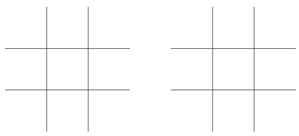

Figure 1: The two-board start position.

Consulting Figure 6, we find that the monoid-value of a single empty board is c. Since we have two such boards in our position, we multiply these two values together and obtain the monoid element

$$c^2 = c \cdot c.$$

Since c^2 is in the set P (equation (3)), the position shown in Figure 1 is a *second* player win. Supposing therefore that we helpfully encourage our opponent to make the first move, and that she moves to the center of one of the boards, we arrive at the position shown in Figure 2.

[2] We mean "non-isomorphic" under a reflection or rotation of the board. In making this count, we're including positions that couldn't be reached in actual play because they have too many completed rows of X's, but that doesn't matter since all those elements are assigned the identity element of Q.

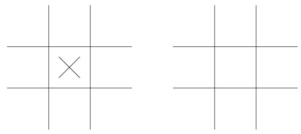

Figure 2: A doomed first move from the two-board start position.

It so happens that if we mimic our opponent's move on the other board, this happens to be a winning move. We arrive at the position shown in Figure 3, each of whose two boards is of value c^2; multiplying these two together, and simplifying via the relations shown in equation (1), we have

$$\begin{aligned} c^4 &= c^3 \cdot c \\ &= ac^2 \cdot c \\ &= ac^3 \\ &= aac^2 \\ &= c^2, \end{aligned}$$

which is a P-position, as desired.

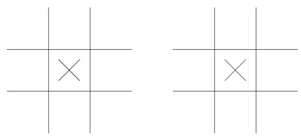

Figure 3: Mimicry works here, but not in general.

So is the general winning strategy of the two-board position simply to copy our opponent's moves on the other board? Far from it: consider what happens if our opponent should decide to complete a line on one of the boards—copying that move on the other board, we'd **lose** rather than win! For example, from the N-position shown in Figure 5, there certainly is a winning move, but it's *not* to the upper-right-hand corner of the board on the right, which loses.

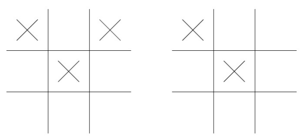

Figure 4: An N-position in which mimickry loses.

We invite our reader to find a correct reply!

6 The iPad game Notakto

Evidently the computation of general outcomes in misere tic-tac-toe is somewhat complicated, involving computations in finite monoid and looking up values from a table of all possible single-board positions.

However, we've found that a human can develop the ability to win from multi-board positions with some practice.

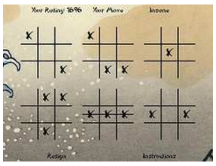

Figure 5: A six-board game of Notakto, in progress.

Notakto "No tac toe" is an iPad game that allows the user to practice playing misere X-only Tic-Tac-Toe against a computer. Impartial misere tic-tac-toe from start positions involving one up to as many as six initial tic-tac-toe boards are supported. The Notakto iPad application is available for free at `http://www.notakto.com`.

7 Final question

Does the 4×4 game have a finite misere quotient?

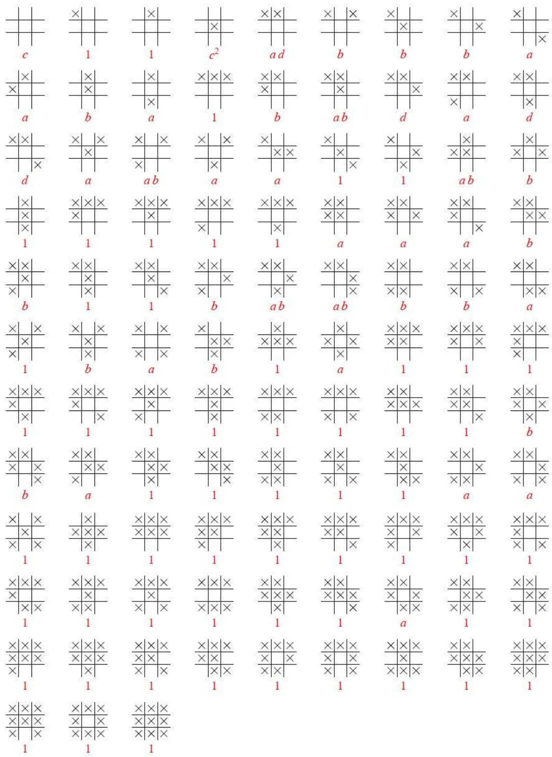

Figure 6: The 102 nonisomorphic ways of arranging zero to nine X's on a board, each shown together with its corresponding misere quotient element from Q.

References

[Chow] Timothy Y. Chow, Online mathoverflow.net question, “Neutral tic tac toe”, 16 May 2010.
http://mathoverflow.net/questions/24693/neutral-tic-tac-toe

[MQ1] http://arxiv.org/abs/math/0501315

[MQ2] http://arxiv.org/abs/math/0609825

[MQ3] http://arxiv.org/abs/0705.2404

[MQ4] http://www.miseregames.org