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Vanishing Area Puzzles

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Games and Puzzles

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In Memoriam Martin Gardner

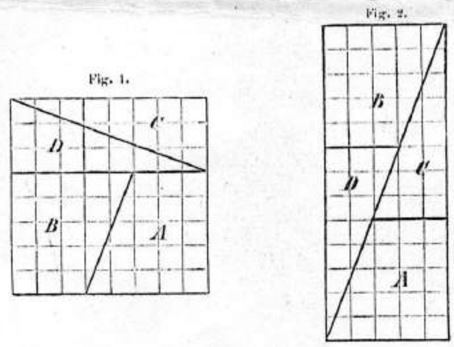
Abstract: *Martin Gardner was very fond of vanishing area puzzles and devoted two chapters to them in his first book. There are actually two distinct types. Sam Loyd's Vanishing Chinaman and similar puzzles have pictures which are reassembled so a part of the picture appears to disappear. But the physical area remains fixed. The second type cuts up an area and reassembles it to produce more or less area, as in the classic chessboard dissection which converts the* 8×8 *square into a* 5×13 *rectangle. Gardner had managed to trace such puzzles back to Hooper in* 1774*. In* 1989*, I was visiting Leipzig and reading Schwenter which referred to an error of Serlio, in his book of* 1535*. Serlio hadn't realised that his dissection and reassembly gained area, but it is clear and this seems to be the origin of the idea. I will describe the history and some other versions of the idea.*

Key-words: Sam Loyd, vanishing area puzzles.

One of Sam Loyd's most famous puzzles is "The Vanishing Chinaman" or "Get off the Earth". Martin Gardner discussed this extensively in [7] and [8].

Figure 1: "Get off the Earth".

However, the term "vanishing area puzzle" is used for two different types of puzzle. In "The Vanishing Chinaman", no actual area vanishes - it is one of the figures in the picture that vanishes, so perhaps we should call this a "vanishing object puzzle". In a true "vanishing area puzzle", an area is cut into several pieces and reassembled to make an area which appears to be larger or smaller than the original area. These are considerably older than vanishing object puzzles. Martin was fond of these puzzles - indeed, his first puzzle book [6] devotes two chapters to such puzzles - still the best general survey of them - and he also wrote several columns about them [7, 8]. The most famous version of these is the "Checkerboard Paradox" where an 8×8 checkerboard is cut into four parts and reassembled into a 5×13 rectangle, with a net gain of one unit of area. This article is primarily concerned with the early history of such puzzles.

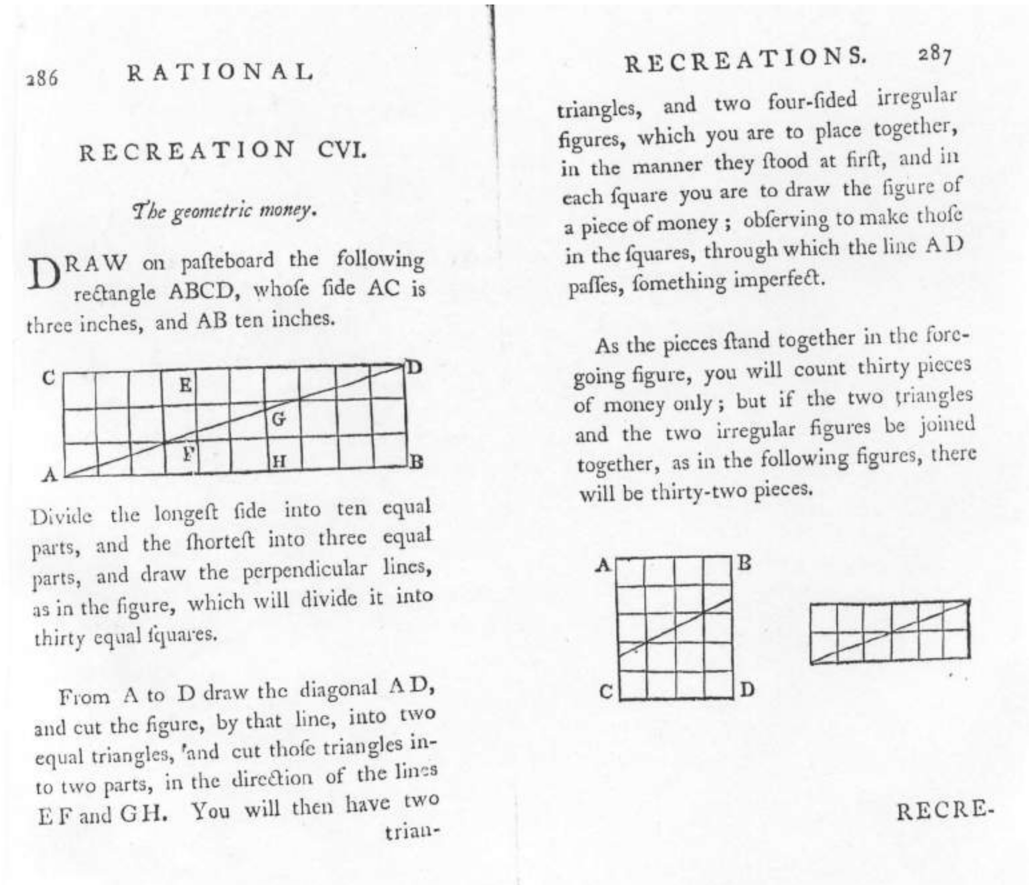


IX. Ein geometrisches Paradoxon. Um ad oculos zu demonstriren, dass das Schachbret nicht nur 64, sondern auch 65 Felder besitzt, schneide man dasselbe aus starkem Papier, zerlege es auf die in Fig. 1

angegebene Weise in vier, zu je zweien congruente Stücke *A*, *B*, *C*, *D* und setze diese zu einem Rechtecke zusammen, welches, wie Fig. 2 zeigt, die Grundlinie 5 und die Höhe 13 besitzt also 65 Felder enthält. — Wir theilen diese kleine Neckerei mit, weil die Aufsuchung des begangenen Fehlers eine hübsche Schüleraufgabe bildet und weil sich an die Vermeidung des Fehlers die Lösung und Construction einer quadratischen Gleichung knüpfen lässt. Schl.

Figure 2: Schlomilch.

Gardner describes some 18 & 19C versions of this puzzle idea, going back to Hooper (1774) [9], and a surprising connection with Fibonacci numbers discovered in 1877 [4]. In 1988, I was visiting Leipzig and looked at some obscure books in their library and discovered references going back to 1537.



286 RATIONAL

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The geometric money.

DRAW on paſteboard the following rectangle ABCD, whoſe ſide AC is three inches, and AB ten inches.

Divide the longeſt ſide into ten equal parts, and the ſhorteſt into three equal parts, and draw the perpendicular lines, as in the figure, which will divide it into thirty equal ſquares.

From A to D draw the diagonal AD, and cut the figure, by that line, into two equal triangles, and cut thoſe triangles into two parts, in the direction of the lines EF and GH. You will then have two triangles, and two four-ſided irregular figures, which you are to place together, in the manner they ſtood at firſt, and in each ſquare you are to draw the figure of a piece of money; obſerving to make thoſe in the ſquares, through which the line AD paſſes, ſomething imperfect.

As the pieces ſtand together in the foregoing figure, you will count thirty pieces of money only; but if the two triangles and the two irregular figures be joined together, as in the following figures, there will be thirty-two pieces.

RECREATIONS. 287

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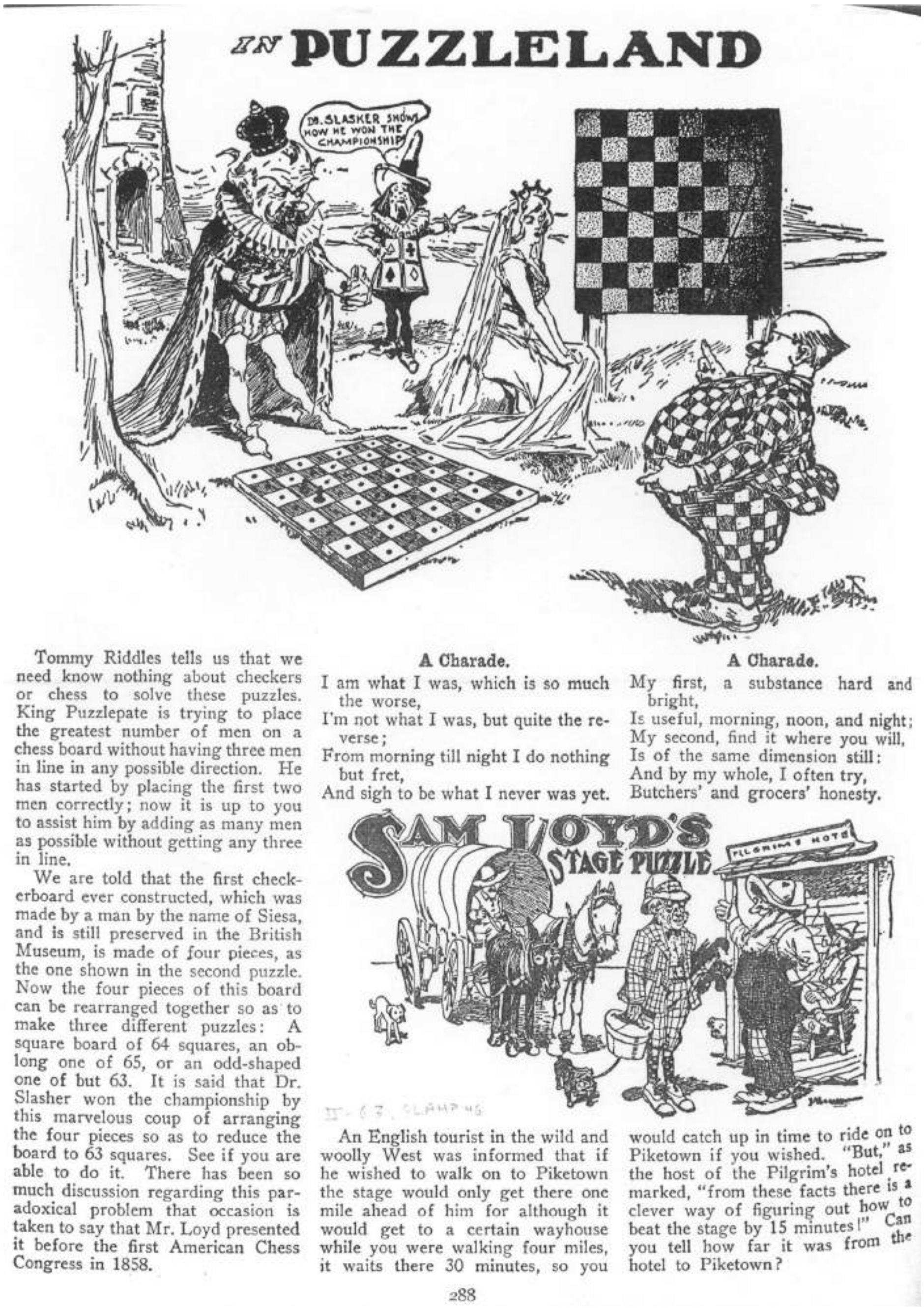
Figure 3: Hooper.

Let us look at the classic 8×8 to 5×13 of Fig. 2. The history of this particular version is obscure. It is shown in Loyd's Cyclopedia [10] (p. 288 & 378).



IN PUZZLELAND

DR. SLASKER SHOWS HOW HE WON THE CHAMPIONSHIP

Tommy Riddles tells us that we need know nothing about checkers or chess to solve these puzzles. King Puzzlepate is trying to place the greatest number of men on a chess board without having three men in line in any possible direction. He has started by placing the first two men correctly; now it is up to you to assist him by adding as many men as possible without getting any three in line.

We are told that the first checkerboard ever constructed, which was made by a man by the name of Siesa, and is still preserved in the British Museum, is made of four pieces, as the one shown in the second puzzle. Now the four pieces of this board can be rearranged together so as to make three different puzzles: A square board of 64 squares, an oblong one of 65, or an odd-shaped one of but 63. It is said that Dr. Slasher won the championship by this marvelous coup of arranging the four pieces so as to reduce the board to 63 squares. See if you are able to do it. There has been so much discussion regarding this paradoxical problem that occasion is taken to say that Mr. Loyd presented it before the first American Chess Congress in 1858.

A Charade.

I am what I was, which is so much the worse,
I'm not what I was, but quite the reverse;
From morning till night I do nothing but fret,
And sigh to be what I never was yet.

A Charade.

My first, a substance hard and bright,
Is useful, morning, noon, and night;
My second, find it where you will,
Is of the same dimension still:
And by my whole, I often try,
Butchers' and grocers' honesty.

SAM LOYD'S STAGE PUZZLE

PILGRIM'S HOTEL

An English tourist in the wild and woolly West was informed that if he wished to walk on to Piketown the stage would only get there one mile ahead of him for although it would get to a certain wayhouse while you were walking four miles, it waits there 30 minutes, so you would catch up in time to ride on to Piketown if you wished. "But," as the host of the Pilgrim's hotel remarked, "from these facts there is a clever way of figuring out how to beat the stage by 15 minutes!" Can you tell how far it was from the hotel to Piketown?

288

Figure 4: LoydCyc288.

For a long time, I tended to ignore this as it seemed smudged. I later saw that he shows that both rectangles have chessboard colouring, and he is the first to indicate this. But when I went to scan Fig. 5, I realised that the smudge is

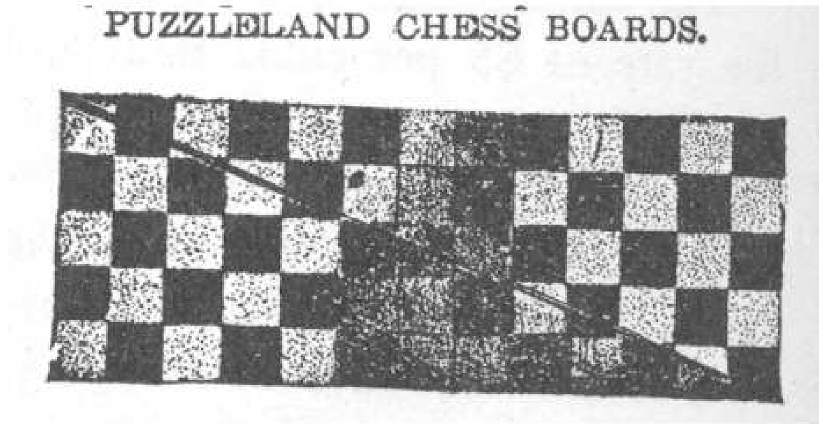


Figure 5: LoydCyc278.

deliberate to conceal the fact that the colouring does NOT match up! Indeed, the corners of a 5×13 board should all be the same colour, but two of them in the solution arise from adjacent corners of the 8×8 chessboard and have opposite colours!

Loyd also poses the related problem of arranging the four pieces to make a figure of area 63, as in Fig. 3.

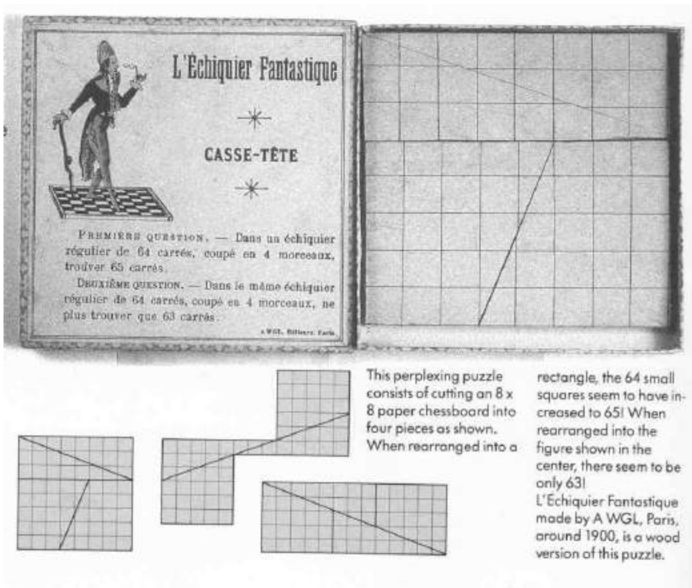


Figure 6: AWGL.

The oldest known version of this is an actual puzzle, dated c1900, [1], shown in [18] and there is a 1901 publication [5].

Loyd asserts he presented "this paradoxical problem" to the First American Chess Congress in 1858, but it is not clear if he means the area 65 version or the area 63 version. Loyd would have been 17 at the time. If this is true, he is ten years before any other appearance of the area 65 puzzle and about 42 years before any other appearance of the area 63 puzzle. I am dubious about this as Loyd did not claim this as his invention in other places where he was describing his accomplishments. In 1928, Sam Loyd Jr. [11] describes the area 63 version as something he had discovered, but makes no claims about the area 65 form, although he often claimed his father's inventions as his own. For example, on p. 5, he says "My "Missing Chinaman Puzzle" of 1896.

The first known publication of the 8×8 to 5×13, Fig. 2, is in 1868, in a German mathematical periodical, signed Schl. [14]. In 1938, Weaver [12] said the author was Otto Schlömilch, and this seems right as he was a co-editor of the journal at the time. In 1953, Coxeter [3] said it was V. Schlegel, but he apparently confused this with another article on the problem by Schlegel. Schlömilch doesn't give any explanation for this "teaser", leaving it as a student exercise!

In 1886, a writer [13] says: "We suppose all the readers . . . know this old puzzle."

By 1877 [4], it was recognised that the paradox is related to the fact that $5 \times 13 - 8 \times 8 = 1$ and that 5, 8, 13 are three consecutive Fibonacci numbers. Taking a smaller example based on the numbers 2, 3, 5 makes the trickery clear.

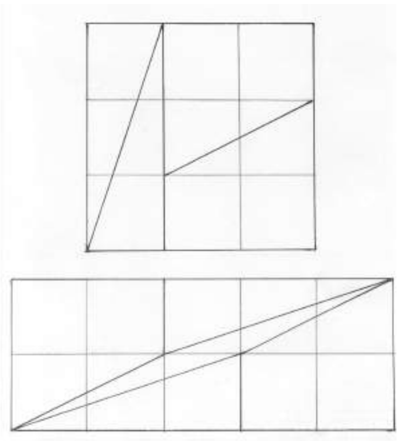


Figure 7: Nine to Ten.

One can also make a 5×5 into a 3×8, but then there is a loss of area from the square form.

However, there are other versions of vanishing area or object puzzles. Since

1900 several dozen have been devised and there are examples where both some area and an object vanish!

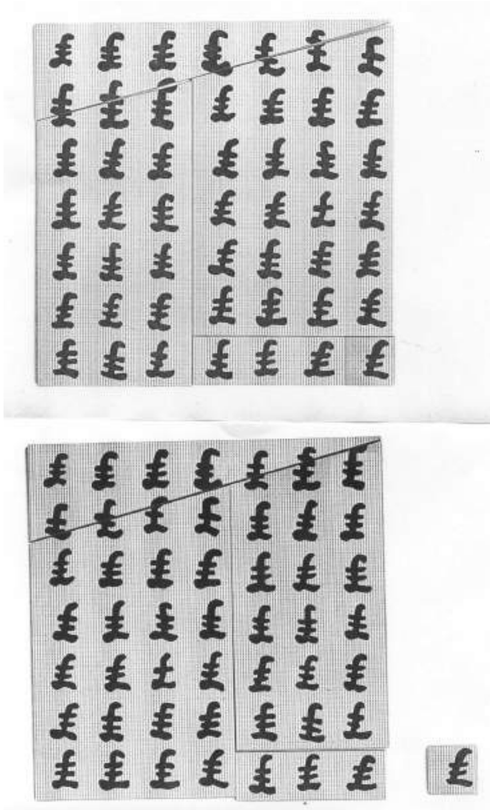


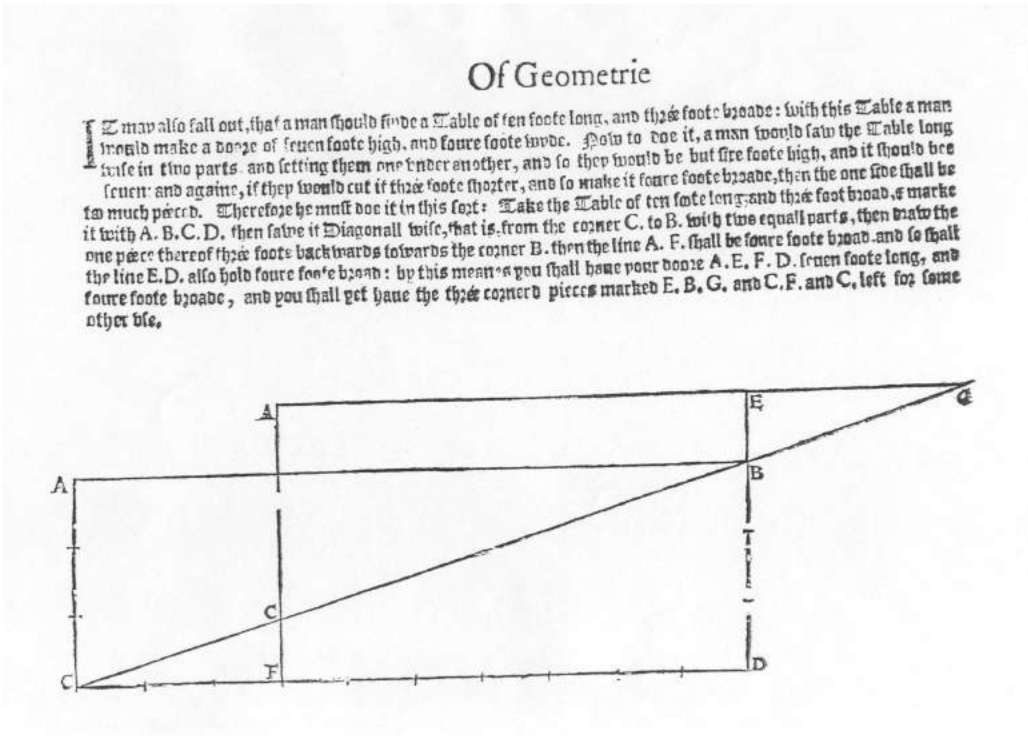
Figure 8: CreditSqueeze.

This is a fairly common magician's trick, taking a 7×7 square array of "£" signs and rearranging to get a 7×7 square array of "£" signs and an extra unit square containing an extra "£" sign. My version, called "Credit Squeeze", using "£", is attributed to Howard Gower, but Michael Tanoff kindly obtained for me an American version, using \$, called "It's Magic DOLLAR DAZE", produced by Abbott's with no inventor named. Lennart Green uses a version of this in his magic shows, but he manages to reassemble it three times, getting an extra piece out each time! Needless to say, this involves further trickery. A version of this is available on the Internet.

But there are earlier examples. Gardner and others tracked the idea back to Hooper [9] in 1783, as seen in Fig. 3. Here we have a 3×10 cut into four pieces which make a 2×6 and a 4×5. However, Hooper's first edition of 1774 erroneously has a 3×6 instead of a 2×6 rectangle and notes there are now 38 units of area. This was corrected in the second edition of 1783 and this version occurs fairly regularly in the century following Hooper.

In 1989, I visited Leipzig and was reading Gaspar Schott [15] where I found a description of a version due to the 16C architect Serlio [17]. I managed to find the Serlio reference, which is in his famous treatise in five books on architecture.

Sebastiano Serlio (1475-1554) was born in Bologna and worked in Rome in 1514-1527 with the architect Peruzzi. He went to Venice and began publishing his Treatise, which appeared in five parts in 1537-1547. This was a practical book and greatly influential. He influenced Inigo Jones and Christopher Wren. Wren's design of the Sheldonian Theatre in Oxford is based on Serlio's drawings of the Roman Theatre of Marcellus. Serlio also describes the "Chinese Lattice" method of spanning a roof using beams shorter than the width. This was studied by Wallis, leading to a system of 25 linear equations for the Sheldonian roof. In 1541, François I summoned Serlio to France and he founded the classical school of architecture in France. He designed the Château of Ancy-le-Franc and died at Fontainebleu.


Of Geometrie

It may also fall out, that a man should finde a Table of ten foote long, and three foote broade: with this Table a man would make a doore of seven foote high, and foure foote wyde. Now to doe it, a man would saw the Table long wise in two parts and setting them one under another, and so they would be but sixe foote high, and it should bee seven: and againe, if they would cut it three foote shorter, and so make it foure foote broade, then the one side shall be too much peeced. Therefore he must doe it in this sort: Take the Table of ten foote long, and three foot broad, & marke it with A. B. C. D. then sawe it Diagonall wise, that is, from the corner C. to B. with two equall parts, then drawe the one peece thereof three foote backwards towards the corner B. then the line A. F. shall be foure foote broad, and so shall the line E.D. also hold foure foote broad: by this meanes you shall have your doore A. E. F. D. seven foote long, and foure foote broade, and you shall yet have the three cornerd pieces marked E. B. G. and C. F. and C, left for some other use.

Figure 9: SerlioRot.

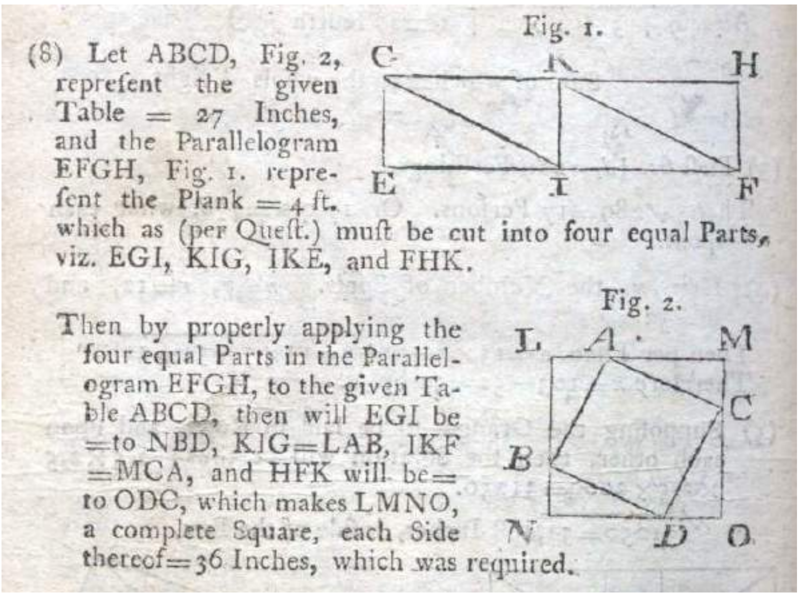
Fig. 9 comes from the 1982 Dover English edition, f. 12v. He is taking a 3×10 board and cuts it diagonally, then slides one piece by 3 to form an area 4×7 with two bits sticking out, which he then trims away. He doesn't notice that this implies that the two extra bits form a 1×3 rectangle and hence doesn't realize the change in area implied.

Already in 1567, Pietro Cataneo [2] pointed out the mistake and what the correct process would be.

I later found a discussion of this in Schwenter [16], citing another architect, so this was well known in the $16-17$C, but the knowledge disappeared despite the fact that Serlio's book has been in print in Italian, Dutch, French and English since that time and Schwenter was fairly well-known.

Since that time, I have found two other late 18C examples, possibly predating Hooper.

Charles Vyse's *The Tutor's Guide* [19], was a popular work, going through at least 16 editions, during 1770-1821. The problem is: "A Lady has a Dressing Table, each side of which is 27 Inches, but she is desirous to know how each Side of the same may = 36 Inches, by having 4 foot of Plank, superficial Measure, joined to the same. The Plan in what Manner the Plank must be cut and applied to the Table is required?" [The plank is one foot wide.] The solution is in: *The Key to the Tutor's Guide* [19] (p. 358).


(8) Let ABCD, Fig. 2, represent the given Table = 27 Inches, and the Parallelogram EFGH, Fig. 1. represent the Plank = 4 ft. which as (per Quest.) must be cut into four equal Parts, viz. EGI, KIG, IKE, and FHK.

Then by properly applying the four equal Parts in the Parallelogram EFGH, to the given Table ABCD, then will EGI be = to NBD, KIG=LAB, IKF =MCA, and HFK will be= to ODC, which makes LMNO, a complete Square, each Side thereof=36 Inches, which was required.

Figure 10: Vyse's Solution.

She cuts the board into two 12" × 24" rectangles and cuts each rectangle along a diagonal. By placing the diagonals of these pieces on the sides of her table, she makes a table 36" square. But the diagonals of these triangles are $12\sqrt{5} = 26.83\ldots$".

Note that $27^2 + 12 \times 24 = 1305$ while $36^2 = 1296$. Vyse is clearly unaware that area has been lost. By dividing all lengths by 3, one gets a version where one unit of area is lost. Note that 4, 8, 9 is almost a Pythagorean triple. I have not seen the first edition of this work, but the problem is likely to occur in the first edition of 1770 and hence predates Hooper. Like Serlio, the author is unaware that some area has vanished!

The 1778 edition of Ozanam by Montucla [12] has an improvement on Hooper.

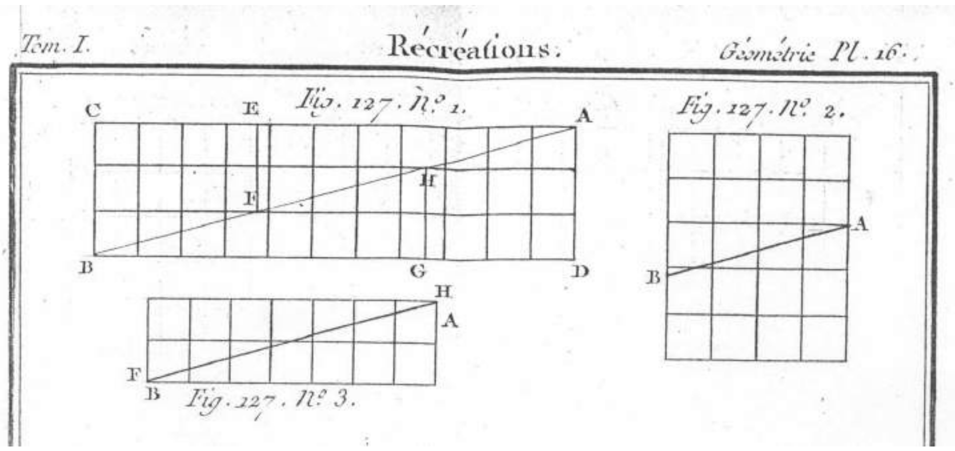


Figure 11: Ozanam.

The image is Fig. 127, plate 16, p. 363 for prob. 21, pp. 302-303 in my copy of the 1790 reissue. This has 3×11 to 2×7 and 4×5. Here just one unit of area is gained, instead of two units as in Hooper. He remarks that M. Ligier probably made some such mistake in showing $172 = 2 \times 122$ and this is discussed further on the later page.

In conclusion, we have found that vanishing area puzzles are at least two hundred years older than Gardner had found. We have also found a number of new forms of the puzzle. Who knows what may turn up as we continue to examine old texts? I think Martin would have enjoyed these results.

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[13] Richard A. Proctor. *Some puzzles.* Knowledge 9, pp. 305-306, Aug 1886.

[14] Schl. [probably Otto Schlömilch]. "Ein geometrisches Paradoxon". Z. Math. Phys. 13, 162, 1868.

[15] Gaspar Schott. *Magia Universalis.* Joh. Martin Schönwetter, Bamberg, Vol. 3, pp. 704-708, 1677.

[1] As Chap 3 in each of these.

[2] The image is taken from my copy of the 2nd ed.

[3] This is a reprint of Loyd's Our Puzzle Magazine, a quarterly which started in June 1907 and ran till 1908. From known issues, it appears that these problems would have appeared in Oct 1908 and Jun 1908, but I don't know if any copies of these issues exist.

[4] Numerous editions then appeared in Paris and Amsterdam, some in one volume; About 1723, the work was revised into 4 volumes, sometimes described as 3 volumes and a supplement, published by Claude Jombert, Paris, 1723. "The editor is said to be one Grandin.". In 1778, Jean Étienne Montucla revised this, under the pseudonym M. de C. G. F. [i.e. M. de Chanla, géomètre forézien], published by Claude Antoine Jombert, fils aîné, Paris, 1778, 4 volumes. The author's correct initials appear in the 1790 reissue] This is a considerable reworking of the earlier versions. In particular, the interesting material on conjuring and mechanical puzzles in Vol. IV has been omitted. The bibliography of Ozanam's book is complicated. I have prepared a detailed 7 pp. version covering the 19 (or 20) French and 10 English editions, from 1694 to 1854, as well as 15 related versions - this is part of my *The Bibliography of Some Recreational Mathematics Books.* The above is an extract.

[16] Daniel Schwenter. *Deliciæ Physico-Mathematicae*. Oder Mathemat- und Philosophische Erquickstunden, Jeremiæ Dümlers, Nuremberg, p. 451, 1636. [5]

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[5] Probably edited for the press by Georg Philip Harsdörffer. Extended to three volumes by Harsdörffer in 1651 & 1653, with vol. 1 being a reprint of the 1636 volume.

[6] This is the first part of his *Architettura*, 5 books, Venice(?), 1537-1547, first published together in 1584. There are numerous editions in several languages, including a 1982 Dover reprint of the 1611 English edition.

[7] The problem is in the 10th ed., ed. by J. Warburton. S. Hamilton for G. G. and J. Robinson, London, 1799, prob. 8, p. 317. The solution, Fig. 10, is in: *The Key to the Tutor's Guide*, 8th ed., G. & J. Robinson, p. 358, London, 1802.