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Doppelgänger Placement Games

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Doppelgänger Placement Games*†

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Abstract: *Are some games Doppelgänger? Can a nice placid game, one that you'd play with your grandmother, have a nasty evil twin that invites fights? Depending on the (lack of) strength of your (mathematical) glasses, it is possible but we show that with good glasses, games with simple rules are not Doppelgänger.*

Key-words: Combinatorial games, graphs, polynomial profile, SNORT, COL.

1 Introduction

A *combinatorial game* is a game of perfect information and no chance, with two players called Left and Right (or Louise and Richard) who play alternately, for example CHESS, CHECKERS and GO. The word 'game' has many meanings in English, for us, a *game* is a set of rules and a *position* is an instance of the game on some board. We are interested in *placement* games; those in which the players place pieces on a board and thereafter the pieces are never moved or removed from play. For example, COL and SNORT are both usually played on a chessboard and in both Louise places a blue piece on a square, Richard a red. However, in COL pieces of the same colour cannot be adjacent, in SNORT two pieces of opposite colour cannot be adjacent. The question of Doppelgänger games arises in [4] where it is shown that SNORT is the rowdy twin to COL. We need a few definitions to make this question meaningful.

Definition 1.1. *Let G be a placement game played on a board (graph) B. Let n be the number of vertices of B. The pieces are placed on the vertices of B. The bi-variate polynomial profile of G on B is the bivariate polynomial*

$$P_{G,B}(x,y) = \sum_{k=0}^{n} \sum_{j=0}^{k} f_{j,k-j} x^j y^{k-j}$$

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where $f_{j,k-j}$ is the number of legal positions of G on B which have j Left pieces and $k-j$ Right pieces. The polynomial profile is obtained by putting $x = y$

$$P_{G,B}(x) = \sum_{k=0}^{n} f_k(G,B)x^k$$

where now $f_k(G,B)$ is the number of positions with exactly k pieces.

Of course, $P_{G,B}(1)$ just counts the total number of positions. Enumerating the number of positions or the number of special positions is a relatively new endeavour, see [4, 6, 7, 8, 9, 11] for connections to Bernoulli numbers of the second kind, Catalan numbers, Dyck paths, amongst others.

For COL on a strip of 3 squares, the total number of positions are

$$\cdot\cdot\cdot,\quad \cdot\cdot B,\quad \cdot B\cdot,\quad B\cdot\cdot,\quad \cdot\cdot R,\quad \cdot R\cdot,\quad R\cdot\cdot,$$

$$R\cdot B,\quad RB\cdot,\quad BR\cdot,\quad \cdot RB,\quad B\cdot R,\quad \cdot BR,\quad B\cdot B,\quad R\cdot R,$$

$$RBR,\quad BRB$$

and

$$\begin{aligned} P_{\text{COL},P_3}(x,y) &= 1+3x+3y+6xy+x^2+y^2+x^2y+xy^2; \\ P_{\text{COL},P_3}(x) &= 1+6x+8x^2+2x^3; \qquad P_{\text{COL},P_3}(1) = 17. \end{aligned}$$

For SNORT,

$$\begin{aligned} P_{\text{SNORT},P_3}(x,y) &= 1+3x+3y+2xy+3x^2+3y^2+x^3+y^3; \\ P_{\text{SNORT},P_3}(x) &= 1+6x+8x^2+2x^3; \qquad P_{\text{SNORT},P_3}(1) = 17. \end{aligned}$$

Over a class of boards $\mathcal{B}$, games G and H are called $\mathcal{B}-$ *Doppelgänger* if $P_{G,B}(x) = P_{H,B}(x)$ for all boards B in $\mathcal{B}$. Expanding upon the question from [4] we pose two questions. The focus of the paper is on the second.

The Doppelgänger questions:
1: Given two games G and H, what is the maximum set of boards $\mathcal{B}$ for which G and H are $\mathcal{B}-$ Doppelgänger?
2: Given a set of boards $\mathcal{B}$ what set of games, $\{G_1, G_2, \ldots G_n\}$, are mutually $\mathcal{B}-$ Doppelgänger?

Informally, the smaller the set of boards being considered the weaker your mathematical glasses are. A wider selection of boards allows for more chances to discriminate between the games.

If $\mathcal{B}$ is the set of $m \times n$ checkerboards then COL and SNORT are $\mathcal{G}$-Doppelgänger. Even more, they are Doppelgänger for all bipartitite graphs but no larger class ([4]) thereby answering our Question 1.

We are interested in games in which the legal, or rather the illegal, placement of pieces depends only on the distances to already played pieces. A *distance placement* game (dp-game) G is described by two sets of numbers: S_G which gives the illegal distances for two of the same pieces; and D_G is the illegal distances for two different pieces. For COL and SNORT we have $S_{\text{COL}} = \{1\} = D_{\text{SNORT}}$ and $D_{\text{COL}} = \emptyset = S_{\text{SNORT}}$.

Note that not all placement games are described by distance sets. NOCOL is COL with the extra condition that every piece must be adjacent to a non-occupied vertex. Also, DOMINEERING [2] is a non-dp but an independent placement game, and NOGO [5] (also known as *anti-atari go* [10]) is a non-independent placement game. Both games are played by "real" people.

Distance games are a subset of *independence placement* games [9]. Independence games have an associated auxiliary graph and the legal positions in the game on the board correspond to independent sets in the auxiliary graph. To show how this works for distance games, let G be a distance game and B be a board (graph) with vertices $v_1, v_2, \ldots, v_n$. We take two copies of the vertices and call them $(v_1, b), (v_2, b), \ldots, (v_n, b)$ and $(v_1, r), (v_2, r), \ldots, (v_n, r)$ and form an auxiliary board $A(B)$. We imagine that only Louise can play on a vertex with a b and Richard on a vertex with an r. We now connect two vertices $(x, y) \sim (v, w)$ in $A(B)$ as follows:

- $x = v$ — since no vertex can be played twice;
- $d(x, v) \in S_P$ and $y = w$ — illegal distance for two of the same pieces;
- $d(x, v) \in D_P$ and $y \neq w$ — illegal distance for two different pieces;

Now it should be clear that: *Let G be a dp-game on a board B. A legal position on B corresponds to an independent set on $A(B)$.* However, for the purposes of this paper, instead of focusing on the negative we will consider the positive. Given a game G and a board B with vertices $1, 2, \ldots, n$ a position will be given by the *word* $b_{i_1}, b_{1_2}, \ldots, b_{i_k}, r_{j_1}, r_{j_2}, \ldots, r_{i_l}$ where the vertices $i_1, \ldots, i_k$ are occupied by blue pieces and $j_1, \ldots, j_l$ are occupied by red.

In the next section we prove our two results. Theorem 2.1 says that, in general, there are no Doppelgänger games and the proof shows that given two dp-games, G and H, there is a relatively small and simple board that tells them apart. Any one actually playing a dp-game probably would play on a grid. We show, Theorem 2.2, that if the game contains an odd distance in its distance sets then it has a Doppelgänger on a grid as well as on bipartite graphs in general.

Aside: If G has few elements in D_G and many in S_G then it is relatively tame since a move leaves the opponent with more places to go than the player. If the $|D_G| > |S_G|$ then the reverse is true and could be called 'rowdy'. Theorem 2.2 can be used to see when a tame game will have a rowdy Doppelgänger. Although it is not required to understand this paper, for more on combinbatorial game theory please consult [1, 3].

2 Doppelgänger

Theorem 2.1. *No two different distance games are Doppelgänger on the set of all possible boards.*

Proof. Let G and H be two distance games with the distance sets $S_G = \{g_1, g_2, \ldots\}$, $S_H = \{h_1, h_2, \ldots\}$, $D_G = \{g'_1, g'_2 \ldots\}$, and $D_H = \{h'_1, h'_2, \ldots\}$. If the distance sets differ, let i and j be the smallest indices such that $g_i \neq h_i$ and $g'_j \neq h'_j$ respectively. There are several cases to consider, and we will show that, for each case, a board exists on which G and H are not Doppelgänger.

Case 1 ($S_G \neq S_H$ and $D_G = D_H$): Assume without loss of generality that $g_i < h_i$. Now consider the games G and H played on the path P_{g_i+1}. Since $diam(P_{g_i+1}) = g_i$, no two pieces will have a distance greater than g_i and we can ignore any elements in the distance sets greater than g_i. Thus essentially $S_G = S_H \cup \{g_i\}$. Then every position with two pieces that is legal in G is also legal on H. On the other hand though, $b_1 b_{g+1}$ and $r_1 r_{g+1}$ are legal positions of H, but not of G. Thus $f_2(H) = f_2(G) + 2$, proving that G and H are not Doppelgänger.

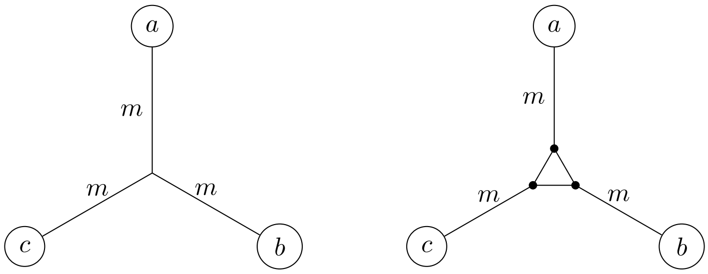
Case 2 ($S_G = S_H$ and $D_G \neq D_H$): The argument is along the same line as in the previous case.

Case 3 ($S_G \neq S_H$ and $D_G \neq D_H$): There are 3 subcases to consider:

Case 3a ($g_i < h_i$ and $g'_j < h'_j$): Let $m = \min\{g_i, g'_j\}$. Then repeat the argument of the first case on the path P_{m+1}.

Case 3b ($g_i < h_i$ and $g'_j > h'_j$ with $g_i \neq h'_j$): Let $m = \min\{g_i, h'_j\}$. Then repeat the argument of the first case on the path P_{m+1}.

Case 3c ($g_i < h_i$ and $g'_j > h'_j$ with $g_i = h'_j$): If $g_i = 2m$ is even, then we consider the board in Figure 1(A). If $g_i = 2m + 1$ is odd, we consider the board in Figure 1(B). Since the diameter of both graphs is g_i, we can again ignore all distances greater than this, thus essentially $S_G = S_H \cup \{g_i\}$ and $D_G \cup \{g_i\} = D_H$.



(A) $g_i = 2m$ (B) $g_i = 2m + 1$

Figure 1: Proof of Theorem 2.1: Boards for Case 3c.

We will look at the positions involving exactly three pieces played. Triples with at most one of the vertices a, b, or c will have all distances less than g_i, thus such a triple is a legal position of G if and only if it is a legal position of H. Let the number of triples with none of these vertices be K_0 and the number of triples involving one be K_1.

Now consider the number of triples involving two of the vertices a, b, and c. Let a_k be the vertex that has distance k from the vertex a where $k \leq m$ (i.e. a_k is on the same 'branch' as a) and similarly for b_k and c_k. For the case $g_i = 2m$, we have $a_m = b_m = c_m$. If $d_a e_b f_{a_k}$ is a legal position where $d, e, f \in \{b, r\}$, then through 'rotation' and due to symmetry of the board we also get the two legal positions $d_b e_c f_{b_k}$ and $d_c e_a f_{c_k}$, and similarly for $d_a e_b f_{b_k}$ and $d_a e_b f_{c_k}$. Thus we can partition the number of such triples into equivalence classes where two positions are equivalent if and only if they are rotations of each other. Each of these equivalence classes has size 3. Let the number of such triples in G be $3K_2$ and in H be $3K_2'$.

In G, the triple abc is illegal since no two of the vertices can have the same colour piece. In H though, the triples $b_a b_b b_c$ and $r_a r_b r_c$ are legal.

# of vertices of $\{a,b,c\}$ used	Number of triples in G	H
0	K_0	K_0
1	K_1	K_1
2	$3K_2$	$3K_2'$
3	0	2

Table 1: Proof of Theorem 2.1: Number of Triples for Case 3c.

Looking at the total number of triples in G and H, we have $f_2(G) = K_0 + K_1 + 3K_2 + 0$ and $f_2(H) = K_0 + K_1 + 3K_2' + 2$ (see Table 1). Considering these modulo 3, we have $f_2(G) \equiv_3 K_0 + K_1$ and $f_2(H) \equiv_3 K_0 + K_1 + 2$, which shows $f_2(G) \neq f_2(H)$. Thus G and H are not Doppelgänger. □

2.1 Bipartite Doppelgänger games

Let G be a dp-game played on a bipartite graph B. Let $V(B) = V_1 \cup V_2$ where V_1 and V_2 are the two sets in the bipartition of the vertices of B. We refine our position notation to

$$(b_{i_1}, b_{i_2}, \ldots), (b_{j_1}, b_{j_2}, \ldots), (r_{k_1}, r_{k_2}, \ldots), (r_{l_1}, r_{l_2}, \ldots)$$

where the $\{i_1, i_2, \ldots\} \cup \{k_1, k_2, \ldots\} \subset V_1$ and $\{j_1, j_2, \ldots\} \cup \{l_1, l_2, \ldots\} \subset V_2$. The *bipartite flip* of a board B, $BF(B)$, is a map from positions to positions defined by taking a position

$$(b_{i_1}, b_{i_2}, \ldots), (b_{j_1}, b_{j_2}, \ldots), (r_{k_1}, r_{k_2}, \ldots), (r_{l_1}, r_{l_2}, \ldots)$$

and returning the position

$$(b_{i_1}, b_{1_2}, \ldots), (b_{l_1}, b_{l_2} \ldots), (r_{k_1}, r_{k_2}, \ldots), (r_{j_1}, r_{j_2}, \ldots).$$

This process changes the colour of any piece on a vertex of V_2. Clearly, taking a position and applying the bipartite flip twice returns us to the original position.

Let G be a dp-game and let G' be the dp-game defined by

$$S_{G'} = \{2i : \text{ if } 2i \in S_G\} \cup \{2i+1 : \text{ if } 2i+1 \in D_G\}$$

and

$$D_{G'} = \{2i+1 : \text{ if } 2i+1 \in S_G\} \cup \{2i : \text{ if } 2i \in D_G\}.$$

On any bipartite board B, if $(b_{i_1}, b_{1_2}, \ldots), (b_{j_1}, b_{j_2}, \ldots), (r_{k_1}, r_{k_2}, \ldots), (r_{l_1}, r_{l_2}, \ldots)$ is an *illegal* position for G , $(b_{i_1}, b_{1_2}, \ldots), (b_{l_1}, b_{l_2} \ldots), (r_{k_1}, r_{k_2}, \ldots), (r_{j_1}, r_{j_2}, \ldots)$ is an illegal position in G' and vice-versa. Thus, the bipartite flip on any bipartite board gives a correspondence between the legal positions of G to those of G'. Therefore we can refer to the *bipartite flip of the game* G, denoted $BF(G)$. Since the corresponding positions of G and $BF(G)$ on any bipartite board B have the same number of pieces then $P_{G,B}(x) = P_{BF(G),B}(x)$. This gives us the proof of Theorem 2.2 concerning bipartite Doppelgänger games.

Theorem 2.2. *On the class of bipartite boards, if* G *is a dp-game and* $S_G \cup D_G$ *contains an odd distance then* G *and* $BF(G)$ *are Doppelgänger.*

Note that the 'odd distance' constraint is necessary to ensure that G and $BF(G)$ are different.

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