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Informations
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Articles
Games and Puzzles
Problems
MathMagic
Mathematics and Arts
Math and Fun with Algorithms
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News

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Articles

OPTIMAL RECTANGLE PACKING FOR THE 70 SQUARE

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Abstract: *Gardner asked whether it was possible to tile/pack the squares* $1 \times 1, \ldots, 24 \times 24$ *in a* 70×70 *square. Arguments that it is impossible have been given by Bitner–Reingold and more recently by Korf–Mofitt–Pollack. Here we outline a simpler algorithm, which we hope could be used to give an alternative and more direct proof in the future. We also derive results of independent interest concerning such packings.*

Keywords: packing problems, 70×70 square.

Introduction

Consider the equation

$$1^2 + 2^2 + \cdots + n^2 = m^2 \tag{1}$$

where $n, m \in \mathbb{N}$. Since the work of Watson [5] in 1918, it is known that the only solutions are the trivial case, namely $1 = 1$, and

$$1^2 + \cdots + 24^2 = 70^2.$$

A natural question arising from this fact is to see whether a configuration (or packing) of the squares $1 \times 1, \ldots, 24 \times 24$ exist which exactly tile a 70×70 square. The problem was popularized by Martin Gardner in articles [2], [3] in *Scientific American*, and several readers sent in their best efforts to fill as much of the square as possible. Twenty-seven readers managed to fit all the squares except the 7×7 square inside the 70×70 square; it has recently been proven [4] with the help of extensive computer computations that this is the 'best possible packing'. In particular, Korf et. al. showed it is impossible to construct a packing with the given squares which wastes less space. Their algorithm models the problem as a *Constraint-Satisfaction Problem*. This computation involved 375 million nodes, and took 16 minutes to compute (see Table 6 in the appendix of [4], as well as Section 4.5 for an overview of their algorithm). Each node is a placement of a set of squares in the enclosing rectangle, without overlap. An earlier article [1] states that applying their technique of backtrack programming shows that there is no such packing, but details are not provided due to the length of the arguments.

Our techniques and results are in a similar vein, and involve significant computations using MATLAB. Our goal is to give a mathematical proof, via a combination of combinatorial arguments and case-by-case analysis. Whilst we cannot yet fully achieve our goal, we outline an alternative approach using direct geometric arguments that we believe will yield a simpler proof in the future. Many of the results are applicable in wider contexts. To analyze the cases we could not directly rule out, we show how to rule out a packing with two fixed edges which were randomly chosen from the remaining possibilities. One could thus try to adapt our techniques and perform a case-by-case analysis to give a complete mathematical proof that no packing is possible. We hope to continue this work in the future.

Results

Assuming a packing exists of the 70×70 square, our goal will be to derive a contradiction. We will call this a *potential packing*, and for short denote it as $\mathcal{P}$. For the reader's convenience and to keep the diagrams a reasonable size, we will color in the 1×1 square in blue and the 2×2 squares in orange.

Constructing the Matrix A

Definition. An *edge* ϵ of $\mathcal{P}$ is a subset of $\mathcal{P}$ consisting of all squares which touch the same edge of the 70×70 square. The *frame* of $\mathcal{P}$ is the union of the four edges.

Let $a_1, \ldots, a_n$ be squares.

Definition. $ord(a_1, \ldots, a_n)$ will denote an *ordered set*; this is a set of squares $\{a_1, \ldots, a_n\}$ so that a_k and a_{k+1} have adjacent sides for every $k = 1, \ldots, n-1$.

Definition. Any permutation of an ordered set $ord(a_1, \ldots, a_n)$ is denoted $\operatorname{adj}(a_1, \ldots, a_n)$.

The notation $\operatorname{adj}(a_1, \ldots, a_n)$ acts as a placeholder for multiple squares next to each other in an unordered fashion.

The following theorem restricts the number of possibilities significantly.

Theorem 1. *(Restriction Theorem) Any edge of $\mathcal{P}$ must satisfy the following:*

(i) The 1, 2, 3, 4, and 5 squares cannot be on any edge;

(ii) We cannot have the 6, 7, and 8-square on the same edge;

(iii) If the 6-square is on an edge, then we must have adj(6,7) on the same edge;

(iv) If n is the smallest square on an edge, then the next smallest square on the edge must be less than $2n-1$;

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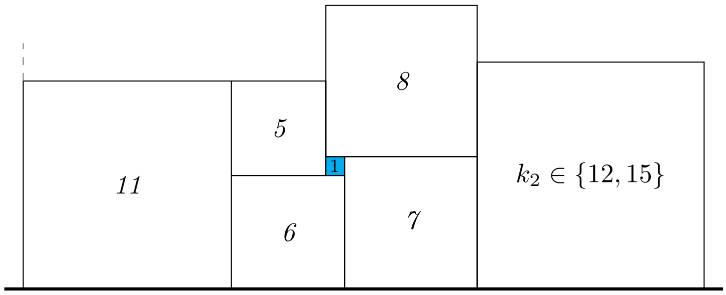
(v) If adj(6,7) is on an edge, then the following construction must be on the same edge.



The next step, undertaken in Section 3, is to construct a matrix A where each row uses the numbers 6–24 and sums to 70; these are all the possible sets of squares that can be on an edge of $\mathcal{P}$. To create this matrix, we first need to know how many squares (at most) are in a row of A; i.e., the number of columns in A. Since squares 1–5 cannot lie on the edge, observing that

$$6 + 7 + 8 + 9 + 10 + 11 + 12 = 64 < 70,$$

it follows that A must have at most seven columns. To handle cases where, for example, one edge has seven squares and another has five, zeros must be included as placeholders to ensure MATLAB can display all possible edges as a matrix. The number of zeros is the maximum number of possible squares minus the minimum number of possible squares. Notice that $24 + 23 + 22 + 1 = 70$, so the minimum number of possible squares on an edge is four. After adding in the other four restrictions from Theorem 1 into our MATLAB code, this resulted in an entire column of zeros. Therefore, Theorem 1 ensures A must have at most six columns; that is, there are *at most* six squares on an edge. Therefore there must be $6 - 4 = 2$ zeros included.

Ruling Out Edges

Our first attempt in MATLAB was to determine the matrix A without any restrictions other than using 1–24 to sum to 70, i.e., not implementing Theorem 1. The closest we could reach to this was using 2–24 with seven columns which gave 9,285 rows. Applying all the restrictions from Theorem 1 to A, we obtain 391 rows.

We chose an element $v = \{9, 14, 23, 24\} \in A$. This is the first entry of the matrix A. In Section 4 we will establish the following theorem:

Theorem 2. *There is no frame with the edge*

$$v = \{9, 14, 23, 24\}.$$

Many other potential edges occurring in A can be ruled out via similar combinatorial arguments to the proof of Theorem 2.

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Constructing All Possible Frames

In Section 5, we will produce an algorithm which produces all possible frames. We fix a row in A, called v, and find all rows in A that have nothing in common with v (excluding zeros). This new matrix is all possible sets of squares that are the opposite side of the 70×70 square; we denote this matrix by A_{op}. Fixing a row in A_{op}, denoted v_{op}, a matrix C is constructed from A such that each row of C has exactly one element in common with v and v_{op} (not the same element). By finding two unique rows in C, then we have a possible frame. This is determined by fixing a row in C, denoted c, and determining all rows in C with no common element with c. Finally, we construct the matrix C_{op} from C by removing c and any row that has a nonzero element in common with c from C.

The matrix A_{op} is a subset of the matrix A. Thus we can only estimate the bound of A_{op} from our bound of A. After some numerical investigation, it appears that the bound for A_{op} appears to be approximately one third the bound of A, and thus the bound of A_{op} is approximately 3,000 before factoring in our restrictions. Applying Theorem 1, the number of rows in A_{op} was reduced to 55. Similarly, for the matrices C and C_{op}, the bounds will roughly be 660 and 224 respectively, before including restrictions. And after applying Theorem 1 the bounds of C and C_{op} become 12 and 2, respectively.

In conclusion, we now have roughly $391 \times 55 \times 12 \times 3$ (roughly 770,000) possible frames to check. Given $\mathcal{P}$, its frame must be one of these possibilities.

Ruling Out Frames

To analyze our results, in Section 6 we rule out specific examples of frames which are obtained in our construction. This helps us to gain some intuition for how a proof would work in general.

Choosing the first edge v where we cannot easily adapt the proof of Theorem 2 (which turns out to be the 80th possible entry of the matrix A), we randomly choose an element of A_{op} and show how no frame with these choices of v and v_{op} is possible.

Theorem 3. *There is no frame with edges*

$$v = \{7, 8, 14, 18, 23\},$$
$$v_{op} = \{9, 10, 11, 12, 13, 15\}.$$

In the course of the proof, we reduce the number of possible frames with these two elements ($12 \times 3 = 36$) to only nine cases, which are all easily handled by considering the possible ways of tiling around the smallest corner square. This leads us to believe our approach will work in general, which we leave as a project for the future.

Near Misses

We conclude with an interesting observation in Section 7. Running a direct calculation to find solutions of Equation 1, MATLAB ran out of memory when checking different n values around $n = 10^6$. The well-known formula

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

cuts down on computation and allows us check more cases. Using MATLAB, there was a round-off error which reached the false conclusion that

$$1^2 + 2^2 + \cdots + 2,542,690^2 = 2,340,882,545^2.$$

We regard this as a "*near-miss*" solution, since the values on both sides of the equation differ by 1,140; which when divided by m^2 becomes minuscule. Defining the weighted error for given integers n, m as

$$E(n,m) := \left\{ \frac{k}{m^2} \mid (1^2 + \cdots + n^2) - m^2 = k \right\}$$

then our above example has a weighted error of a mere 2.08×10^{-16}. With this definition, we consider a pair (n, m) to be a *near-miss* solution if $|E(n,m)| < 10^{-15}$. Searching with MATLAB, we seem to find many more examples of near misses, which are recorded in the following table.

n	m	$E(n,m)$
2,542,690	2,340,882,545	2.08039×10^{-16}
3,179,535	3,273,293,063	1.11158×10^{-16}
3,344,320	3,531,028,388	-5.00474×10^{-17}
4,832,360	6,133,076,209	-7.49974×10^{-17}
5,988,346	8,460,572,575	-1.32213×10^{-16}
6,471,528	9,504,946,458	-1.08474×10^{-16}
7,050,120	10,807,722,436	-1.35917×10^{-16}
7,671,515	12,267,642,825	5.74769×10^{-18}

It is interesting to ask if there are infinitely many pairs (n, m) yielding near misses with less than a given error. If so, does the error approach zero as $n \to \infty$?

Proof of Theorem 1

The proof will be divided into a case-by-case analysis. To begin, we outline some definitions we will use in the proof.

Definition. Let $a, b \in \mathcal{P}$ be squares with $\operatorname{adj}(a, b)$. We say there is a *flush* of a, b (or an a, b-flush) if there exists a square $c \neq a, b$ such that $\operatorname{adj}(a, c)$ and $\operatorname{adj}(b, c)$.

We write $|a|$ for the length of an $a \times a$ square.

Definition. A set of squares $\Pi = \{a_i : i = 1, \ldots, n\} \subseteq \mathcal{P}$ is called a *path* if Π satisfies:

1. $ord(a_1, \ldots, a_n)$,
2. $\sum_{i=1}^{n} |a_i| = 70$, and
3. $a_1 \in \epsilon$ and $a_n \in \epsilon_{op}$ for some edges ϵ, ϵ_{op}.

Definition. Let a and b be squares. We say b is *on top of* a if there exists a path Π with $a, b \in \Pi$, so that $a = a_i \in \Pi$ and $b = a_{i+1} \in \Pi$ for some fixed i.

Note that the definition of "on top of" is deliberately flexible: performing a rotation allows us to view any given path as starting at the bottom of the square and proceeding vertically upwards.

Definition. Let a and b be squares with b on top of a, where $|b| > |a|$ and $\mathrm{adj}(a, b)$ such that there exists an a, b-flush. Then b will *hang over* a.

If the flush is the left then the hang over is on the right and vice-versa.

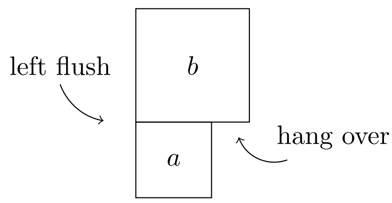

Figure 1: This illustrates an a, b-flush, and that b is hanging over a.

Proving Theorem 1(i)

Here we prove Theorem 1(i), which states that squares 1–5 cannot be on the edge. We begin with a preliminary lemma to reduce the size of the proof. Owing to its length, the rest of the proof will be split up into a series of claims. Up to rotation we can assume an edge is on the bottom of the square. Without loss of generality the notation *ord* corresponds to squares being placed from left to right.

Lemma 1. Suppose the $1, 2, \ldots, n-1$-squares are not on an edge. Then the n-square is not a corner square.

Proof. Suppose the squares $1, 2, \ldots, n-1$ are not on an edge. Further, suppose for contradiction that the n-square is a corner square. Then there are adjacent squares a_1, a_2 on the corresponding edges. However, since squares $1, \ldots, n-1$ are not on an edge and the n-square was already used, then $|a_1|, |a_2| > |n|$ which guarantees an overlap and thus a contradiction. $\square$

Throughout there will be diagrams of square packings. To help see the larger packings, we color the 1-square with blue and color the 2-square with orange throughout the paper. We also use the symbol $\times$ in the diagrams to illustrate where a contradiction occurs.

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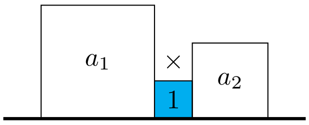
Claim. The 1-square cannot be on the edge.

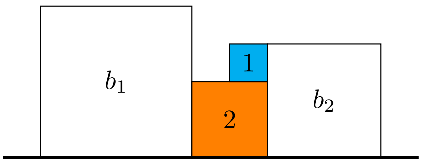
Proof. Suppose the 1-square is on the edge. Let squares a_1, a_2 be squares on the same edge where (without loss of generality) $1 < |a_2| < |a_1|$ are on the right and left of the 1-square, respectively. Then, any remaining square placed on top of a_1 will yield wasted space above the 1-square. The 1-square cannot be a corner square by a similar argument as Lemma 1. $\rightarrow\leftarrow$



□

Claim. The 2-square cannot be on the edge.

Proof. Suppose the 2-square is on the edge. By Lemma 1, the 2-square is not a corner square. Let b_1, b_2 be squares on the edge such that (without loss of generality) $2 < |b_2| < |b_1|$. Place b_1 and b_2 on the left and right of the 2-square, respectively. Then the set of squares on the 2-square must sum to two. However, the 1-square is the only square less than the 2-square remaining and this yields wasted space. $\rightarrow\leftarrow$



□

Claim. The 3-square cannot be on the edge.

Proof. Suppose the 3-square is on the edge. By Lemma 1, the 3-square is not a corner square. Let c_1, c_2 be squares on the edge where $3 < |c_2| < |c_1|$. Place c_1 and c_2 on the left and right (without loss of generality) of the 3-square, respectively. Then the only squares that can be placed on top of the 3-square is adj(1,2). Then for any $|c_2| > 4$, we get wasted space over the 1-square for either permutation of adj(1,2). Thus, c_2 must be the 4-square, and the 1-square must be on the right. Similarly, for any $|c_1| > 5$, there will be wasted space over the 2-square.

Now consider if the 4-square is a corner square. Then any square placed on to the 1,4-flush must be $\geqslant 6$ (since squares 1–5 have been used). However, this guarantees overlap with the 2-square. This implies that there must be a square c_3 where $|c_3| \geqslant 6$, on the right of the 4-square on the edge. Thus any set of squares placed on top of the 1,4-flush must sum to five, but there are no remaining sets that sum to five. $\rightarrow\leftarrow$

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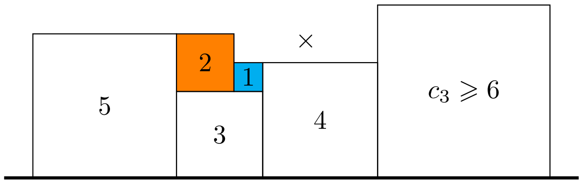


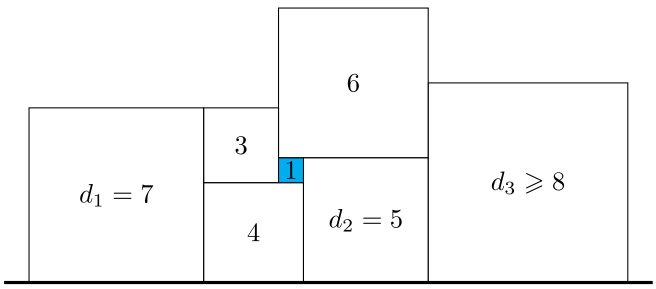
□

Claim. The 4-square cannot be on the edge.

Proof. Suppose the 4-square is on the edge. By Lemma 1, the 4-square is not a corner square. Let d_1, d_2 be squares on the edge where $4 < |d_2| < |d_1|$. Place d_1 and d_2 on the left and right (without loss of generality) of the 4-square, respectively. So the set of squares on the 4-square must sum to four. The only set of squares that satisfies this sum is adj(1,3), so place this on top of the 4-square. Notice that for any $|d_2| > 5$, there will be wasted space over the 2-square. So d_2 must be the 5-square, and to avoid wasted space we must have adj(1,5).

Since squares 3–5 have been used then any edge square placed adjacent to the 5-square will be larger than five, and so the set of squares on the 1,5-flush must sum to six (this also holds if the 5-square is a corner square). The 6-square is the only square that can be placed on the 1,5-flush; make this placement. Note that $|d_1| \geqslant 7$. Consider when $|d_1| > 7$, then the 2-square is the only remaining square that can be placed on the 3-square. However, this guarantees wasted space adjacent to the 2-square. Thus d_1 must be the 7-square.

Consider if the 5-square is a corner square. Then any square placed on top of the 6-square and on this new edge must be $\geqslant 8$, and will therefore hang over the 6-square. However, the 2-square is the only remaining square that can be placed on the 3-square, and this guarantees wasted space adjacent to the 2-square. Thus, there must be another square d_3 on the right of the 5-square. This further implies that any set of squares placed on the 6-square must hang over the 6-square on the right.



Since the set of squares on the 6-square will hang over on the right, then the set of squares adjacent to the 5,6-flush must sum to 11. The only remaining sets are adj(2,9) and the 11-square. Consider when d_3 is the 9-square with the 2-square placed on top. Then we must have the 8-square placed on the 2,6-flush otherwise there will be wasted spaced to the right of the 2-square. Further, any square placed on the right of the 9-square will be larger than nine, so the

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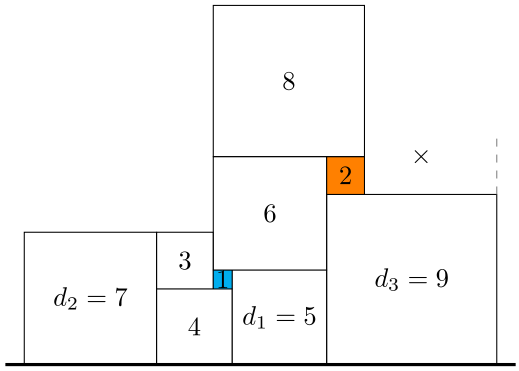
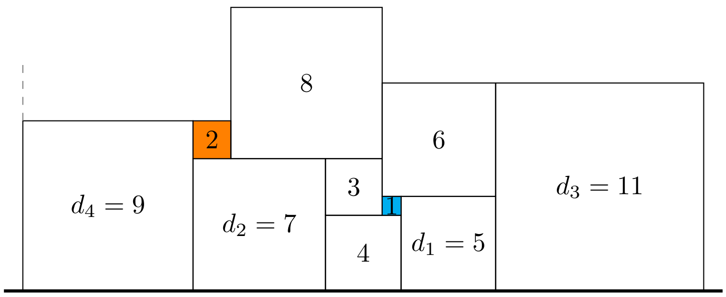
(remaining) set of squares on the 9-square must sum to nine—this also holds if the 9-square is a corner. However, there are no remaining squares less than nine. So d_3 cannot be the 9-square and therefore must be the 11-square.

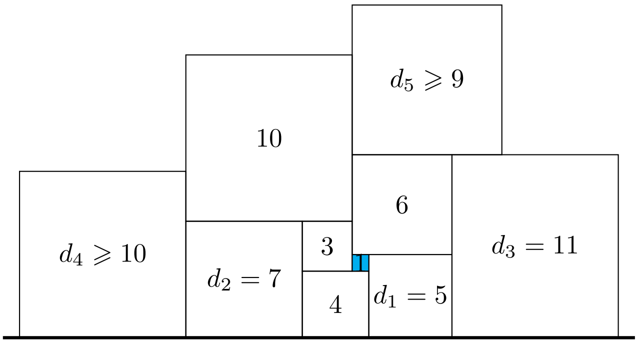


Next place another square d_4 on the left of the 7-square. Further, since squares 3–7 have been used then $|d_4| \geqslant 8$. It is possible that the 7-square is a corner square; we will modify our arguments if necessary on a case-by-case basis. Therefore the set of squares placed on top of the 3,7-flush must sum to ten. The only remaining sets that satisfy this are adj(2,8) and the 10-square.

Consider when adj(2,8) are placed on the 3,7-flush. Then the 2-square must be on the left, otherwise there would be wasted space over the 2-square. Note that $|d_4| \geqslant 9$, but if $|d_4| > 9$, then there will be wasted space over the 2-square; so d_4 must be the 9-square. Again, since squares 1–9 have been used, any square placed on the left of the 9-square on the edge will be greater than nine. So the set of squares on the 2,9-flush must sum to eleven—this also holds if the 9-square is a corner. But there are no remaining sets that sum to eleven. Thus, we cannot have adj(2,8) on the 3,7-flush.



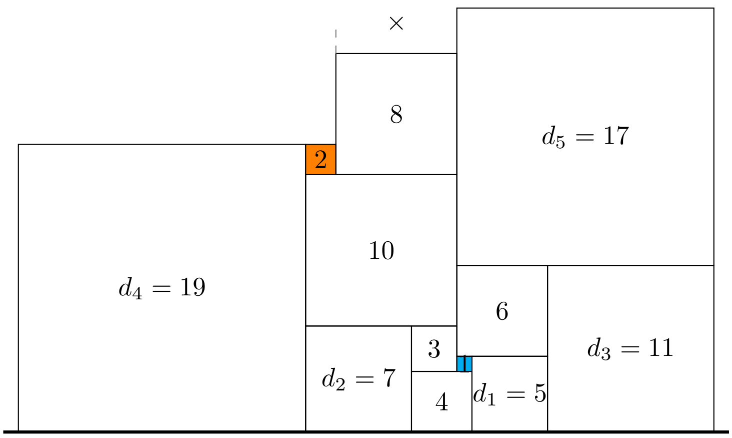
Therefore, the 10-square must be placed on the 3,7-flush. Note the following argument will also hold if the 7-square is a corner. Place another square d_5 on top of the 6,11-flush. If d_5 is the 2-square, then this will yield wasted space adjacent to the 2-square; so $|d_5| \geqslant 8$. Then we have two cases to consider: the set of squares on the 10-square either sums to ten or will hang over 10-square.



Note that the 8-square can no longer be on the edge since it is the smallest remaining square that can be placed on the edge but there are no remaining sets of squares that sum to eight. Similarly, the 9-square also cannot be on the edge since the set of squares on the 9-square would need to sum to nine, and there are no such remaining sets.

1. Suppose the set of squares on the 10-square sums to ten; adj(2,8) is the only remaining set that sums to ten. The 2-square must be on the left otherwise there will be wasted space over the 2-square (for any square placed on the 6-square). Note that the set of squares adjacent to the 2,7,10-flush must sum to 19, otherwise there will be wasted space over the 2-square. Further, the 19-square is the only remaining square that sums to 19; so d_4 must be the 19-square.

   Recall the 9-square cannot be on the edge. Thus if a square is on the right of the 11-square must be greater than 11. Therefore the set of squares on the 6,11-flush must sum to 17 (this holds if the 11-square is a corner square); the 17-square is the only remaining square that satisfies this sum. However, any square placed on the 2-square will yield wasted space over the 8-square. Thus, we cannot have adj(2,8) on the 10-square, and therefore the set of squares on top of the 10-square must hang over the 10-square.



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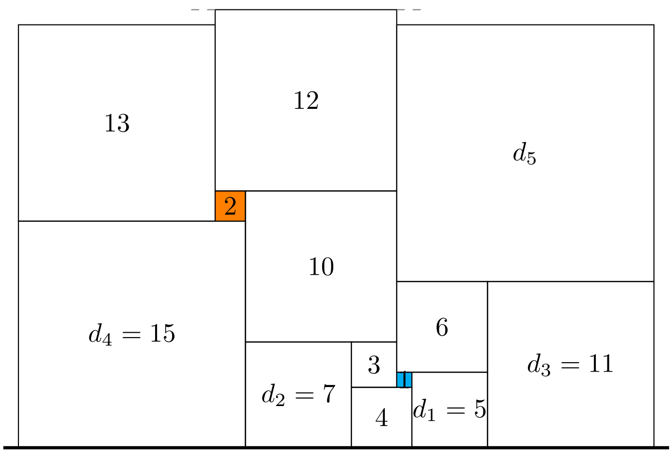
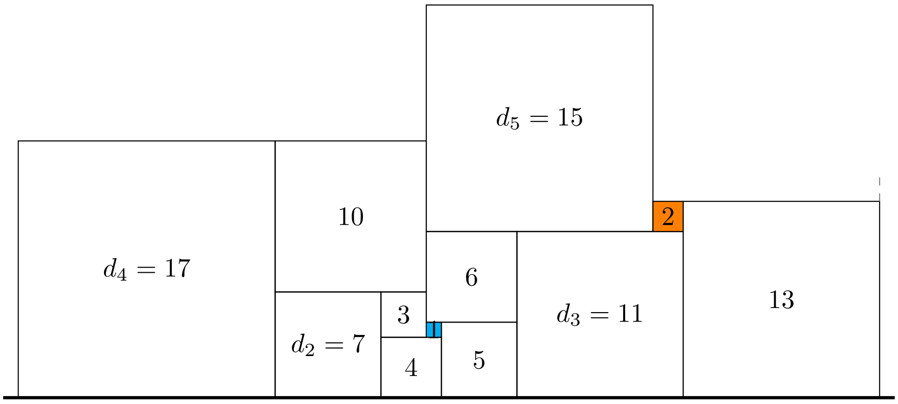
2. Suppose the set of squares on the 10-square hangs over the 10-square. Note that the 7-square cannot be a corner square or this would contradict our assumption. Further, the set of squares adjacent to the 7,10-flush must sum to 17. The only sets that satisfy this are adj(2,15), adj(8,9), and the 17-square. However, we determined before that the 8 and 9-squares cannot be edge squares. Further, we need a set of squares that sums to 17 to be placed on the 6,11-flush.

   (a) Suppose d_4 is the 15-square and place the 2-square on top of it. Note that there are no remaining sets of squares that sums to 15, so the set of squares on top of the 15-square must sum to 15 (this also holds if the 15-square is a corner square). That is, the remaining set of squares on the 15-square must sum to 13; the 13-square is the only square that satisfies this sum. Note any square placed on the 6-square will have a height above the 10-square. So the set of squares on the 2,10-flush must sum to 12, and the 12-square is the only square that satisfies this. Further, any square placed on the 12-square will hang over the 12-square and yield wasted space on the respective side. Thus, d_4 cannot be the 15-square.



   (b) Suppose d_4 is the 17-square and adj(8,9) is placed on the 6,11-flush. Further, suppose the 9-square is on the left. Then any square placed on the 9-square will hang over on either side, yielding wasted space. Similarly, if the 8-square is on the left, any square placed on top will hang over on the left. This guarantees wasted space over the 10-square. So adj(8,9) cannot be on top of the 6,11-flush. Then adj(2,15) must be on the 6,11-flush; the 2-square must be on the right otherwise there will be wasted space over the 2-square. Then we must have the 13-square on the right of d_3 or there will be wasted space over the 2-square. Since there is remaining set that sums to 13, then we must have the set of squares on the 2,13-flush must sum to 15. But there are also no remaining sets of squares that sum to 15. Therefore, there will be wasted space over the 2,13-flush. Thus, d_4 cannot be the 17-square.



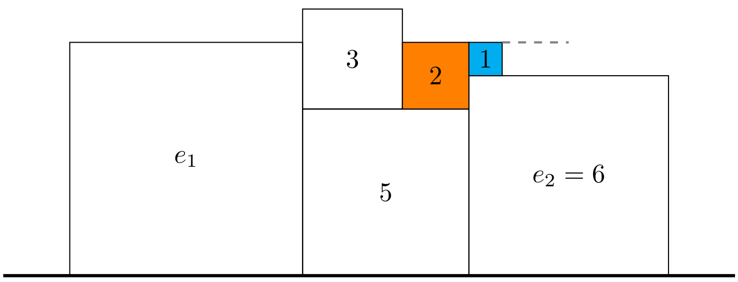
In summary, we cannot have the set of square on the 10-square sum to ten nor can they hang over the 10-square. We have a contradiction, and thus the 4-square cannot be on the edge.

□

Claim. The 5-square cannot be on the edge.

Proof. Suppose the 5-square is on the edge. By Lemma 1, the 5-square is not a corner square. Let e_1, e_2 be squares on the edge where $5 < |e_2| < |e_1|$. Place e_1 and e_2 on the left and right (without loss of generality) of the 5-square, respectively. So the set of squares on the 5-square must sum to five. The only sets that satisfies this sum are adj(2,3) and adj(1,4).

1. Suppose adj(2,3) is placed on top of the 5-square with the 2-square on the right. If $|e_2| > 7$ then any set of squares placed on the 2-square will yield wasted space. If e_2 is the 6-square, then any set of square placed on the 2-square will hang over the 2-square. So place the 1-square on the 6-square, making a 1,2-flush. However, any square placed on the 2-square will hang over the 1-square, yielding wasted space. Thus, e_2 must be the 7-square.



Note e_1 cannot be the 6-square since $5 < |e_2| < |e_1|$. Consider when e_1 is the 9-square, then any set of squares on the 3-square will hang over the 3-square. The only configuration that will not (immediately) yield wasted space is to place the 1-square on the 2-square, and place the 4-square on the 1,3-flush. However, we must have the set of squares on the 2,7-flush sum to nine, otherwise there will be wasted space next to the 7-square. So

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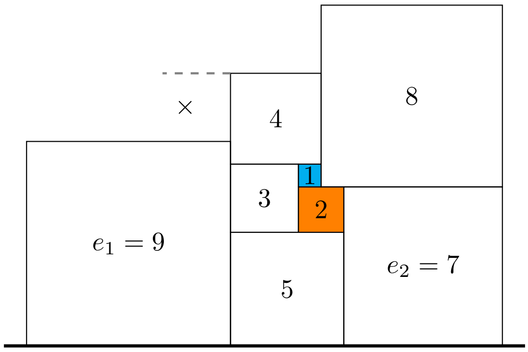
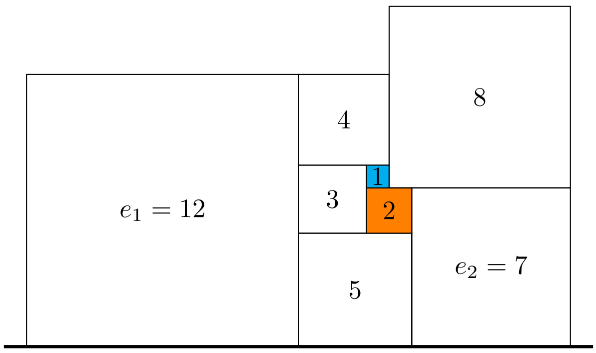
we must have adj(1,8) on the 2,7-flush. However, then any square placed on the 4-square will hang over the 4-square and leave wasted space under the overhang. Thus e_1 cannot be the 9-square.



Now, we have the remaining cases: $|e_1| > 9$ and $|e_1| = 8$. Suppose $|e_1| > 9$ then we see from the case e_1 was the 9-square, that we must have the 1-square on the 2-square making a 1,3-flush and the 4-square on top of the flush. Now any square placed on the 4-square will hang over the 4-square, so e_1 must be the 12-square since no other sets sum to twelve. Again from the previous case, we know that the set of squares on the 2,7-flush must sum to nine and so the 8-square must be placed on the 2,7-flush.



Since there are no remaining sets of squares that sum to eight, then the set of squares on top of 8-square will hang over the 8-square. But any square placed on the 4-square will have a height above the 8-square, so the overhang must be on the right. This implies that e_2 cannot be a corner square. Therefore, we need the squares adjacent to the 7,8-flush to sum to 15. The only squares that satisfy this are adj(6,9) and the 15-square.

Consider when the 9-square is on the right of the 7-square, with the 6-square placed on top of the 9-square. Then any square on the right of the 9-square will be larger than the 9-square. But there are no remaining squares that can be placed adjacent to the 6-square. So we cannot have adj(6,9) on the 7,8-flush.

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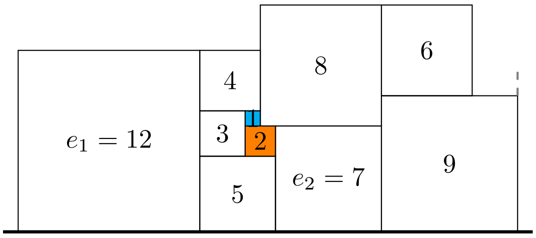
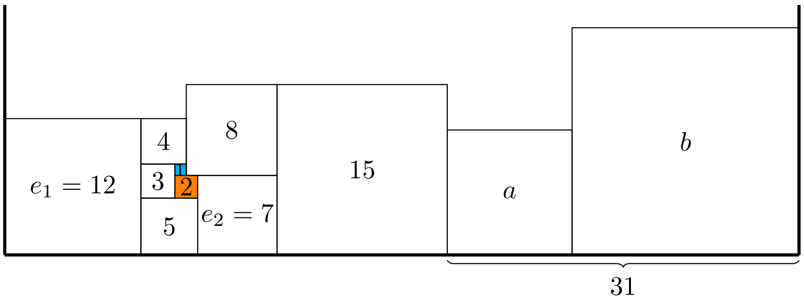


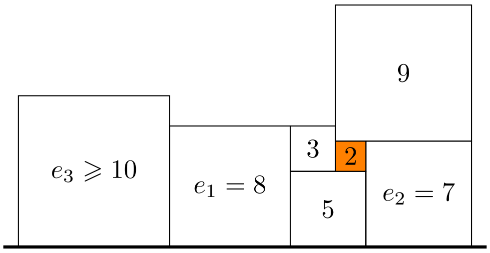
We must have the 15-square on the right of the 7-square. Consider the total length of these squares on the edge, $12 + 5 + 7 + 15 = 39$. The remaining length on the edge is $70 - 39 = 31$. From the remaining squares, the possible combinations are adj(6,11,14), adj(9,22), adj(10,21), adj(11,20), adj(13,18), and adj(14,17). Note that adj(6,11,14) is ruled out since no remaining set of squares sum to 16. So all of these sets can be represented as $\mathrm{adj}(a, b)$ where $9 \leqslant |a| < |b|$.

Consider the smaller square a in each of these sets. For any placement on the edge, the set of squares on a must sum to $|a|$. Note that the 6 and 9-squares are the two smallest remaining squares, so $|a| \geqslant 15 = 6 + 9$. However, a is at most the 14-square. Therefore this guarantees wasted space over a in any of these sets. Thus, we cannot have adj(7,15).

For example, if adj(14,17) were placed on the edge, then there must be another set of squares to be placed on the 14-square. But the next smallest squares are the 6 and 9-square, so this guarantees wasted space over the 14-square (the image below is simply a visual aide; further, the argument holds for any placement of a and b on the remaining edge).

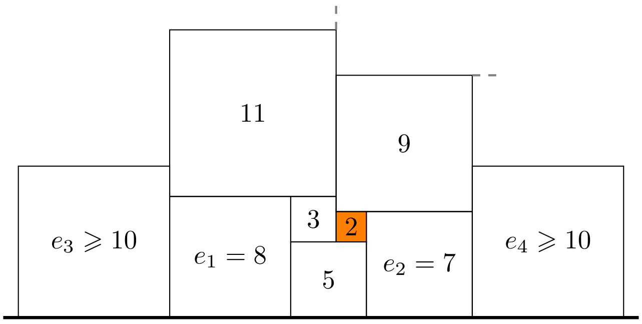


Thus, e_1 cannot be the 12-square. In summary, we have ruled out $|e_1| = 6$ and $|e_1| \geqslant 9$. So e_1 must be the 8-square. Again, we must have the set of squares on the 2,7-flush sum to nine. But the 9-square is the only square that satisfies this sum. So place the 9-square on the 2,7-flush. We place another square on the left of the 8-square denoted e_3. Note that $|e_3| \geqslant 10$, since any set of squares on the 6-square would yield wasted space and squares 7–9 have been used. Then the set of squares on the 3,8-flush must sum to 11. The only sets that satisfy this are adj(1,10), adj(1,4,6), and the 11-square.



Consider the cases when adj(1,10) and adj(1,4,6) are placed on the 3,8-flush. Then any permutation will guarantee wasted space over the 1-square. Therefore the 11-square must be on the 3,8-flush. Note that if the 8-square is a corner square then these first two cases would have the same wasted space. Note that there are no remaining sets of squares that sum to eight or nine. Then any set of squares placed on top of the 9-square will hang over the 9-square on the right. This implies the 7-square is not a corner square, and the set of squares adjacent to the 7,9-flush must sum to 16. Place a square, e_4, on the right of e_2. The only remaining sets that sum to 16 are adj(1,15), adj(4,12), adj(6,10), and the 16-square.

We therefore have the following cases: the set of squares on the 11-square either sums to 11 or hangs over the 11-square. Now, consider the particular case when the 4-square is placed on the 9-square and the 1-square is placed on the 11-square; this creates a 1,4-flush. But any squares on the right of the 4-square and left of the 1-square will yield wasted space above the 1,4-flush. Further, if the 1-square is placed on the 9-square there will be wasted space over the 1-square. Therefore, any square placed on the 9-square will have a height above the 11-square.



(a) Suppose the set of squares on the 11-square sums to 11. Then the only sets that satisfy this are adj(1,4,6) and adj(1,10). Consider when adj(1,4,6) is placed on the 11-square, then we have either ord(1,4,6), ord(1,6,4) or ord(4,1,6). However, ord(4,1,6) guarantees wasted space over the 1-square, and ord(1,4,6) guarantees wasted space over the 1-square for any square placed on top of the 4-square. Further, for ord(1,6,4), any square placed on the 6-square will hang over the left or right of the 6-square which will guarantee wasted space over the 1 or 4-square accordingly.

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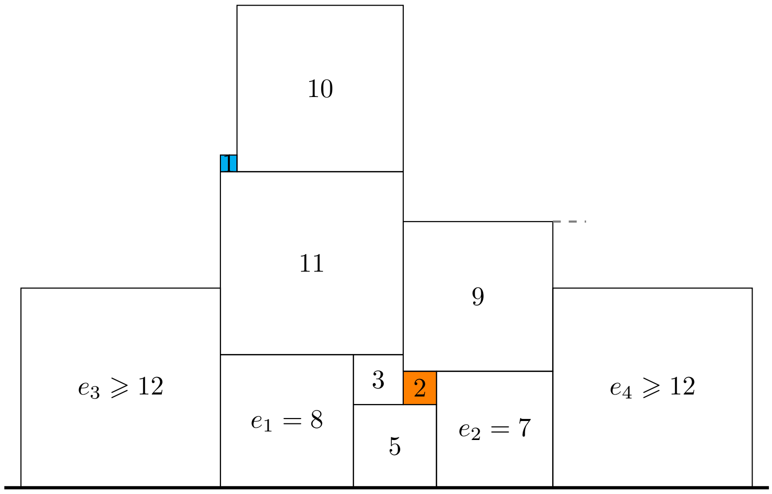
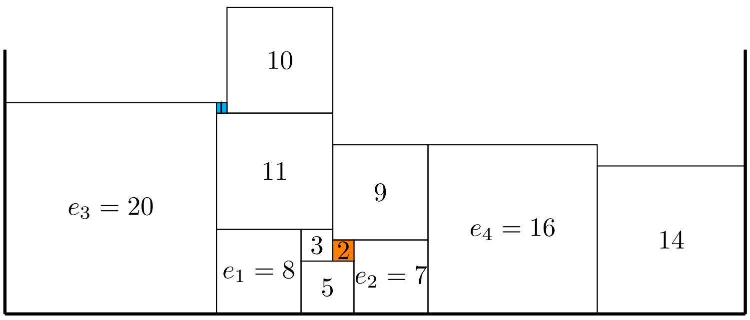
So place adj(1,10) on the 11-square, with the 1-square on the left since we've shown that the 1-square cannot be on the right. Then the set of squares on the 8,11,1-flush must sum to 20, to avoid wasted space over the 1-square. The only sets that satisfy this are adj(4,16), adj(6,14), and the 20-square. But we cannot have adj(4,16) since this removes the remaining sets on the right of the 7-square.



Suppose e_3 is the 14-square and place the 6-square on top of it. Then any square placed adjacent to the 6-square will hang over the 14-square. And there are no remaining sets that sum to 14. Thus, e_3 cannot be the 14-square, and so e_3 must be the 20-square.

Consider if the e_4 is the 16-square. Then the total length of squares on the edge is $20 + 8 + 5 + 7 + 16 = 56$; so there remaining squares must sum to 14. However, there are no remaining sets that sum to 14. Thus, e_3 cannot be the 16-square (the image shows the 14-square on the right of the placed squares without loss of generality; the argument holds if the 14-square is on the left as well).



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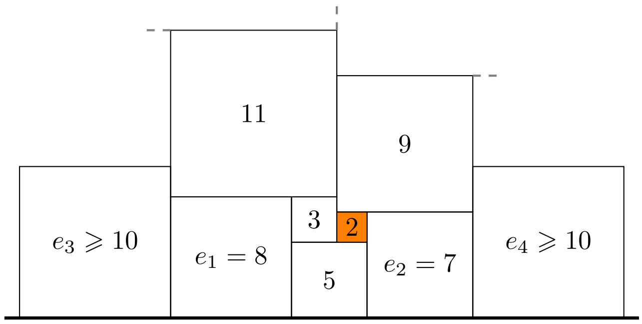
Let e_4 be the 12-square and place the 4-square on top of it. Then any squares placed on the remaining space on the 12-square will hang over the 12-square. However any square placed on the right of the 12-square will be greater than 12. Thus, e_4 cannot be the 12-square.

Thus we cannot have the set of squares on the 11-square sum to 11.

(b) Suppose the set of squares on the 11-square hangs over the 11-square. Then the set of squares adjacent to the 8,11-flush must sum to 19. The only sets that satisfy this are adj(1,18), adj(4,15), adj(6,13), adj(1,4,14), and the 19-square.



i. Suppose e_3 is the 18-square and place the 1-square on top of it. Now, the only sets of squares that adjacent to the 7,9-flush that sum to 16 are adj(6,10) and the 16-square. We again consider the total length of the squares on the edge used: $18+8+5+7 = 38$. So we have $70-38 = 32$ remaining length on the edge. If e_4 is the 16-square then there is a length of 16 remaining, but adj(6,10) is the only set of squares that would fit this space, but the 6-square will yield wasted space above it.

Further, if e_4 is the 10-square then there is a length of $70 - (18 + 8 + 5 + 7 + 10) = 22$ remaining on the edge. We must have the set of squares on the 10-square sum to ten to avoid wasted space over the 10-square. Place ord(4,6) on the 10-square with the 6-square on the left (otherwise there will be wasted space over the 4-square). If the 10-square is a corner square then there will be wasted space over the 4-square. This implies that there must be a square to the right of the 10-square; the 14-square is the only square that can be placed to the right of the 10-square without wasted space. This however leaves a remaining length of $22 - 14 = 8$. Therefore e_3 cannot be the 18-square (the image shows the remaining length on the right without loss of generality).

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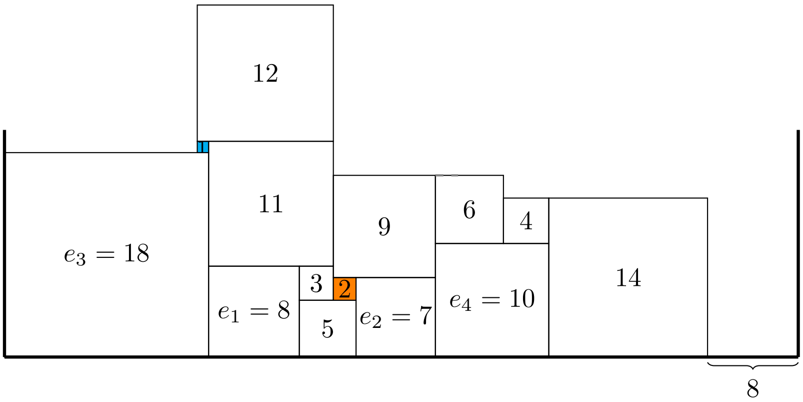
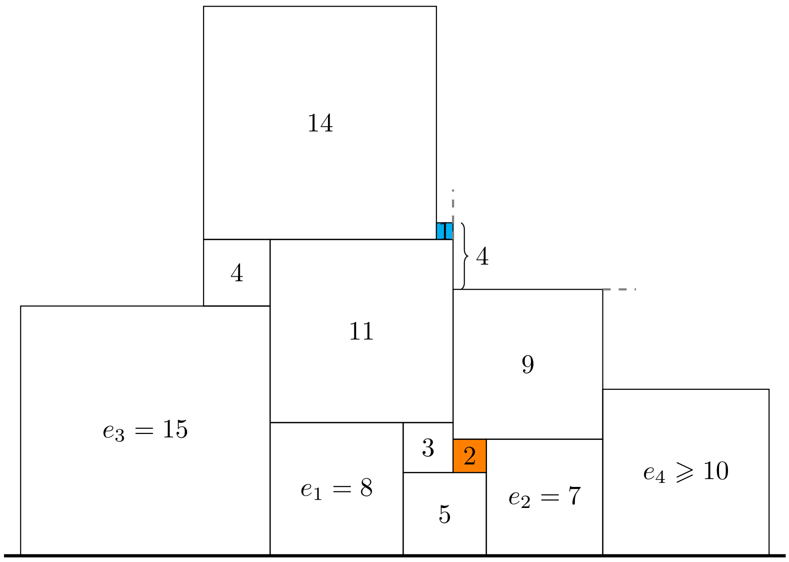


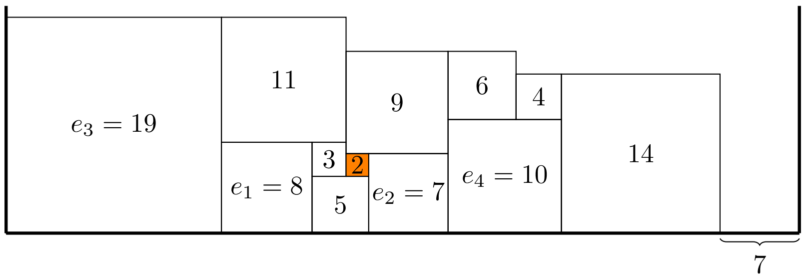
ii. Suppose e_3 is the 15-square and place the 4-square on top of it. Then any square placed on the left of the 4-square will be greater than the 4-square (the 1-square would yield wasted space). Also any set of squares placed on the 9-square will have a height above the 11-square. We need the set of squares on the 4,11-flush to sum to 15. The only set that satisfies this is adj(1,14). This however will yield wasted space over the 1-square for either permutation. Thus, e_3 cannot be the 15-square.



iii. Suppose e_3 is the 13-square and place the 6-square on top of it. Then suppose the 12-square is placed on the left of the 13-square. Then there are no sets that sum to 12 that can be placed on top of the 12-square. So the square on the right of the 13-square must be greater than 13. That is, the set on top of the 13-square must sum to 13. Since the 6-square is already placed on the 13-square then we need a set that sums to 7, but there is no such set of squares remaining. Thus, e_3 cannot be the 13-square.

iv. Suppose e_3 is the 14-square, place the 4-square on top of the 14-square, and place the 1-square on top of the 4-square. Then any square placed next to the 1-square will hang over the 4-square leaving wasted space. So, e_3 cannot be the 14-square.

v. Suppose e_3 is the 19-square. Consider the length of squares on the edge so far, we have $19 + 8 + 5 + 7 + |e_4| = 39 + |e_4|$. Observe the set of squares placed on the 9-square will hang over the 9-square. Therefore the set of squares on the 7,9-flush must sum to 16. That is, we must have one of adj(6,10), adj(4,12), adj(1,15), or the 16-square on the 7,9-flush. Note the 6-square cannot be on the edge since no remaining sets sum to six. Suppose e_4 is the 10-square then, similarly to the first case, we must have adj(4,6) on the 10-square and the 14-square must be to the right of the 10-square. This leaves a length of 7 remaining on the edge, which guarantees wasted space.



Suppose e_4 is the 12-square. Then the remaining total gap is 19. The only sets of squares that can fill this is adj(1,18) and adj(6,13). However neither of these sets can be placed on an edge.

Suppose e_4 is the 15-square. There is then a total gap of 16 remaining on the edge. Since the 6-square cannot be on the edge then the 16-square must fill this gap. We consider when the 16-square is to the right of the 15-square. Then the set of squares on top of the 15-square must sum to 15. The only possible set is adj(1,14). We must have ord(1,14) since no remaining sets sums to nine. However, this yields wasted space over the 16-square. An easier, analogous argument shows we cannot have the 16-square on the left of the 19-square.

Now supposing e_4 is the 16-square, then the remaining length on the edge is $70 - 55 = 15$. The 15-square is the only square that can be placed in the remaining edge. The only sets remaining that sum to 15 are adj(1,14) and adj(1,4,10).

Consider when adj(1,4,10) is placed on the 15-square. Then the 1-square must be on the right to avoid wasted space. Further, the 15-square must be a corner square since we have used the entire edge. Therefore the 10-square must be on the corresponding edge. However, any square placed to the right of the 10-square (on the new edge, call this ϵ_2) must be larger

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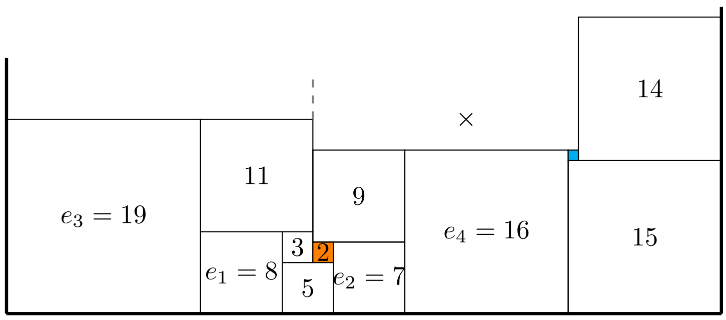
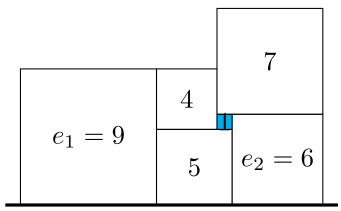
than the 10-square. So we must have adj(4,6) on the 10-square. However, this guarantees wasted space over the 1-square. Therefore adj(1,14) must be on the 15-square.

Suppose the 13-square is placed on the 1-square. Then there are no remaining sets that sum to 13. So there will be wasted space above the 9,16-flush. So, the 13-square cannot be placed on the 1-square. Now, the remaining sets that sum to 26 are adj(4,22), adj(6,20), and adj(4,10,12). If adj(6,20) is placed on the 9,16,1-flush, then the 6-square must be placed on the left or there will be wasted space over the 6-square. However, if the 6-square is on the left, then any square placed on the 6-square will hang over the 6-square on the left and intersect the square over the 11-square. Therefore we cannot have adj(6,20) on the 9,16,1-flush. Similarly, adj(4,22) and adj(4,10,12) will yield wasted space adjacent to the 4-square for any permutation. Thus, e_3 cannot be the 19-square.



Thus, we have exhausted all cases where the set on top of the 11-square hangs over the 11-square. Therefore, we cannot have adj(2,3) on the 5-square.

2. Now, place adj(1,4) on the 5-square, with the 4-square on the left (without loss of generality). Since this is a new configuration of squares on the 5-square, we will "reset" all e_i squares from before. Let e_1 and e_2 be placed as before, and e_2 must be the 6-square to avoid wasted space over the 1-square. Then the square on the right of e_2 will be greater than the 6-square, so the set on top of the 1,6-flush must sum to seven. The 7-square is the only square that satisfies this sum. There are no remaining squares that sums to four, so the set of squares on the 4-square will hang over the 4-square. Therefore we need the squares adjacent to the 4,5-flush sum to nine. The 9-square is the only square that satisfies this sum, so e_1 must be the 9-square.



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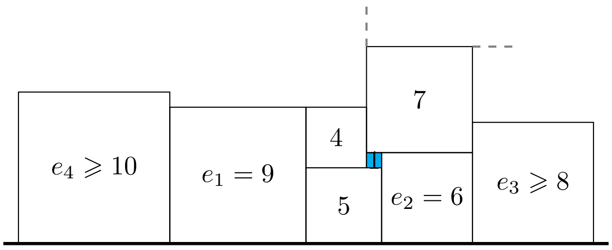
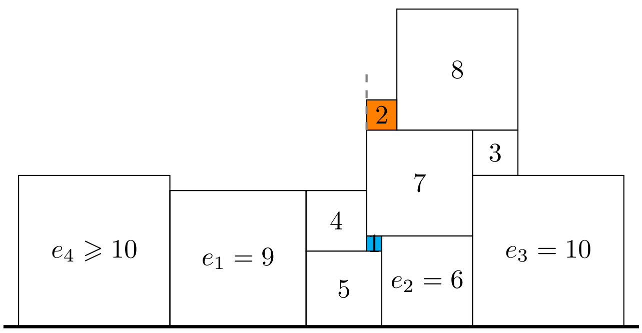
Place squares e_3, e_4 on the right and left of of the 6 and 9-square, respectively (note $|e_3|, |e_4| \geqslant 8$). If e_4 is the 8-square, then there are no remaining sets that sums to eight—so any set placed on top of the 8-square will yield wasted space. Therefore e_4 cannot be the 8-square and so $|e_4| \geqslant 10$. So we need the set of squares on the 4,9-flush to sum to 13 – this also holds if the 9-square is a corner square. But since there are no remaining sets that sum to four, then the squares placed on top of the 4-square will have a height above the 7-square. Further, since there are no squares that sum to seven, then the set of squares on top of the 7-square will hang over the 7-square. This implies the 6-square cannot be a corner square. Note the set of squares that sum to 13 are adj(2,3,8), adj(2,11), adj(3,10), and the 13-square. We need two distinct sets to sum to 13: one to go on the 4,9-flush and one to go on the 6,7-flush.



(a) Suppose e_3 is the 8-square, place the 3-square on top of the 8-square, and place the 2-square on top of the 3-square. Then any square placed next to the 3-square will yield wasted space next to the 2-square.

(b) Suppose e_3 is the 11-square, place the 2-square on top of e_3. Then we need the set of squares on top of the 2,7-flush to sum to nine to avoid wasted space over the 2-square. Here there are no remaining sets that sum to nine.

(c) Suppose e_3 is the 10-square, place the 3-square on top of e_3. Then we need the set of squares on top of the 3,7-flush to sum to ten to avoid wasted space over the 3-square. Then only set that satisfies this is adj(2,8). However, for either permutation, there will be wasted space above the 2-square.



Therefore e_3 must be the 13-square.

Now, we need a set of squares that sums to 13 to be placed on the 4,9-flush. The sets are: adj(2,3,8), adj(2,11), and adj(3,10).

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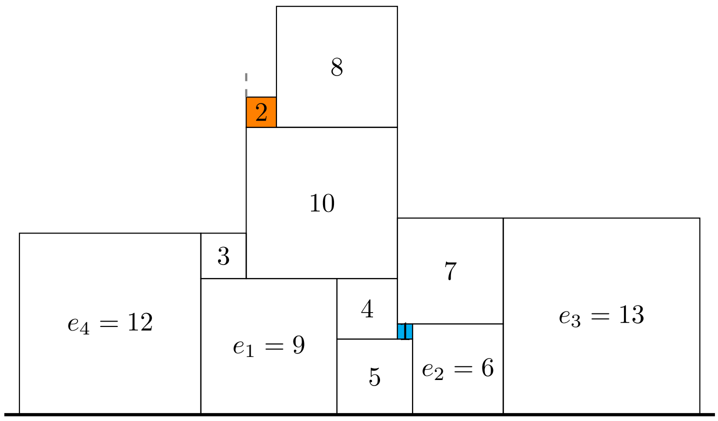
(a) If adj(2,3,8) is placed on the 4,9-flush, then the 8-square must be placed on the right to avoid wasted space. However there will be wasted space over the 2-square for either permutation.

(b) If adj(2,11) is placed on the 4,9-flush, then the 2-square must be on the left to avoid wasted space. Further the set of squares adjacent to the 2,9-flush must sum to 11 to avoid wasted space over the 2-square and the only set that satisfies this is adj(3,8). However, we established earlier that $|e_4| \geqslant 10$. So we cannot have adj(2,11) on the 4,9-flush.

(c) If adj(3,10) is placed on the 4,9-flush, then the 3-square must be on the right to avoid wasted space over the 3-square. Further, e_4 must be the 12-square to avoid wasted space over the 3-square (there are no other sets that sum to 12). Note that the remaining squares less than the 10-square are the 2 and 8-square. So any squares placed on top of the 3 or 7-square will have a height above the 10-square. Therefore the set of squares on top of the 10-square must sum to ten. The only set that satisfies this is adj(2,8). However, for either permutation there will be wasted space over the 2-square. This is because the next smallest square is the 11-square and placed on either side will have a height above the 2-square.



Therefore, e_3 cannot be the 13-square.

Thus, the sets adj(1,4) and adj(2,3) cannot be placed on top of the 5-square and hence the 5-square cannot be an edge square. □

Remove 6, 7, and 8 From an Edge

Claim. We cannot have the 6, 7, and 8-square on the same edge.

Proof. Suppose the 6, 7, and 8-squares are on the edge. Then there are three cases to consider: all adjacent, one disjoint, all disjoint.

1. We consider the case where all squares are disjoint (this case will help prove the other cases). Since squares 6, 7, and 8 are disjoint on an edge, then each of them has squares $\geqslant 9$ on either side or (at most two) are

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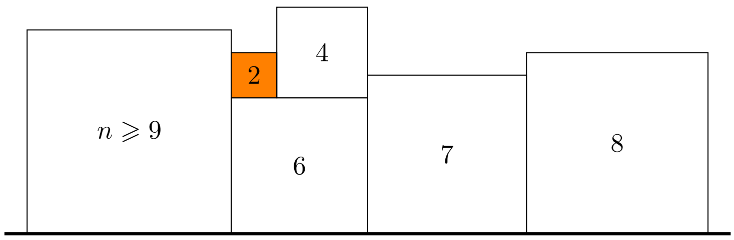
corner squares. Therefore we need unique sets of squares to be placed on the 6, 7, and 8-square that sums to six, seven, and eight respectively. Consider the total length of these squares, namely, $6+7+8=21$, and the total length of squares 1–5, namely, $1+2+3+4+5=15$. Since $21>15$ then there is no combination of squares 1–5 that can uniquely be placed of squares 6, 7, and 8. Thus, squares 6, 7, and 8 cannot be disjoint on an edge.

2. Suppose adj(6,7,8). There are six different permutations of adj(6,7,8). However by symmetry, we may say $ord(6,7,8) \equiv ord(8,7,6)$, $ord(6,8,7) \equiv ord(7,8,6)$, and $ord(7,6,8) \equiv ord(8,6,7)$. So we only need to consider three permutations.

   (a) Suppose ord(6,7,8) is on an edge. Then place a square on the left of the 6-square, call this n; note $|n| \geqslant 9$. Then any squares placed on top of the 6-square must sum to six; this holds if the 6-square is a corner square as well. Then we have the possible combinations: adj(1,5), adj(2,4), and adj(1,2,3).

      i. Suppose adj(2,4) is placed on top of the 6-square. Then the set of squares placed on top of the 7-square must sum to seven. Note that the remaining squares less than seven are the 1, 3, and 5-squares; so there are no combination of these squares that sum to seven. This implies that there are no squares that can be placed on top of the 7-square without wasted space.



      ii. Consider the two cases that involve the 1-square. If the 1-square is not on the right, then there will be wasted space over the 1-square. Place the 1-square on the right. Then, any set of squares that are placed on top of the 1,7-flush must sum to eight.

      - If adj(1,2,3) was placed on top of the 6-square then the remaining squares less than the 8-square are the 4 and 5-square. Therefore there is no combination of these that sum to eight.
      - If adj(1,5) was placed on top of the 6-square then the remaining squares less than 8 are the 2, 3, and 4-square, so there is no combination of these that sum to eight. Therefore there will be wasted space over the 1,7-flush for any set of squares placed on the 1,7-flush.

      Thus, both cases that involve the 1-square cannot be placed on top of the 6-square.

   Thus we cannot have ord(6,7,8) on an edge.

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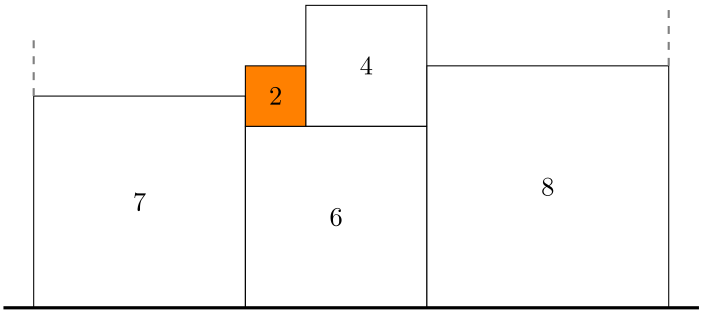
(b) Suppose ord(7,6,8) is on the edge. Then, any set of squares placed on top of the 6-square must sum to six. The only combinations that satisfy that are adj(2,4), adj(1,5), and adj(1,2,3).

i. Suppose adj(2,4) is placed on top of the 6-square, in either permutation. Then the set of squares placed on top of the 7-square must sum to seven. Note that the remaining squares less than the 7-square are the 1, 3, and 5-squares; so there are no combination of these squares that sum to seven. Therefore, there are no squares that can be placed on top of the 7-square without wasted space.

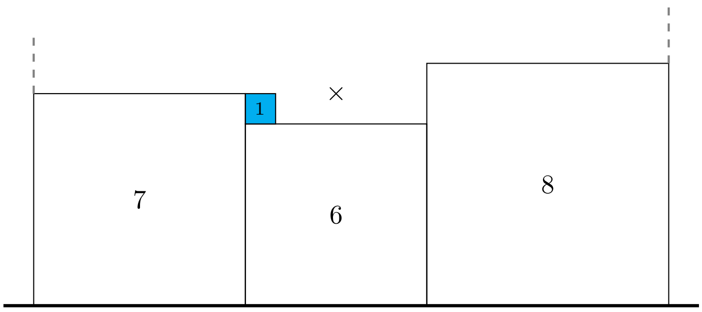


Thus, we cannot have adj(2,4) placed on top of the 6-square.

ii. Consider the two cases that involve the 1-square. If the 1-square is not on the left, then there will be wasted space over the 1-square. So place the 1-square on the left. Then, any set of squares that are placed on top of the 1,7-flush must sum to eight.

- If adj(1,2,3) was placed on top of the 6-square then the remaining squares less than the 8-square are the 4 and 5-square. So there is no combination of these that sum to eight.
- If adj(1,5) was placed on top of the 6-square then the remaining squares less than the 8-square are the 2, 3, and 4-square. So there is no combination of these that sum to eight.

Therefore, we cannot have adj(1,2,3) nor adj(1,5) placed on top of the 6-square, for any permutation.



Thus, we cannot have ord(7,6,8) on an edge.

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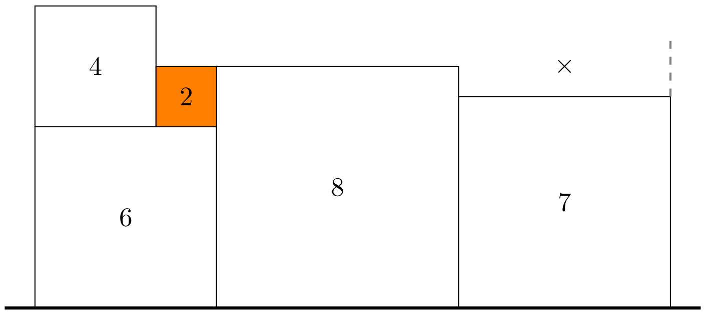
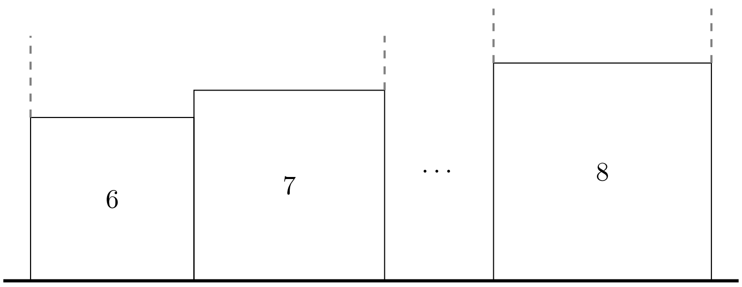
(c) Suppose ord(6,8,7) is on an edge. Note that any square placed on the right of the 6-square will be $\geqslant 9$, and similarly for the right of the 7-square. Any set of squares placed on top of the 6 and 7-square must therefore sum to six and seven respectively. However, if the 1-square is placed on the 6-square this guarantees wasted space over the 6-square; so place adj(2,4) on the 6-square. Further, the only remaining squares less than the 7-square are the 1,3, and 5-squares. Therefore there are no combinations that sum to seven, and will thus yield wasted space over the 7-square.



Thus, we cannot have ord(6,8,7) on an edge.

We have covered the cases of adj(6,7,8); now we will cover the cases that involve one disjoint.

3. Suppose adj(6,7) with the 8-square disjoint are all on an edge. Without loss of generality let the 6-square be on left of the 7-square, and place the 8-square placed arbitrarily on the base disjoint from adj(6,7). Note that either the 6 or 7-square is a corner square, or the squares placed on either side of adj(6,7) is $\geqslant 9$; this holds for the 8-square as well.



Note that we must have unique sets of squares to be placed on the 6 and 8-squares that sum to six and eight respectively. The sets of squares that sum to six are adj(2,4), adj(1,5), and adj(1,2,3).

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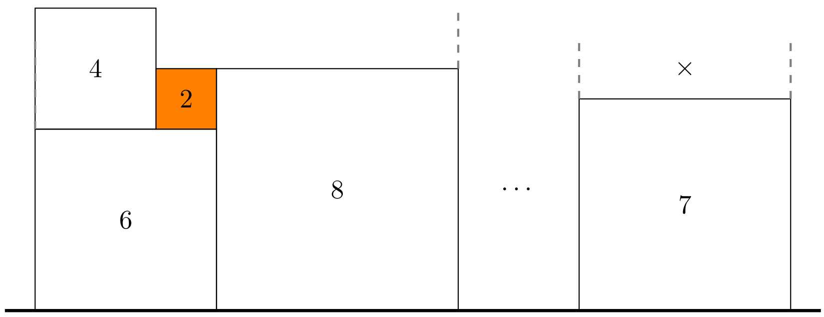
(a) Suppose adj(2,4) is placed on top of the 6-square. Then we need unique sets of squares to be placed on the 7 and 8-square. That is, we need unique sets of squares on the 6, 7, and 8-square, which cannot happen by the all disjoint case.

(b) Consider the two cases that involve the 1-square. If the 1-square is not on the right, then there will be wasted space over the 1-square. So place the 1-square on the right. Then, any set of squares that are placed on top of the 1,7-flush must sum to eight; adj(3,5) is the only set satisfying this sum. However this leaves no remaining sets to be placed on the 8-square.

Thus, we cannot have adj(6,7) with the 8-square disjoint.

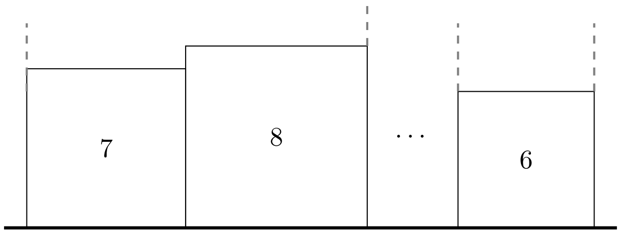
4. Suppose adj(6,8) with the 7-square disjoint are all on an edge. Without loss of generality let the 6-square be on left of the 8-square, and place the 7-square placed arbitrarily on the base disjoint from adj(6,8). Note that either the 6 or 8-square is a corner square, or the squares placed on either side of adj(6,8) is $\geqslant 9$; this holds for the 7-square as well.

   Then any set of squares placed on top of the 6-square must sum to six. The possible combinations of squares that sum to six are adj(1,2,3), adj(1,5), and adj(2,4). However if the 1-square is placed on the 6-square, then there will be wasted space for any permutation of any set placed on the remainder of the 6-square; so place adj(2,4) on the 6-square. Note any set of squares placed on top of the 7-square must sum to seven. The only remaining squares less than the 7-square are the 1, 3, and 5-square. Therefore there are no remaining sets that sum to seven and will yield wasted space over the 7-square.



Thus, we cannot have adj(6,8) with the 7-square disjoint.

5. Suppose adj(7,8) with the 6-square disjoint are all on the edge. Without loss of generality let the 6-square be on left of the 8-square, and place the 7-square placed arbitrarily on the base disjoint from adj(6,8). Either the 7 or 8-square is a corner square, or the squares placed on either side of adj(7,8) is $\geqslant 9$; this holds for the 6-square as well.



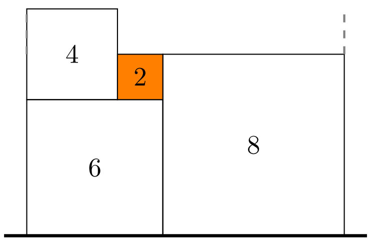
We must have the set of squares placed on top of the 7-square sum to seven, and similarly, the set of squares on the 6-square must sum to six. The only unique sets that satisfy this are: adj(1,5) on the 6-square and adj(3,4) on the 7-square. By placing adj(3,4) on the 7-square, the set of squares on top of the 8-square must then sum to eight. But the 2-square is the only remaining square less than the 8-square. Thus, there will be wasted space over the 8-square. Thus, we cannot have adj(7,8) with the 6-square disjoint.

Therefore, all cases of the 6, 7, and 8-square placed on the edge have been eliminated. □

Proof of Theorem 1(iii)

Proof. Suppose the 6-square is on the edge but not adj(6,7). Then either the 7-square is on the same edge but disjoint or the 7-square is not on that edge. Consider the case when the 6 and 7-squares are on an edge but disjoint. This case yields wasted space since it is logically equivalent to having adj(6,8) and 7-square disjoint or having (6,8,7) on an edge.

Suppose the 7-square is not on an edge. Then the 6-square will have squares on either side which are $\geqslant 8$ (note the 6-square cannot be a corner square by Lemma 1). If the 8-square is not adjacent to the 6-square then we have $ord(n_1, 6, n_2)$ but $|n_1|, |n_2| > 8$ which guarantees wasted space over the 6-square. So we must have adj(6,8). Furthermore, if the 1-square is placed on top of the 6-square, there will be wasted space. Then we must have adj(6,8) on an edge and adj(2,4) on the 6-square. Moreover the 2-square must be placed adjacent to the 8-square to avoid wasted space. Now we need a set of squares that sum to ten to be placed on top of the 2,8-flush. The remaining sets that sum to ten are adj(1,9), adj(3,7), and the 10-square.



We begin by considering when adj(1,3) is placed on the 4-square. Then the 10-square is the only set that sums to ten. So the 10-square must be placed on the 2,8-flush. Then any square placed on the 3-square will hang over the

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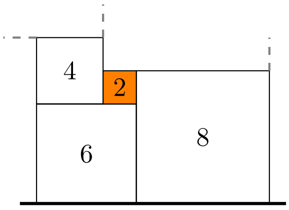
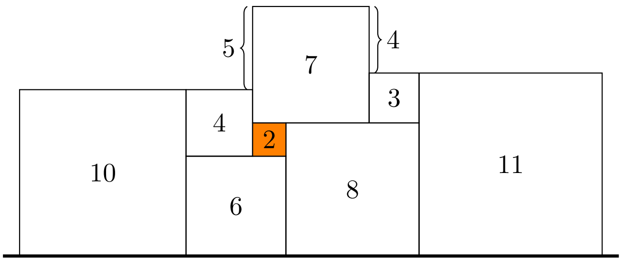
3-square and yield wasted space over the 1-square. Thus adj(1,3) cannot be placed on the 4-square. Thus, any set of squares on the 4-square will hang over the 4-square on the left (since any square placed on the 2-square will have a height above the 4-square). That is, we need two unique sets of squares that sum to ten. Note that we cannot have adj(3,7) on the left of the 4,6-flush by assumption (this would be adj(6,7)).



1. Suppose the adj(6,9) is on the edge, with the 1-square on the 9-square. Then the 5-square must be placed on the 1,4-flush, otherwise there will be wasted space next to the 1-square. If there is a square adjacent to the 9-square on the edge, it will be $\geqslant 10$. This implies that the remaining set of squares on the 9-square must sum to eight (this logically equivalent if the 9-square is a corner square as well). However, there are no remaining sets of squares that sum to eight. Thus we cannot have adj(1,9) on the left of the 4,6-flush.

2. Suppose adj(6,10) is on the edge, and adj(1,9) is placed on the 2,8-flush. Then the 1-square must be placed on the right to avoid wasted space. Then any square placed on the 1-square would hang over to the right of the 1-square. (This implies the 8-square is not a corner since it would otherwise be a contradiction.) So, we need a set of squares that sums to nine to be placed adjacent to the 1,8-flush. However, there are no remaining sets that sum to nine. Thus we cannot have adj(1,9) on the 2,8-flush.

3. Suppose adj(6,10) is on the edge, and adj(3,7) is placed on the 2,8-flush. If the 3-square is placed on the left then any set of squares placed on the 3-square will hang over the 3-square and yield wasted space next to the 3-square. So place the 3-square on the right. Again, any set of squares placed on the 3-square will hang over the 3-square, which implies the 8-square is not a corner square. Note the 11-square is the only remaining set of squares that sums to eleven and therefore we must have adj(8,11) on the edge.



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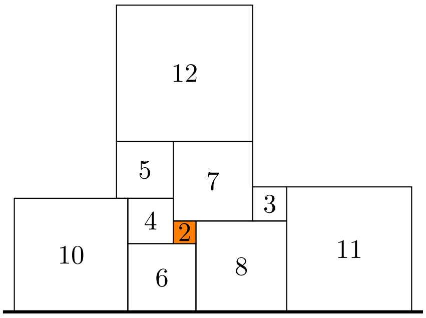
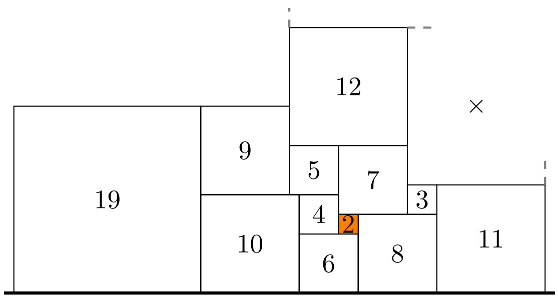
Here we see that any square placed on the 3-square will have a height above the 7-square. Further, any set of squares placed on the 7-square will hang over the 7-square (on the left). Therefore the 5-square is the only square that can be placed on the on the 4-square without wasted space. Now, there are no remaining sets of squares that sum to five, so the set of squares on the 5,7-flush must sum to 12. The 12 square is the only square that satisfies this sum, so place the 12-square on the 5,7-flush.



Suppose the 10-square is a corner square. Then the remaining squares on the 4,10-flush must sum to nine. The 9-square is the only remaining square satisfying this sum. However, since the 9-square is the only remaining set, then there will be wasted space for any set of squares placed on the 9-square. Thus, the 10-square is not a corner square.

Suppose the 9-square is placed on the edge adjacent to the 10-square. Then the set of squares on the 9-square must sum to nine. But there are no remaining sets that satisfy this, therefore there must be a square $\geqslant 13$ adjacent to the 10-square on the edge. A similar argument holds for the 11-square; that is, if the 11-square is not a corner square, there must be a square $\geqslant 13$ to the right of the 11-square.

We have just determined that the 10-square is not a corner square but is adjacent to a square that is at least the 13-square. Therefore the set of squares on the 10-square must sum to nine, so place the 9-square on the 10-square. Then any set of squares placed on the 9-square will hang over the 9-square. So we need the set of squares on the 9,10-flush to sum to 19. The only sets that satisfy this are adj(1,18) and the 19-square. However, if adj(1,18) is placed on the 9,10-flush then any set of squares placed on the 1,9-flush would yield wasted space next to the 1-square. Therefore we place the 19-square adjacent to the 10-square on the edge.



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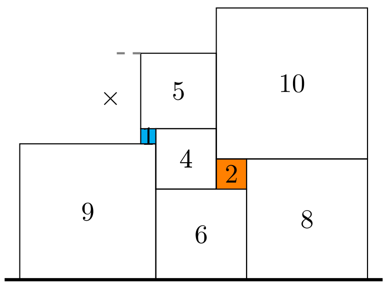
Note that squares 2–12 have been used. So any square placed on the 9-square will have a height above the 12-square. Similarly any set of squares placed on the 12-square will hang over the 12-square. Thus the remaining set of squares adjacent to the 7,12-flush must sum to 16. The only sets that sum to 16 is adj(1,15) and the 16-square. However since the set of squares on the 3,11-flush must sum to 14 then this is a contradiction for both cases.

Thus, we cannot have adj(3,7) on the 2,8-flush.

4. Suppose the 10-square is placed on the 2,8-flush. Consider when adj(1,3) is placed on the 4-square. Then the 1-square must be on the left to avoid wasted space. Then any set of squares placed on the 4-square will hang over the 4-square and therefore yield wasted space over the 1-square. Therefore adj(1,3) cannot be on the 4-square. Further this implies any set of squares placed on the 4-square will hang over the 4-square. So we need a set squares that sums to ten to be placed adjacent to the 4,6-flush. The only remaining set that satisfies this is adj(1,9) (recall the 7-square cannot be on the edge by assumption). Place the 9-square on the edge to the left of the 6-square, and the 1-square on top of the 9-square. Then the 5-square must be placed on the 1,4-flush to avoid wasted space on the left of the 1-square. Note any set of squares placed on the 5-square will hang over the 5-square. However, there are no remaining sets of squares that sum to six to be placed adjacent to the 1,5-flush. Thus, we cannot have the 10-square placed on the 2,8-flush.



Therefore, we have exhausted all cases with the 7-square both on an edge (disjoint from the 6-square) and the 7-square not on an edge. Thus, we conclude that if the 6-square is on an edge, then we must have adj(6,7) on an edge. □

Proof of Theorem 1(iv)

Suppose $ord(a, n, b) \subseteq \epsilon$ is an ordered subset of a packing $\mathcal{P}$, where n is the smallest square lying on ϵ. We will show that

$$\min\{|a|, |b|\} \leqslant 2|n| - 1.$$

To begin, we need a preliminary lemma which will reduce the number of cases to consider.

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Lemma 2. Suppose $ord(a, n, b)$ is as above, and let $\Gamma = \{a_1, a_2, \dots, a_k\}$ be a partition of $|n|$ into distinct integers; that is, $|n| = a_1 + \cdots + a_k$ and $a_i \neq a_j$ for $i \neq j$. Furthermore, suppose that the corresponding squares adj$(a_1, \dots, a_k)$, where $|a_i| < |a_{i+1}|$, all lie on top of n in $\mathcal{P}$.

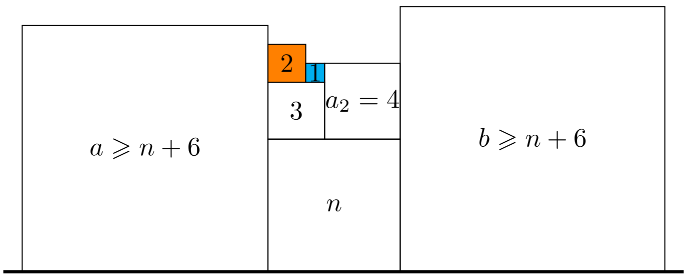
1. If $|a|, |b| \geqslant |n| + 2$ then $1 \notin \Gamma$.

2. If $|a|, |b| \geqslant |n| + 3$, then $2 \notin \Gamma$.

3. If $|a|, |b| \geqslant |n| + 6$, then $3 \notin \Gamma$.

Proof. Suppose $1 \in \Gamma$ and $|a|, |b| \geqslant |n| + 2$. Since $|a|, |b| \geqslant |n| + 2$, then this guarantees wasted space over the 1-square, for any permutation of adj$(a_1, \dots, a_k)$. A similar argument holds (2).

Now, suppose $|a|, |b| \geqslant |n| + 6$ and $3 = a_1 \in \Gamma$ for contradiction. Then we have $\Gamma = \{3, a_2, \dots, a_k\}$ where $a_i > 3$ for each $i = 2, \dots, k$. We must have a set of squares on the 3-square that sums to three. Adj(1,2) is the only set satisfying this sum; place adj(1,2) on the 3-square. Let adj(3,a_2), if $|a_2| > 4$ then this guarantees wasted space over the 1-square for either permutation. Therefore let a_2 be the 4-square and makes a flush with the 1-square. Then the 5-square is the only square that can be placed on the 1,4-flush. However, this will guarantee wasted space over the 2-square.



Note that if $|n| > 7$ then we can improve the restriction of (3) to: $|a|, |b| \geqslant |n|+5$, then $3 \notin \Gamma$.

□

Example: Consider when n is the 7-square and the smallest square on an edge, ϵ. Further suppose that ord(10,7,9) in on ϵ. We need to determine the sets of squares that sum to seven: the distinct partitions of seven are $1+6$, $2+5$, $3+4$, and $1 + 2 + 4$. By Lemma 2, we can remove the cases that use the 1-square; that is, the remaining partitions are $2 + 5$ and $3 + 4$. So the remaining sets of squares that can be placed on the 7-square are adj(2,5) and adj(3,4).

Proof. (of Theorem 1.1 (iv)). Suppose n is the smallest square on the edge. We want to show that we cannot have the second smallest square be greater than or equal to $2n - 1$. Recall the 6-square is the smallest square we can have on the edge. Further, note that there must be a minimum of four squares on the edge. So the maximum that n can be is the 10-square, otherwise if n were the 11-square, then the smallest set of squares is $(11, 21, 22, 23)$ which sum is greater that 70. This leaves us with five cases to check: $n = 6, 7, 8, 9$, and 10.

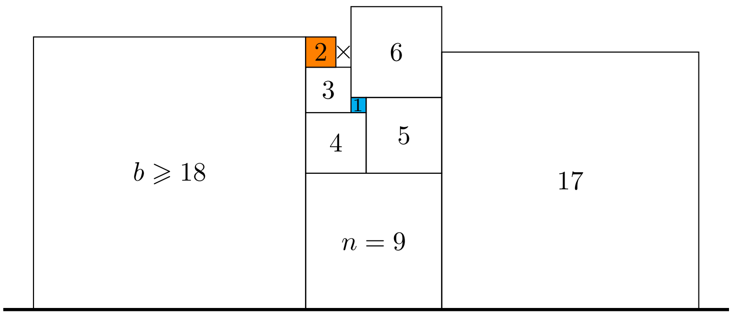
1. Suppose n is the 6-square. Since the 6-square is on an edge, then by Theorem 1(iii) we must have adj(6,7) on the edge, and this is case holds.

2. Suppose n is the 7-square, let a be the 13-square and b such that $|b| \geqslant 14$ on the right and left (without loss of generality) of the 7-square respectively. Then the only squares that can be placed on top of the 7-square are adj(1,6), adj(2,5), adj(3,4), and adj(1,2,4). These are all ruled out by Lemma 2.

3. Suppose n is the 8-square where a is the 15-square placed on the right and b such that $|b| \geqslant 16$ on the left (without loss of generality). The only squares we have that can go on the 8-square are adj(1,7), adj(2,6), adj(3,5), adj(1,2,5), and adj(1,3,4). However, by Lemma 2 these are all ruled out.

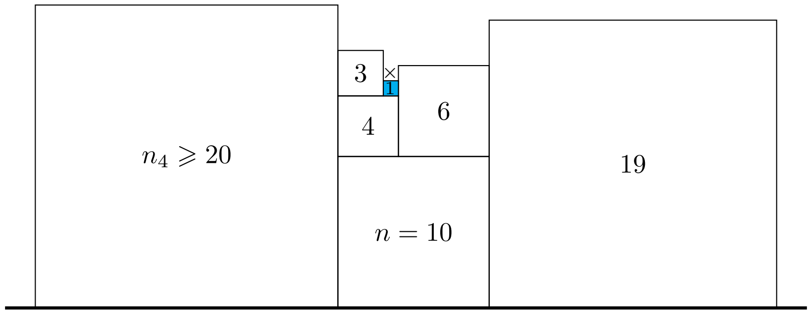
4. Suppose n is the 9-square where a is the 17-square placed on the right and b such that $|b| \geqslant 18$ on the left (without loss of generality). We have the following possibilities that can go on the 9-square are adj(1,8), adj(2,7), adj(3,6), adj(4,5), adj(1,2,6), adj(1,3,5), and adj(2,3,4). However, by Lemma 2 we have only to check adj(4,5).

   Suppose adj(4,5) is placed on top of the 9-square with the 4-square on the left. The only squares that can be placed on the 4-square are the 1, 2, and 3-squares. Placing the 2-square on top of the 4-square will yield wasted space. Place adj(1,3) on top of the 4-square with the 3-square on the left (otherwise there is wasted space over the 1-square). Notice, we have a 1,5-flush, and the set of squares placed on the 1,5-flush must sum to six. The 6-square is the only remaining square that satisfies this sum; so place the 6-square on the 1,5-flush. The only square that can placed on top of the 3-square is the 2-square, and that would yield wasted space on the left/ right of the 2-square.



5. Suppose n is the 10-square where a is the 19-square placed on the right and b such that $|b| \geqslant 20$ on the left (without loss of generality). Then the possible sets of squares that can go on the 10-square are adj(1,9), adj(2,8), adj(3,7), adj(4,6), adj(1,2,7), adj(1,3,6), and adj(2,3,5). Again applying Lemma 2 we are only left with the case adj(4,6).

   Suppose adj(4,6) is placed on top of the 10-square; consider when the 6-square is on the right. Then, the only square that can be placed on the 4-square without wasted space is adj(1,3). However, this yields wasted space over the 1-square for either permutation.



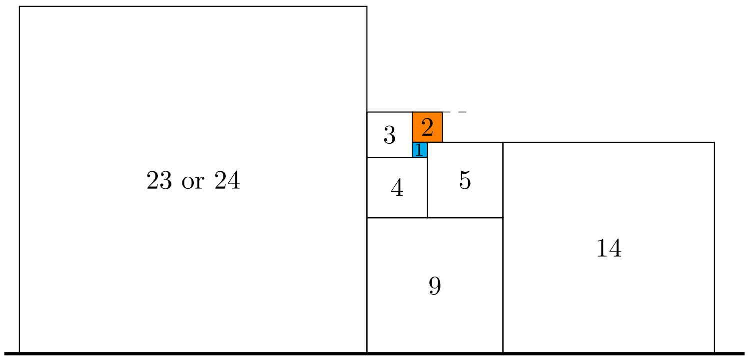
Therefore, with the constraints of the 70×70 square, we cannot have n and $2n - 1$ be the smallest squares on the edge. □

Proof of Theorem 2

Proof. Suppose $v = \{9, 14, 23, 24\}$ is an edge. The 9-square must be either a corner square or a non-corner square.

1. Suppose the 9-square is a corner square. Then we must have adj(9,14) by Theorem 1(iv). Further, the square on the adjacent edge (i.e., c or c_{op}) that is adjacent to the 9-square must be the 6, 7, or 8-square. Similar to the proof of Lemma 2, we are guaranteed wasted space for the 9-square adjacent to the 6, 7, and 8-squares.

2. Suppose the 9-square is not a corner square. Again by Theorem 1(iv) we must have adj(9,14). Note that we also have either adj(9,23) or adj(9,24). Then by Lemma 2 adj(4,5) is the only set of squares that can be placed on the 9-square. Further, adj(1,3) will be the only squares that can be placed on the 4-square, without wasted space. However, any set of squares placed on the 3-square will hang over the 3-square. The 2-square is the only square that can be placed under the overhang from the 3-square. However, any set of squares placed on the 2,3-flush will hang over the 2-square, which guarantees wasted space. Thus, the 9-square must be a corner; hence a contradiction.



Therefore combining these cases, we conclude $v = \{9, 14, 23, 24\}$ is not an edge. Thus establishing the theorem. □

Constructing the Frame

To construct a potential packing $\mathcal{P}$, we first need to determine the edges. In this section, we present the MATLAB code that generates all possible frames.

The Matrix A

Here we determine the matrix A : all possible combinations of squares that sum to 70 taking Theorem 1(i) into account. If we do not include these restraints initially, then MATLAB cannot handle the calculation.

```
1  clear, clc, format compact;
2  sumto=70; % number we want to sum to
3  n=24; % largest number to be used
4  m=7; % number of elements per row
5  %----------------------------------------
6  zro=linspace(0,0,3);
7  f=[zro,6:n]; % f is a vector with elements to be used
8  A=nchoosek(f,m); % A is a matrix of all possible combos
9  Asum=sum(A,2); % adds a column stating the row's sum
10 A=[A, Asum]; % list of all combos; last element in row is
       its sum.
11 R=rows(A); % R is number of rows
12 k=1;
13 for i=1:R % deletes rows that don't sum to 70
14     if A(i,m+1)==sumto
15         B(k,:)=A(i,:);
16         k=k+1;
17     end
18 end
19 A=unique(B, 'rows'); % Eliminates repeated rows
20 A(:,m+1)=[]; % Removes the sum column
```

The purpose of the first line is to keep the display cleaner. In lines 2–4 we define the variables *sumto, n, m* where *sumto* $:= 70$ since we want integer numbers that sum to 70, $n := 24$ since we use the numbers 1–24, and $m := 7$ since there will be at most seven squares on an edge (due to the restraints stated earlier). In line 6, we have *linspace(0,0,3)* which creates a vector with three elements "from zero to zero" (i.e., a vector with three zeros only). The number of zeroes is the max possible squares on an edge minus the minimum possible squares on an edge. The zeros act as placeholders for the matrices computed. On line 7 we make a vector $f = [zro, 6 : n]$ which means it has the elements from *zro* and 6–24 (recall $n := 24$). On line 8 we create the matrix A using the *nchoosek* function, which creates all possible combinations of the elements in f using $m := 7$ columns. Lines 9 and 10 create an eighth column in the matrix A which is the sum of each of the rows accordingly. In lines 11 and 12, we create parameters for the loop which begins on line 13: R is the number of rows in A

and k will be a counter in the loop. The *for loop* removes every row that does not have 70 as the eighth element—since the last column is the sum of each row. Finally, line 19 removes any repetition that may have occurred and line 20 removes the entire eighth column—it is no longer required and removing it will reduce later computation.

Next, the remaining constraints from Theorem 1 are included. In particular, we eliminate rows that do not satisfy at least one Theorem 1(ii)–(v). Using the same $R = rows(A)$ from above, we run another for loop.

```
R=rows(A);
for i=R:-1:1
 for j=1:5
  if A(i,j)~=0 && A(i,j+1)>=2*A(i,j)-1
   A(i,:)=[]; % We cannot have (n,2n-1,...)
   break;
  end
  if A(i,j)==6 && A(i,j+1)~=7
   A(i,:)=[]; % must have adj(6,7)
   break;
  end
  if A(i,j)==6 && A(i,j+1)==7 && A(i,j+2)==8
   A(i,:)=[]; % Remove 6,7,8 from an edge.
   break;
  end
  if ismember(6,A(i,:))==1 && ismember(11,A(i,:))==0
   A(i,:)=[];
   break;
  end
 end
end
```

Each if-statement is a different restriction. MATLAB performs this by removing the cases that do not satisfy the claim; e.g., if there is a row in A that has $(6, 9, \ldots)$ then that is removed since it does not satisfy Theorem 1 (iii). Therefore from here on, A is the matrix of all possible ways to sum to 70 given our restraints from Theorem 1. (Note that we use a for-loop where $i : R \to 1$. It was later determined that a while loop is more easily manageable and yields the same result. This is why we use a for-loop first but while-loops later.)

The Matrix $\mathbf{A_{op}}$

Next, we fix a row $v \in A$ and remove all rows which have no common element with v (not including zeros).

DOI: https://doi.org/10.2478/rmm-2018-0001

```
1  q1=1;
2  v=A(q1,:); % we fix a vector v in A
3  A(q1,:)=[]; R=R-1; % we remove v from A
4  A2=A; R2=rows(A2);
5  p=1;
6  while p<=R2 % Here we create A2
7      Int=intersect(A2(p,:),v);
8      Int=Int(Int>0); %removes zero element
9      r3=length(Int);
10     if r3>=1 % if there is a common element
11         A2(p,:)=[];
12         R2=R2-1;
13         p=p-1;
14     end
15     p=p+1;
16 end
```

Note that line 1 states $q1 = 1$; this is for future potential loops. By adding this we can later add a while loop to loop through all different choices of v, but for now it determines our fixed v.

The Matrix C

Next, we fix a row $v_{op} \in A_{op}$ and create the matrix C which each row has one common element with v and v_{op}. This will also done for the other fixed edges.

```
q2=1; vop=A2(q2,:); % we fix a vector vop on A2
A2(q2,:)=[]; R2=R2-1;
C=A; R3=rows(C);
j=1;
while j<=R3 % creating matrix C
    Int1=intersect(v(v>0),C(j,:));
    L1=length(Int1);
    Int2=intersect(vop(vop>0),C(j,:));
    L2=length(Int2);
    if L1~=1 || L2~=1 % want L1=1 or L2=1 or both
        C(j,:)=[];
        R3=R3-1;
        j=j-1;
    end
  j=j+1;
end
C=unique(C, 'rows');
```

The matrix C is now the matrix where each row corresponds to an edge of $\mathcal{P}$ such that there is exactly one square in common with v and one (different) square in common with v_{op}.

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Throughout the following theorems, let $\mathcal{P}$ be a packing with edges $\{\epsilon_i\}$. Define $a_i = \min \epsilon_i$ to be the smallest square on each edge, and without loss of generality assume $|a_i| < |a_{i+1}|$. Let

$$S = \{1, 2, \ldots, a_4\} - \{a_1, a_2, a_3, a_4\} \subseteq \mathcal{P}$$

be the set of all possible squares that could be placed on top of all a_i-squares.

Theorem 4. *If $\mathcal{P}$ is a packing, then $\mathcal{P}$ must satisfy*

$$\sum_{i=1}^{4} |a_i| \leqslant \sum_{n \in S} |n|.$$

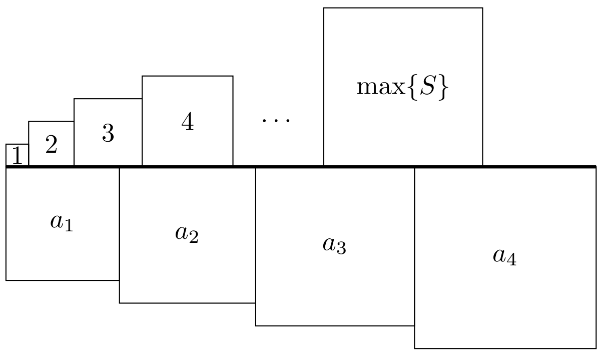
Proof. Consider a packing $\mathcal{P}$ such that

$$\sum_{i=1}^{4} |a_i| > \sum_{n \in S} |n|$$

for contradiction. Then we can place the squares $\text{adj}(1, 2, \ldots, \max\{S\})$ such that there is a common flush. Similarly, on the opposing side of the flush place $\text{adj}(a_1, a_2, a_3, a_4)$. By assumption $|adj(1, 2, \ldots, \max\{S\})| < |adj(a_1, a_2, a_3, a_4)|$, so any permutation of either sets of squares shows that there will be wasted space over a_i for some $i = 1, 2, 3, 4$. Therefore a packing $\mathcal{P}$ must satisfy

$$\sum_{i=1}^{4} |a_i| \leqslant \sum_{n \in S} |n|,$$

as needed.



□

Example Let

$$\partial \mathcal{P} = \{9, 14, 23, 24\} \cup \{10, 17, 21, 22\} \cup \{11, 13, 22, 24\} \cup \{12, 17, 18, 23\}$$

be the set of all edge squares of a packing $\mathcal{P}$. The smallest squares on each edge are $\{9, 10, 11, 12\}$. Then $S = \{1, \ldots, 8\}$, so we have $\sum_{n \in S} |n| = 1 + \cdots + 8 = 36$. Further, $\sum |a_i| = 9 + 10 + 11 + 12 = 42$. That is,

$$\sum_{n \in S} |n| = 36 < 42 = \sum |a_i|.$$

Thus by Theorem 4, $\mathcal{P}$ will yield wasted space.

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Corollary 1. If $\mathcal{P}$ is a packing, then $\mathcal{P}$ must satisfy

$$4(|a_1| + |a_2| + |a_3| + |a_4|) \leqslant |a_4|(|a_4| + 1).$$

Proof. Consider the inequality of Theorem 4 and recall $|a_i| < |a_{i+1}|$. Then,

$$\begin{aligned}
\sum_{i=1}^{4} |a_i| &\leqslant \sum_{n \in S} |n| \\
|a_1| + |a_2| + |a_3| + |a_4| &\leqslant (1 + \cdots + |a_4|) - (|a_1| + |a_2| + |a_3| + |a_4|) \\
2(|a_1| + |a_2| + |a_3| + |a_4|) &\leqslant 1 + \cdots + |a_4| \\
2(|a_1| + |a_2| + |a_3| + |a_4|) &\leqslant \frac{|a_4|(|a_4| + 1)}{2} \\
4(|a_1| + |a_2| + |a_3| + |a_4|) &\leqslant |a_4|(|a_4| + 1),
\end{aligned}$$

as needed. □

In MATLAB it is more practical to use Corollary 1 as opposed to Theorem 4.

Example Let $\partial\mathcal{P}$ be as in the above example. We can see that

$$4(9 + 10 + 11 + 12) = 4(42) = 168 > 156 = 12(13).$$

So, by Corollary 1, $\partial\mathcal{P}$ will again yield wasted space.

The Matrix $\mathbf{C_{op}}$

We run a large while loop fixing each row in C. Once $c \in C$ is fixed, we then create the matrix $C_{op} \subseteq C$ by again removing all rows in C with common, non-zero elements of c. After the matrix C_{op} is created, we then apply Corollary 1 and remove rows in C_{op} that do not satisfy

$$4(|a_1| + |a_2| + |a_3| + |a_4|) \leqslant |a_4|(|a_4| + 1),$$

where a_i are the minimum squares on each of the edges and $|a_i| < |a_{i+1}|$. Finally, after including Corollary 1, we create the matrix (for display purposes) Frm which has rows v, v_{op}, and c; we also display the matrix C_{op} to each corresponding Frm (*count* is used to count the number of frames, i.e., *count* adds 1 for every c_{op} generated).

DOI: https://doi.org/10.2478/rmm-2018-0001

```
% Next we create Cop: vectors opposite of c and loop
    through it
q3=1;
R3=rows(C);
count=0;
while q3<=R3
   c=C(q3,:);
   C(q3,:)=[];
   R3=R3-1;
   Frm=[v;vop;c]
   Cop=C;
   R4=rows(Cop);
   q4=1;
   while q4<=R4 %creates Cop
        cop=Cop(q4,:);
        Intc=intersect(cop,c(c>0)); % vector of union
        Rc2=length(Intc);
        if Rc2>0
             Cop(q4,:)=[];
             R4=R4-1;
             q4=q4-1;
        end
   q4=q4+1;
   end
   % next we remove frames that do not satisfy amax(amax
       +1),4sum(ai), ai=min of edges
   R4=rows(Cop);
   q4=1;
   while q4<=R4
        cop=Cop(q4,:);
        ais=[min(v(v>0)),min(vop(vop>0)),min(c(c>0)),min(
            cop(cop>0))];
        amax=max(ais);
        if amax*(amax+1)<4*sum(ais)
             Cop(q4,:)=[];
             R4=R4-1;
             q4=q4-1;
        end
   q4=q4+1;
   end
   if rows(Cop)>1 % all else will fail Cmid
   Frm=[v;vop;c]
   Cop
   Copsize=size(Cop);
   count =count+Copsize(1);
   end
   q3=q3+1;
end
count
```

Proof of Theorem 3

Proof. Running our code from the preceding section with $v = \{7, 8, 14, 18, 23\}$ and $v_{op} = \{9, 10, 11, 12, 13, 15\}$, there are 56 frames to check. Each "Frm" is a matrix where the first and second rows are v and v_{op}, respectively. The third row is a different c for each Frm. The matrix/vector following is Cop matrix, i.e., possibilities for the last edge.

The following Lemma will be important as it cuts the number of frames down from 56 to 9.

Lemma 3. Given our fixed edges v and v_{op}, the 10 and 14-squares cannot be corner squares.

Proof. Suppose firstly the 10-square is a corner square. Consider the square that will be placed adjacent to the 10-square in the adjacent edge (i.e., c or c_{op}). Since the 7-square is in v, then the 6-square cannot be adjacent to the 10-square. Further, squares 7–15 have been placed in either v or v_{op}. Therefore, the next smallest square that can be placed adjacent to the 10-square in the adjacent edge is the 16-square. Therefore, the 9-square is the only square that can be placed adjacent to the 10-square in v_{op}. However, this guarantees wasted space over the 9-square. Thus, the 10-square cannot be an edge.

Now, suppose the 14-square is a corner square. By the same reasoning, we know that the 16-square is next possible square that can be placed adjacent to the 14-square in the adjacent edge. This along with Theorem 1(iii) implies we must have adj(7,8,14) on v_{op}. Then the possible sets of squares to be placed on the 7-square are adj(1,6), adj(2,5), adj(3,4), and adj(1,2,4). Consider when we have adj(7,14). Adj(1,2,4) is guaranteed wasted space adjacent to the 2-square for all permutations, and placing adj(1,6) on the 7-square would leave a 1×2 (or longer) gap above the 6-square. Therefore the set of squares on the 8-square must sum to eight. Suppose adj(3,4) is placed on the 7-square, then we must have adj(1,2,5) on the 8-square. However, this guarantees wasted space over the 1-square for all permutations. In a similar way, we rule out placing adj(2,6) on the 8-square. Suppose adj(2,5) is placed on the 7-square. Then we must have adj(1,3,4) placed on the 8-square. We must have adj(1,2) to avoid wasted space over either the 1 or 2-square. This guarantees wasted space over the 1,2-flush.

This shows we cannot have adj(14,7). However, all cases were exhausted without the permutation of the 7 and 8-square taken into account except for adj(1,6) on the 7-square; we only need to consider this case. Let ord(14,8,7) be on v_{op}, and place adj(1,6) on the 7-square. Then we must have adj(1,8) to avoid wasted space over the 1-square. Then we need a set of squares that sum to nine to be placed on the 1,8-flush. The only sets are adj(2,3,4) and adj(4,5). But adj(2,3,4) is ruled out by Lemma 2. Place adj(4,5) on the 1,8-flush. Then either permutation will leave wasted space over the 4-square.

Therefore, we have exhausted all possibilities and so the 14-square cannot be a corner square.

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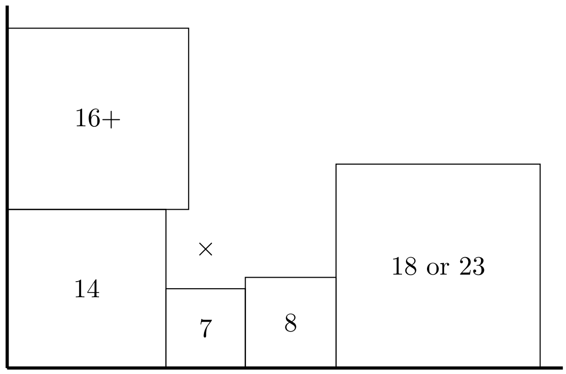


□

To account for this Lemma in the computer code, we add the following scripts *before* the code for generating *Cop* and Frm, and then run the code to generate the new set of possible frames.

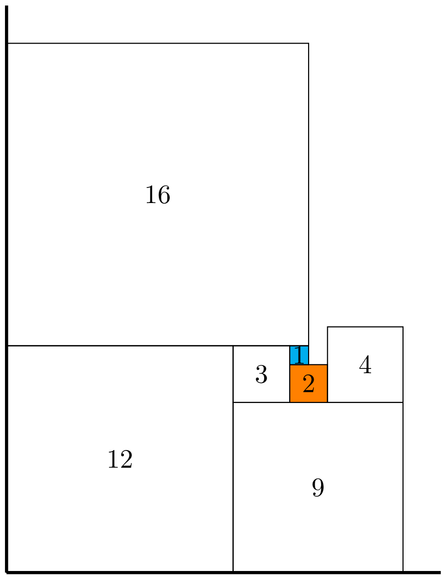
```
q3=1;
R3=rows(C);
while q3<=R3
   c=C(q3,:);
   if ismember(14,c)==1 || ismember(10,c)==1
    C(q3,:)=[];
    R3=R3-1;
    q3=q3-1;
   end
   q3=q3+1;
end
```

Note that this takes into consideration if a given v, v_{op}, and c yield zero c_{op} options, then these cases are ruled out immediately. This yields the following frames:

Frm1 =	0 0 7 8 14 18 23	Cop =	0 0 0 12 18 19 21
	0 9 10 11 12 13 15		0 0 0 13 17 18 22
	0 0 0 11 16 20 23		
Frm2 =	0 0 7 8 14 18 23	Cop =	0 0 0 12 16 18 24
	0 9 10 11 12 13 15		0 0 0 15 16 18 21
	0 0 0 11 16 20 23		
Frm3 =	0 0 7 8 14 18 23	Cop =	0 0 0 12 16 19 23
	0 9 10 11 12 13 15		
	0 0 0 11 18 20 21		
Frm4 =	0 0 7 8 14 18 23	Cop =	0 0 0 11 17 18 24
	0 9 10 11 12 13 15		0 0 0 13 17 18 22
	0 0 0 12 16 19 23		0 0 0 15 17 18 20
Frm5 =	0 0 7 8 14 18 23	Cop =	0 0 0 12 18 19 21
	0 9 10 11 12 13 15		
	0 0 8 9 16 17 20		

Frames 1, 2, and 3 are all ruled out (for all choices of c_{op}) as the 11-square is in the corner of c and v_{op}, but we must have adj(11,16) or adj(11,20) which yields wasted space to the right of the 11-square, for any c_{op}.

For frame 4, we have the 12-square in the corner of c and v_{op}, where adj(12,$\geqslant$ 16) $\subset c$. We can see that the 9, 10, and 11-squares are the only possible squares that can be adjacent to the 12-square in v_{op}. However, placing the 10 or 11-square adjacent to the 12-square leaves a $1 \times k_1$ or $2 \times k_2$ for $k_1, k_2 \geqslant 4$. Therefore we must have adj(9,12). Then the only tiling that can be placed above the 9-square is placing the 3-square adjacent to the 9 and 12-squares, then placing adj(1,2) on the 3-square. Finally the 4-square is the only remaining square that can be placed on the 9-square which makes the set of squares on the 9-square sum to nine. However, this guarantees wasted space over the 2-square.



Finally, consider fame 5. Note the 8 and 9-squares are a corner squares and the next smallest square to be adjacent to the 8-square in c is the 16-square. That is, we must have adj(8,$\geqslant$ 16), which contradicts Theorem 1(iv). Therefore frame 5 is also ruled out.

Therefore, we have exhausted all possible frames for $v = \{7, 8, 14, 18, 23\}$ and $v_{op} = \{9, 10, 11, 12, 13, 15\}$. Thus these cannot both be edges opposite each other. $\square$

Near Misses

Here is the MATLAB code to determine near-misses (Table 2.5).

```
clear; clc; format long;
t=1;
st=2*10^6; % start value
en=4*10^6; % end value
for n=st:en
y=sqrt(n*(n+1)*(2*n+1)/6);
        if rem(y,1)==0
                S(t,1)=n;
                S(t,2)=y;
                t=t+1;
        end
end
S
```

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Games and puzzles

MAXIMUM GENUS OF THE JENGA LIKE CONFIGURATIONS

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Abstract: *We treat the boundary of the union of blocks in the Jenga game as a surface with a polyhedral structure and consider its genus. We generalize the game and determine the maximum genus among the configurations in the generalized game.*

Keywords: Jenga, combinatorics, geometric topology.

Introduction – Jenga and its maximum genus

Jenga is a game of physical skill marketed by Hasbro [1] in Europe and Takara Tomy [3] in Japan. The game starts from building blocks packed in three columns and 18 levels. Here we quote its rules from Wikipedia [4].

> Jenga is played with 54 wooden blocks. Each block is three times longer than its width. ⋯ Moving in Jenga consists of taking one and only one block from any level (except the one below the incomplete top level) of the tower, and placing it on the topmost level to complete it. ⋯ The game ends when the tower falls, or if any piece falls from the tower other than the piece being knocked out to move to the top. The winner is the last person to successfully remove and place a block.

In this paper, we treat the boundary of the union of blocks in the game as a surface with a polyhedral structure, which is called a *polyhedral closed surface*, and consider its genus. In the initial configuration of the game the genus of the surface is 0. As the game progresses, the configuration of the surface changes and its genus may increase. Based on this observation, one may ask the following question:

By how much does the genus in the game increase?

Namely, when the Jenga game with k levels is started for a given natural number $k \geqslant 2$, how can the maximum genus of the surface be described in terms of k? Of course we assume that the players do not make any mistakes.

We generalize the game and consider the same question for the generalized game. Let n and k be two natural numbers. We consider a game that starts from building blocks packed in n columns and k levels. We call the game an (n, k)-Jenga game or an (n, k)-game for short. Basically, we adopt the rule for the original game (three columns). The significant point to note in the rules quoted above is

"except the one below the incomplete top level ".

We call a configuration of blocks which can appear in the (n, k)-game under the above rule the *Jenga like configuration.* In this paper, we determine the Jenga like configuration in the (n, k)-game that gives the maximum genus and compute the maximum genus for given n and k.

Main Theorem. (Theorems 3 and 5, Propositions 7 and 8) *For given n and k $(\geqslant 2)$ the maximum genus is realized by the (n, k)-configuration and its genus is given by the formula*

$$g(n,k) = \begin{cases} \dfrac{n(n-2)(k-2)}{2} & (n \text{ is even}) \\[2ex] \dfrac{n(n-1)(k-2)}{2} & (n \text{ is odd}). \end{cases}$$

The definition of the (n, k)-configuration is given in Section 4. For even n the (n, k)-configuration has $\frac{k}{2}$ levels. For odd n the number of levels of the (n, k)-configuration is determined by the quotient of nk divided by $n + \frac{n-1}{2}$. See the definition of the (n, k)-configuration for details. We derive the formula using the Gauss-Bonnet formula for a polyhedral closed surface or Descartes' theorem. To show the maximality of $g(n, k)$ among the (n, k)-game, we consider an algorithm to deform the (n, k)-configuration into the given configuration without increasing the genus.

Preliminaries

Let X be a polyhedron in $\mathbb{R}^3$. Namely X is a subset in $\mathbb{R}^3$ obtained by gluing finitely many convex polygons along their vertices or along their edges. Let $F(X)$, $E(X)$ and $V(X)$ be the sets of all faces, edges and vertices in X, respectively. In this paper, we use a surface with a polyhedral structure defined as follows:

Definition (Polyhedral closed surface)**.** Let Q be a polyhedron in $\mathbb{R}^3$. If Q satisfies the following two conditions, then Q is called a *polyhedral closed surface.*

- Each edge of Q is an edge of exactly two faces of Q.
- For each vertex $v \in V(Q)$, the link $\mathrm{lk}(v)$ of v is connected. Here the link $\mathrm{lk}(v)$ is defined by

  $\mathrm{lk}(v) := \{e \in E(Q)\ | e \in E(f) \text{ for some } f \in F(Q),\ v \in V(f) \text{ and } v \notin V(e)\}.$

Definition (Angular defects)**.** If Q is a polyhedral closed surface, the genus of Q is denoted by $g(Q)$. For each vertex $v \in V(Q)$, $\kappa(v)$ denotes the *angular defect* of v, that is,

$$\kappa(v) := 2\pi - \sum_{f \in F(Q), v \in V(f)} [\text{angle of } f \text{ at } v].$$

The following is the main tool for us.

Theorem 1 (Gauss-Bonnet formula for closed polyhedral surface, Descartes' theorem)**.** *For a polyhedral closed surface, Q, the following equality holds:*

$$\sum_{v \in V(Q)} \kappa(v) = 2\pi(\#V(Q) - \#E(Q) + \#F(Q)) = 4\pi(1 - g(Q)).$$

See Chapter 1 in [2] for these topics for example.

Genus and angular defects in the Jenga game

We first describe how to play the (n, k)-game. The game is played with $n \times k$ wooden blocks, where the length of each block is n times its width. The initial configuration has k-levels and each level has n blocks without gaps. The game is played by taking one and only one block from any level (except the one below the incomplete topmost level) of the tower, and placing it on the topmost level. Using the rules quoted above, with emphasis on "except the one below the incomplete top level ", we have the following two fundamental observations.

- The second level from the top always has n blocks.
- The sum of the numbers of blocks on the third level from the top and on the top level is greater than or equal to n.

These facts are important for the (n, k)*-configuration.*

In the (n, k)-game, each configuration has a structure of a polyhedral closed surface, and the genus of the configuration can be defined canonically. For example, the configuration in Figure 1 in the $(5, 5)$-game has genus 3.

The genus of a given configuration of the (n, k)-game can be computed by using Theorem 1. Essentially, following three types of vertices, Type I, Type II and Type III are needed.

Figure 1: A configuration with genus 3 in the $(5, 5)$-game.

A Type I vertex is a type of vertex that appears in the initial configuration of the $(3, 4)$-game (see Figure 2 (1)).

A Type II vertex is a type of vertex that newly appears when a block is removed from the second level of the initial configuration of the $(3, 4)$-game (see Figure 2 (2)).

A Type III vertex is a type of vertex that newly appears when a block is removed from the third level of the configuration described in the Type II vertex (see Figure 2 (3)).

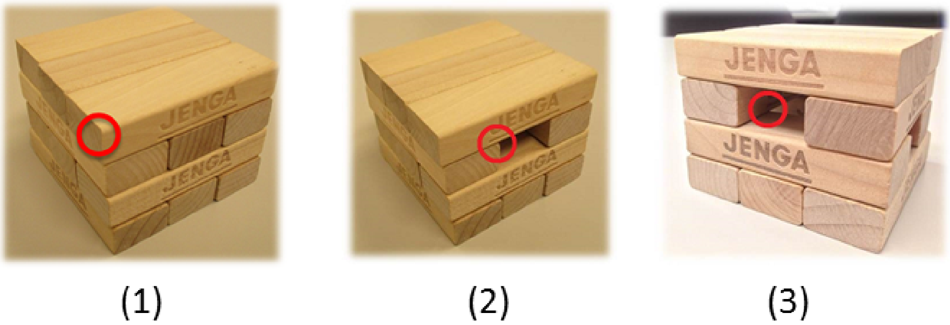

Figure 2: Vertices of Type I, Type II and Type III.

These three types of vertices can be defined in a following rigorous way.

Definition. A Type I vertex is a vertex in a polyhedron in $\mathbb{R}^3$ whose neighborhood is isometric to a neighborhood of the origin of the region

$$\{(x, y, z) \in \mathbb{R}^3 \mid x \geqslant 0 \text{ and } y \geqslant 0 \text{ and } z \geqslant 0\}.$$

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A Type II vertex is a vertex in a polyhedron in $\mathbb{R}^3$ whose neighborhood is isometric to a neighborhood of the origin of the region

$$\{(x, y, z) \in \mathbb{R}^3 \mid [x \leqslant 0 \text{ or } z \leqslant 0] \text{ and } y \geqslant 0\}.$$

A Type III vertex is a vertex in a polyhedron in $\mathbb{R}^3$ whose neighborhood is isometric to a neighborhood of the origin of the region

$$\{(x, y, z) \in \mathbb{R}^3 \mid [x \leqslant 0 \text{ and } z \geqslant 0] \text{ or } [y \geqslant 0 \text{ and } z \leqslant 0]\}.$$

It can be seen that the Type I vertex, v_I, has an angular defect

$$\kappa(v_I) = 2\pi - \frac{\pi}{2} \times 3 = \frac{\pi}{2}. \tag{1}$$

Similarly, the angular defects of the Type II and Type III vertices, v_{II} and v_{III}, respectively, are given by

$$\kappa(v_{II}) = 2\pi - \left(\pi + \frac{\pi}{2} \times 3\right) = -\frac{\pi}{2} \tag{2}$$

and

$$\kappa(v_{III}) = 2\pi - \left(\pi \times 2 + \frac{\pi}{2} \times 2\right) = -\pi. \tag{3}$$

Definition of the (n,k)–configuration

Hereafter, the *box description* will be used to represent Jenga like configurations, as illustrated in Figure 3. In this illustration, a gray or black square represents a block. Specifically, if a given configuration has x levels, then in the box description, $n \times x$-cells are used, which are gray or black if the corresponding position contains a block. Note that each alternate levels have their perspective rotated by 90 degrees.

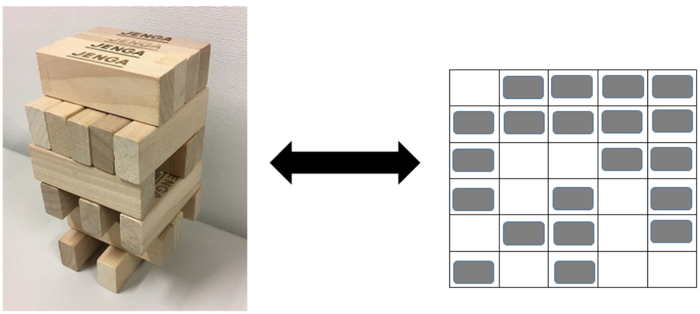

Figure 3: Box description for a given configuration in the (n, k)-game.

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Definition ((n,k)-configuration)**.** Let n be an integer grater than 1 and k an integer grater than 2. We define (n,k)*-configuration* as follows.

(1) Suppose that n is an odd integer. Then the (n,k)*-configuration* for odd case is the Jenga like configuration defined as follows (See Figure 4 and Figure 5).

- Let x and l be non-negative integers uniquely determined by the conditions

$$nk = n + \frac{n-1}{2} + \frac{n+1}{2}(x-3) + l \text{ and } 1 \leqslant l \leqslant \frac{n-1}{2}.$$

- It has x levels.
- The top most level has $\frac{n-1}{2}$ blocks without gaps.
- The second level from the top has n blocks.
- The bottom most level has l blocks with at least one gap.
- The rest of middle $x-3$ levels has $\frac{n+1}{2}$ blocks with one gap.

(2) Suppose that n is an even integer. Then the (n,k)*-configuration* for even case is the Jenga like configuration defined as follows (See Figure 6 and Figure 7).

- It has $2k-1$ levels.
- The top most level has $\frac{n}{2}$ blocks without gaps.
- The second top most level has n blocks.
- The rest of $2k-3$ levels has $\frac{n}{2}$ blocks with at least one gap.

The polyhedral closed surface corresponding to the (n,k)-configuration is denoted by $Q(n,k)$, and we also call $Q(n,k)$ the (n,k)-configuration.

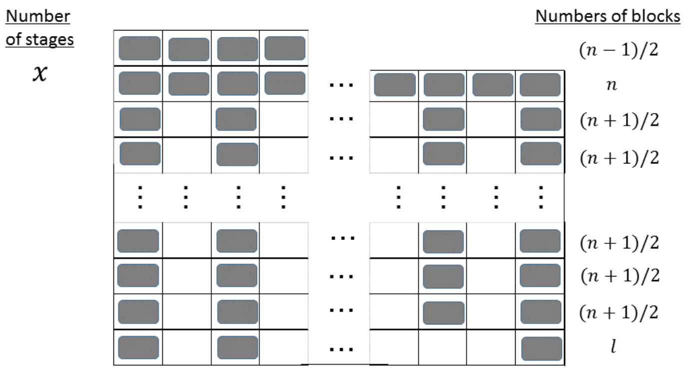

Figure 4: (n,k)-configuration for odd n.

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Figure 5: $(5, 3)$-configuration.

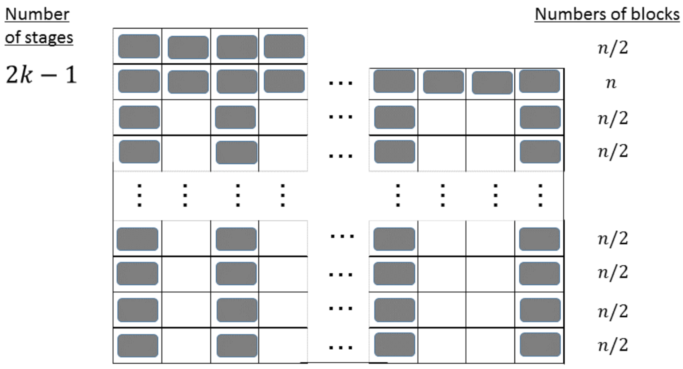

Figure 6: (n, k)-configuration for even n.

Figure 7: $(6, 3)$-configuration.

Remark. *In the (n, k)-configuration for odd case we impose the following condition for the configuration of the blocks in the bottom most level to count the vet rices of Type I, II and III in Proposition 2.*

1. *If $l = 1$, then we put the block at the middle box, the $\frac{n+1}{2}$-th box from left (and right) in the box description.*

2. *If $l = 2$, then we put two blocks at the box described in (1) and the right most box.*

3. *If $l \geqslant 3$, then we put blocks at the blocks described in (2) and the boxes from left with one gap.*

Remark. *The (n, k)-configuration is in fact a Jenga like configuration. Namely when one starts the (n, k)-game one can reach to the (n, k)-configuration by removing the blocks from the bottom most level in order. Moreover the (n, k)-configuration is physically stable*[1] *under the condition in Remark for the bottom most level.*

Let $N_I = N_I(n, k)$ be the number of Type I vertices in the (n, k)-configuration. Analogously, N_{II} and N_{III} are the numbers of Type II and Type III vertices, respectively, in the (n, k)-configuration. Let $g(n, k)$ be the genus of the (n, k)-configuration, $Q(n, k)$. From Theorem 1 and computation of the angular defects (1), (2) and (3), we have the following:

Lemma 1. The genus $g(n, k)$ of the (n, k)-configuration $Q(n, k)$ is given by

$$g(n, k) = -\frac{N_I}{8} + \frac{N_{II}}{8} + \frac{N_{III}}{4} + 1.$$

Remark. *In the subsequent sections we compute N_I, N_{II} and N_{III}, however, in the computation we ignore the vertices of the topmost level because they do not contribute the genus $g(n, k)$ of $Q(n, k)$.*

Computation of the genus – odd case

In this section, we compute N_I, N_{II} and N_{III} and derive the formula for $g(n, k)$ when n is odd under the condition in the previous remark.

Proposition 2. If n is odd and $l \geqslant 2$, then N_I, N_{II} and N_{III} are given by the following formulae:

I. $N_I = 4 + 4l$

II. $N_{II} = 4(x - 3)(n - 1) + 4(l - 1)$

III. $N_{III} = (x - 4)(n - 1)^2 + 2(l - 1)(n - 1)$

Proof. We count the vertices of Type I, II and III on each *floor*, where for each non-negative integer i, the ith floor is the intersection of the ith level and the $(i + 1)$th level. For convenience, we call the intersection of the first level and the ground level the 0*th floor*.

I. The formula for N_I is clear.

[1] Of course we assume that players are prudence enough and they do not make any mistake.

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II. Let $N_{II,i}$ be the number of Type II vertices on the ith floor. It can be seen that (as seen in the cutaway of the ith floor in Figure 8)

$$N_{II,i} = \begin{cases} 0 \qquad (i = 0, x-1) \\ 2(n-1) + 4(l-1) \qquad (i = 1) \\ 2(n-1) \qquad (i = x-2) \\ 4(n-1) \qquad (2 \leqslant i \leqslant x-3), \end{cases}$$

and hence, we have

$$N_{II} = \sum_i N_{II,i} = 4(x-3)(n-1) + 4(l-1).$$

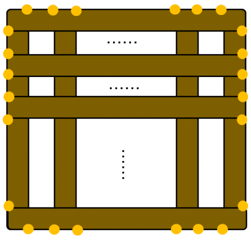

Figure 8: Configuration of Type II vertices in the ith floor for $i = 1, 2, \ldots, x-3$ (Yellow circle = Type II vertex).

III. Let $N_{III,i}$ be the number of Type III vertices on the ith floor. It can be seen that (as seen in the cutaway of the ith floor in Figure 9)

$$N_{III,i} = \begin{cases} 0 \qquad (i = 0, x-1, x-2) \\ 2(l-1)(n-1) \qquad (i = 1) \\ (n-1)^2 \qquad (2 \leqslant i \leqslant x-3), \end{cases}$$

and, hence, we have

$$N_{III} = \sum_i N_{III,i} = (x-4)(n-1)^2 + 2(l-1)(n-1).$$

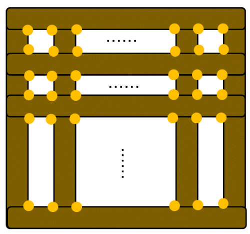

Figure 9: Configuration of Type III vertices in the ith floor for $i = 1, 2, \ldots, x-3$ (Yellow circle = Type III vertex).

□

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Remark. *The formulae in Proposition 2 are not correct for $l = 1$. We use a trick for $l = 1$ to resolve the case $l \geqslant 2$ in the proof of Theorem 3.*

Theorem 3. *When n is odd, $g(n, k)$ is given by*

$$g(n,k) = \frac{n(n-1)(k-2)}{2}.$$

Proof. We first prove this for the case $l \geqslant 2$ of the (n, k)-configuration with odd k. By Lemma 1 and Proposition 2, we have

$$g(n,k) = \frac{(n^2-1)(x-4) + 2l(n-1)}{4}. \tag{4}$$

By counting the number of blocks, we have

$$nk = \frac{n-1}{2} + n + \frac{(x-3)(n+1)}{2} + l = \frac{(n+1)x}{2} + l - 2. \tag{5}$$

By substituting (5) into (4) we have $g(n,k) = \dfrac{n(n-1)(k-2)}{2}$.

Now we consider the case $l = 1$. In this argument, we deviate from rules of the game for a while. We deform the configuration by moving a block within the first level as shown in Figure 10. Note that this operation does not change the

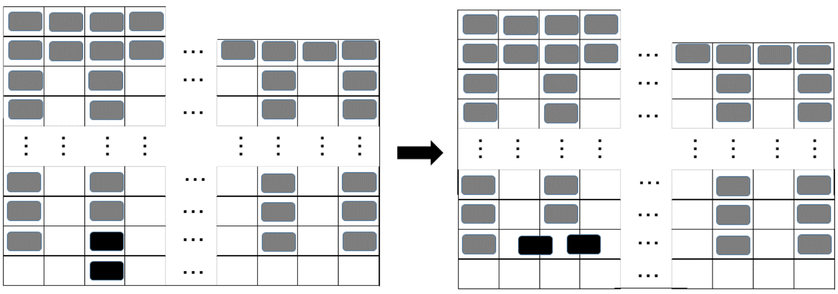

Figure 10: A configuration with $l = 1$ with x levels and a configuration with $l = \frac{n+1}{2} + 1$ with $x - 1$ levels.

genera of two configurations in Figure 10. By substituting $l \to \dfrac{n+1}{2} + 1$ and $x \to x - 1$ in Equation (4) the genus of the right configuration can be computed as

$$\frac{1}{4}(n^2x - 4n^2 - x + 2n + 2),$$

which is equal to Equation (4) with $l = 1$. □

Computation of the genus – even case

In this section, we compute N_I, N_{II} and N_{III}, and derive the formula for $g(n, k)$ when n is even. The following can be proved in almost same way as Proposition 2.

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Proposition 4. If n is even, then N_I, N_{II} and N_{III} are given by the following formulae:

I. $N_I = 4 + 2n$

II. $N_{II} = 8(n-2)(4k-7)$

III. $N_{III} = 2(n-2)^2(k-2)$

Theorem 5. *When n is even, $g(n,k)$ is given by*

$$g(n,k) = \frac{n(n-2)(k-2)}{2}.$$

Proof. The formula can be obtained by Proposition 1 and Proposition 4. □

The maximality of $g(n,k)$

In this section we show that the genus $g(n,k)$ of the (n,k)-configuration is the maximum genus among all genera appearing in the (n,k)-game. Specifically, for the given configuration Q, we show that $g(Q) \leqslant g(n,k)$. To show it, we provide an algorithm for deforming the (n,k)-configuration $Q(n,k)$ into Q without increasing the genus. In each step of the algorithm we deviate from the rules of the game. Namely we may treat a configuration which is not a Jenga like configuration and use operations which are forbidden in our rules.

We first define the following three fundamental operations.

(S) Sliding a block within a level (see Figure 11).

(L) Removing a block and loading it onto the topmost level (see Figure 12).

(I) Removing a block and inserting it into any other level (see Figure 13).

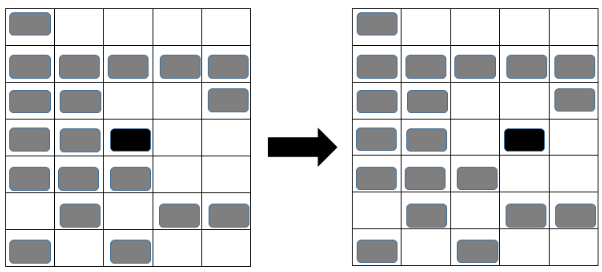

Figure 11: The operation (S) performed on the black block.

For the given configuration Q, the number of levels in Q is denoted by $s(Q)$.

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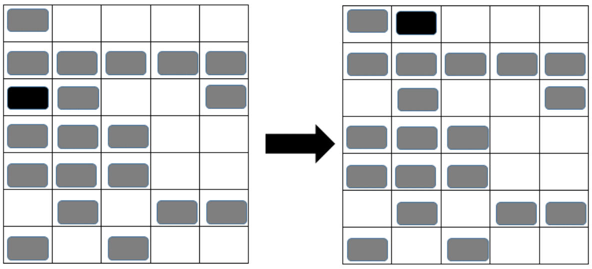

Figure 12: The operation (L) performed on the black block.

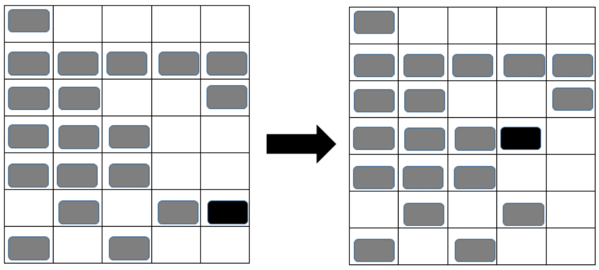

Figure 13: The operation (I) performed on the black block.

Proof for even n

Let $Q'(n,k)$ be the configuration obtained by removing the upper two levels and the first level from $Q(n,k)$. Similarly we consider the configuration Q' removing from Q. See Figure 14.

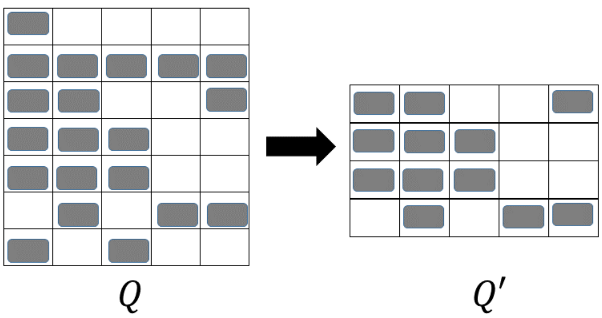

Figure 14: Q and Q'.

We first deform the configuration $Q'(n,k)$ into the configuration Q' by applying (S), (L) and (I) a finite number of times by the following algorithm.

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(A1) For the sth level of $Q'(n,k)$ with $1 \leqslant s \leqslant s(Q')$, whose number of blocks is greater than or equal to that of the sth level of Q', we apply sufficiently many (L) and (S) so that the resulting configuration of the sth level is the same as that of Q'. We apply these operations for all conceivable values of s. Let $Q'_1(n,k)$ be the configuration obtained by the above operation for $Q'(n,k)$.

(A2) We apply the same operations for all conceivable sth level of $Q'_1(n,k)$ with $s(Q'(n,k))+1 \leqslant s \leqslant s(Q')$. Let $Q'_2(n,k)$ be the configuration obtained by the above operation for $Q'_1(n,k)$.

(A3) For the sth level of $Q'_2(n,k)$ with $1 \leqslant s \leqslant s(Q')$, whose number of blocks is less than that of sth level of Q', we apply finitely many (I) and (S) operations by using blocks between the $(s(Q')+1)$th level and the $s(Q'_2(n,k))$th level of $Q'_2(n,k)$, so that the resulting configuration of the sth level is the same as that of Q'. We apply this operation for all conceivable values of s. If the blocks in $Q'_2(n,k)$ become insufficient, then we may use blocks in $Q(n,k) \setminus Q'(n,k)$. Let $Q'_3(n,k)$ be the configuration obtained by the above operations for $Q'_2(n,k)$.

Example 6. Here we demonstrate the algorithm by using an example. Consider the $(6,5)$-game. Let Q be a configuration appearing in the game with the associated configuration Q', as shown in Figure 15.

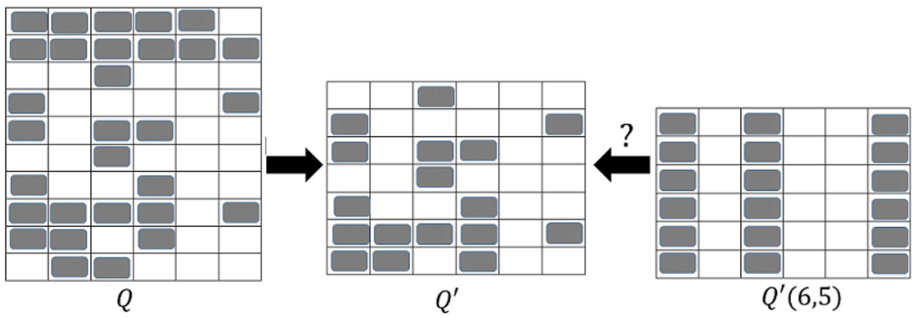

Figure 15: Q, Q' and $Q'(6,5)$.

We apply step (A1) of the algorithm to the first, third, fourth, fifth and sixth levels in $Q'(6,5)$ to obtain $Q'_1(6,5)$ (see Figure 16).

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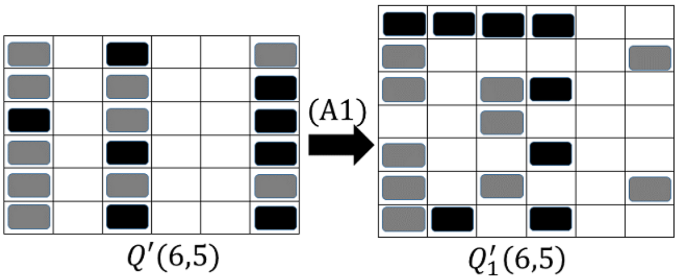

Figure 16: $Q'(6,5)$ and $Q'_1(6,5)$.

Next, we apply step (A2) to the seventh level in $Q'_1(6,5)$ to obtain $Q'_2(6,5)$, as shown in Figure 17.

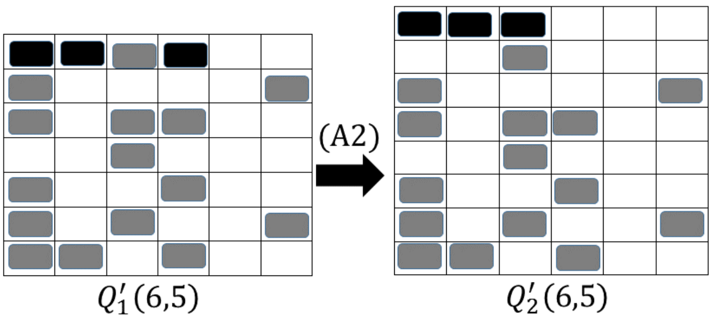

Figure 17: $Q'_1(6,5)$ and $Q'_2(6,5)$.

Finally, we apply step (A3) of the algorithm to the second level in $Q_2(6,5)$ to obtain $Q'_3(6,5)$ (see Figure 18).

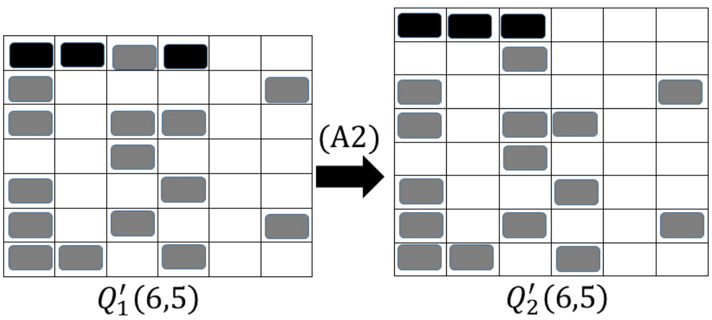

Figure 18: $Q'_2(6,5)$ and $Q'_3(6,5)$.

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Let $\hat{Q}(n,k)$ be the configuration obtained by returning the upper two levels and first level of $Q(n,k)$ to $Q'_3(n,k)$. Note that $\hat{Q}(n,k)$ is in the same configuration as Q up to the first and topmost levels.

Proposition 7. For even n and any Q we have $g(Q) \leqslant g(n,k)$.

Proof. Because the number of blocks in each level of $Q'(n,k)$ is $\frac{n}{2}$ and each piece is arranged with a 1-by-1 gap, the genus does not increase with the operations (A1), (A2), and (A3) of the algorithm for $Q'(n,k)$. Moreover, the increasing genus from $Q'_3(n,k)$ to $\hat{Q}(n,k)$ is not greater than the decreasing genus from $Q(n,k)$ to $Q'(n,k)$, which implies that $g(\hat{Q}(n,k)) \leqslant g(Q(n,k))$. Note that $\hat{Q}(n,k)$ and Q are in the same configuration up to the first and $(s(\hat{Q}(n,k))-1)$th levels. Because the number of blocks in the first level is $\frac{n}{2}$ and that in the $(s(\hat{Q}(n,k))-1)$th level is n, we have $g(Q) \leqslant g(\hat{Q}(n,k))$ and, hence, $g(Q) \leqslant g(Q(n,k)) = g(n,k)$. □

Proof for odd n

Now we assume that n is odd. We will use same notations as in Subsection 7.1. Although we can apply the algorithm for odd n, the last argument in the proof of Proposition 7 is not true in general. In fact, if l (the number of blocks in the first level of $\hat{Q}(n,k)$) is less than $\frac{n+1}{2}$, then we have to estimate the genera of $\hat{Q}(n,k)$ and Q more carefully. To do so, we introduce the following operations for $Q(n,k)$ (recall that we set $x = s(Q(n,k))$ for odd n):

1. Apply the operation (I) $\frac{n-1}{2}$times to the $(x-2)$th level of $Q(n,k)$ by using all the blocks in the xth level. The resulting configuration is denoted by $Q^{(2)}(n,k)$. Note that $s(Q^{(2)}(n,k)) = s(Q(n,k))-1$, $Q^{(2)}(n,k)$ has n blocks in the $s(Q^{(2)}(n,k))$ and $(s(Q^{(2)}(n,k)) - 1)$th level, and $g(Q^{(2)}(n,k)) < g(n,k)$.
2. If the number of blocks in the first level of Q is greater than the number of blocks in the first level of $Q^{(2)}(n,k)$ then, apply (I) to the first level of $Q^{(2)}(n,k)$ by using the blocks in the top level so that the resulting configuration $Q^{(3)}(n,k)$ has the same number of blocks as Q in the first level.

Proposition 8. For odd n and any Q we have $g(Q) \leqslant g(n,k)$.

Proof. If l is greater than or equal to the number of blocks in the first level of Q, then we can apply the same argument in the proof of Proposition 7. Otherwise, if l is less than the number of blocks in the first level of Q, then we consider the configuration $Q^{(3)}(n,k)$ in the above operation. Note that because the decreasing genus in the first step $[Q(n,k) \to Q^{(2)}(n,k)]$ is greater than its increasing genus in the second step $[Q^{(2)}(n,k) \to Q^{(3)}(n,k)]$, we have $g(Q^{(3)}(n,k)) < g(Q(n,k))$. By applying the algorithm, we can deform $Q^{(3)}(n,k)$ to Q and show that $g(Q) \leqslant g(Q^{(3)}(n,k))$. So, $g(Q) < g(Q(n,k)) = g(n,k)$. □

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Articles

$2 \times 2 \times 2$ COLOR CUBES

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Abstract: *Using six colors, one per side, cubes can be colored in* 30 *unique ways. In this paper, a row and column pattern in Conway's matrix always leads to a selection of eight cubes to replicate one of the 30 cubes. Each cube in the set of* 30 *has a* $2 \times 2 \times 2$ *replica with inside faces of matching color. The eight cubes of each replica can be configured in two different ways.*

Keywords: coloring problems, 30 color cubes.

Introduction

Often mathematics is a delight when connections are discovered. One of these connections is that each of the distinct cubes painted with six colors can be replicated with eight other cubes from the set of thirty. Although the eight cubes in the solution do not change they can be arranged in two distinct ways [1].

How Many Different Color Cubes?

How many different ways can a cube be painted with six different colors, one per face? Different means that a cube cannot rotate to match another cube. Place the cube on a flat surface; the bottom face takes one color. The top face may be painted with five different colors. This leaves four colors for the four side faces. Fix a side face to prevent rotation equivalents which takes one color. Three side faces and three colors are left. The way the remaining colors are ordered is important resulting in six different possibilities (a permutation of three taken three at a time). Therefore, each of the top faces has six side possibilities resulting in 30 different color cubes.

Arrangement of Cubes

John Conway (University of Cambridge) set the 30 color cubes up in a 6×6 matrix (see appendix). I numbered the cubes in that matrix. The rows start with Cube 1, Cube 6, Cube 11, Cube 16, Cube 21, and Cube 26 on the left. In addition, the rows of the matrix can be labeled A – F and the columns a – f. No cubes are located along the main diagonal (Aa, Bb, Cc, Dd, Ee, and Ff). Conway had reasons for placing cubes in certain positions in the matrix. For example, cubes that are reflections of one another are strategically placed in the matrix. One of the benefits of using Conway's matrix is that the cubes for every $2 \times 2 \times 2$ color cube fall into a pattern.

$2 \times 2 \times 2$ First Solution

Each of the 30 color cubes can be replicated with eight other cubes of the 30 color cube set. An additional condition is that cubes meeting on the inside of the $2 \times 2 \times 2$ structure have the same color often called the domino condition.

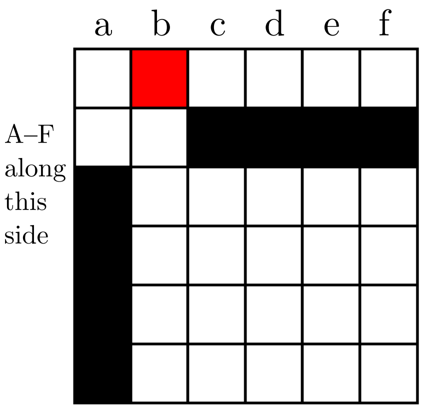

Figure 1: Conway's matrix.

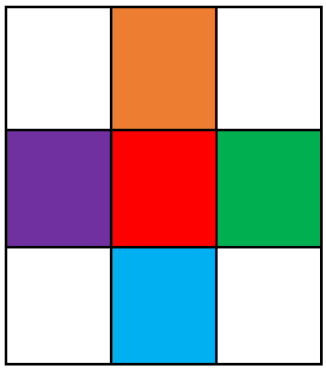

Figure 2: Cube 1, row A, column b (Ab) (top view).

By switching the row and column designations where the cube is positioned, the cubes used to construct a $2 \times 2 \times 2$ replica are located. For example, Cube 1 (Ab), shown in Figure 2, is in the first row, A and second column, b (red square in Figure 1). The cubes will be located in the second row, B and first column, a (black squares in Figure 1). Ba is not in the solution; the switched row/column designation cube is never in the solution. Aa and Bb are on the diagonal and contain no blocks. All other cubes in row B (Bc, Bd, Be, Bf) and column a (Ca, Da, Ea, Fa) form the solution. In general, a cube located in row X, column y (Xy) designates cubes from row Y, column x with Cube Yx not in the solution.

The configuration of the solution for the $2 \times 2 \times 2$ replica of Cube 1 (Ab) is shown below in Figure 3.

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Da	**Bc**	**Top layer**	**Be**	**Fa**	**Bottom layer**
Bf	**Ea**		**Ca**	**Bd**	

Front

Figure 3: Positions of cubes in $2 \times 2 \times 2$ solutions (top view).

The view is looking down on the cubes. Figure 4 shows the cubes and their orientation. Notice the top layer faces join with the color yellow and the bottom layer join with the color red. The rear top blocks, Da and Bc, have a bottom color of blue that corresponds with the top of Be and Fa. The front top blocks, Bf and Ea, have a bottom color of tan that corresponds with the top of Ca and Bd. The domino condition is satisfied and the $2 \times 2 \times 2$ block has the same face colors as Block 1 pictured in Figure 2.

Orientation of the eight color cubes in the first $2 \times 2 \times 2$ solution are shown in Figure 4.

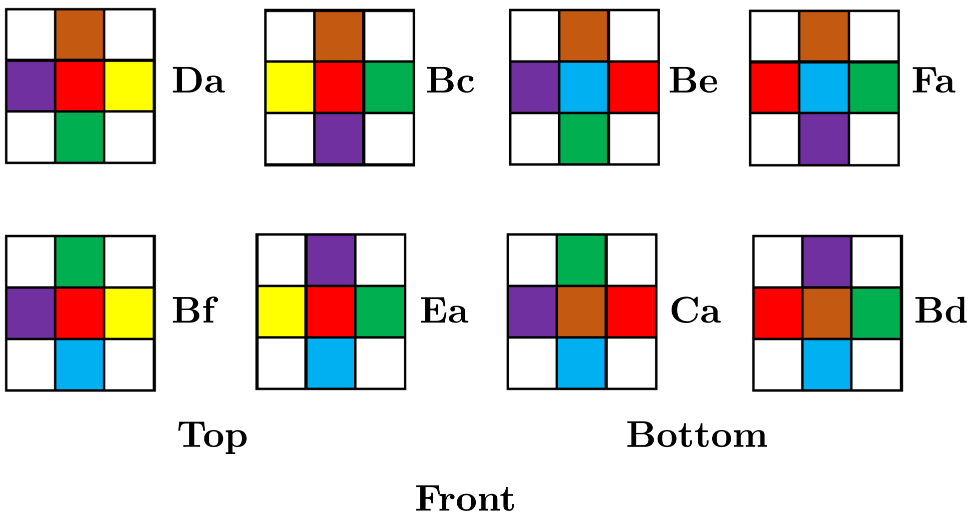

Figure 4: Individual color cubes oriented in the first solution (top view).

$2 \times 2 \times 2$ Second Solution

A second solution can be obtained by transforming the cubes of the first solution. The original top becomes the bottom and the bottom becomes the top. The second solution will produce a $2 \times 2 \times 2$ cube rotated 180 degrees from the original $2 \times 2 \times 2$ cube (Figure 5). Any $2 \times 2 \times 2$ replica will produce a second solution that is rotated 180 degrees. After the blocks are in the proper position the transformations in Table 1 are necessary to get the $2 \times 2 \times 2$ Cube 1 (Ab). The transformations in Table 1 are not a generalized process to obtain the second solution. Cubes are rotated backward, forward, left, or right followed by a clockwise or counterclockwise rotation. Looking at a model of the solution cube (Figure 5) makes the transformation process easier.

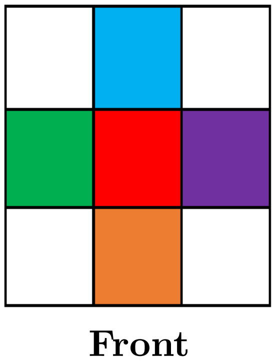

Figure 5: Cube 1 rotated 180 degrees (top view).

Original top → New bottom (looking from the top)	
Da roll right 90°; rotate clockwise 90°	Bc roll left 90°; rotate counterclockwise 90°
Bf roll right 90°; rotate counterclockwise 90°	Ea roll left 90°; rotate clockwise 90°

Original bottom → New top (looking from the top)	
Be roll left 90°; rotate clockwise 90°	Fa roll right 90°; rotate counterclockwise 90°
Ca roll left 90°; rotate counterclockwise 90°	Bd roll right 90°; rotate clockwise 90°

Table 1: A listing of the transformation leading to a second solution.

Figure 6 shows the orientation of the individual cubes in the second $2 \times 2 \times 2$ solution. Notice the cubes join faces with the same color. Be and Ca have a bottom color of purple, and Fa and Bd have a bottom color of green. The original top cubes are now the bottom cubes with yellow as a bottom color, the same bottom color as the first solution. The original bottom cubes are now the top cubes with red as the top color, the same top color as the original solution.

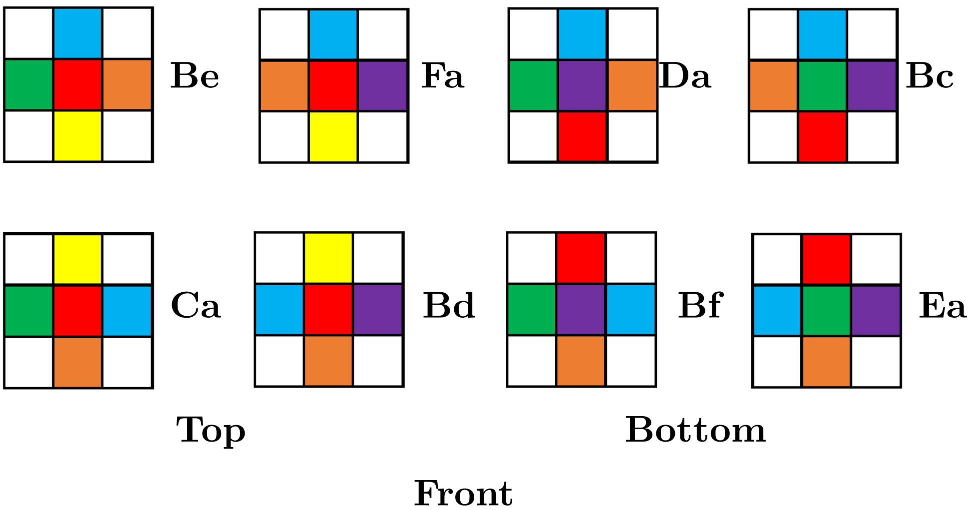

Figure 6: Individual color cubes oriented in the first solution (top view).

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Conclusion

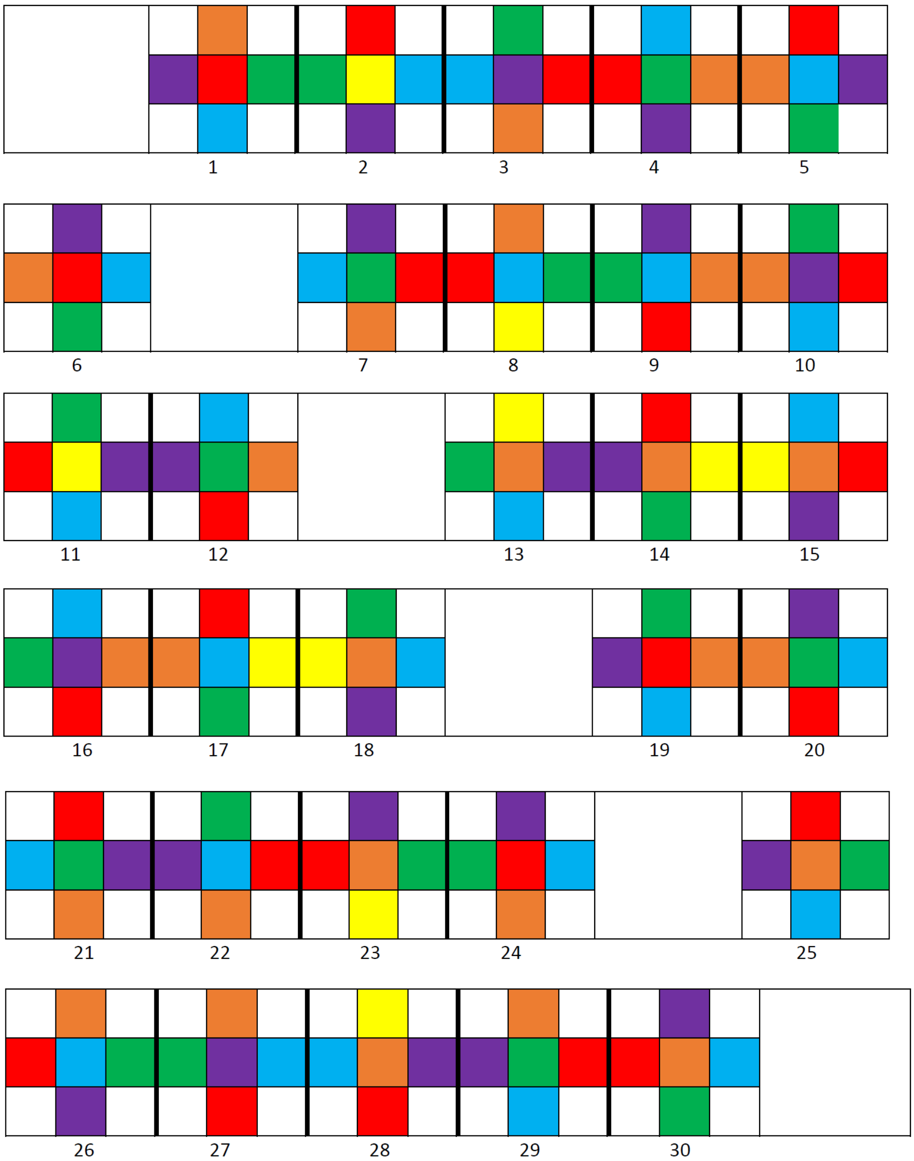
All thirty color cubes have two $2 \times 2 \times 2$ replica cubes with the domino condition for inside faces. The individual cubes in a $2 \times 2 \times 2$ replica cube always follow a row and column pattern in Conway's matrix. And, the first solution transforms into the second solution resulting in a rotation of 180 degrees while the top layer of cubes always switch with the bottom layer. The transformations are slightly different for each solution, but they are selected from a small set of possibilities.

References

[1] Gardner, M. "Mathematical games: Puzzling over a problem-solving matrix, cubes of many colors and three-dimensional dominoes", *Scientific American*, 239(1), 20–31, 1978.

DOI: https://doi.org/10.2478/rmm-2018-0003

Appendix



Conway's matrix ([1], page 22)

Mathematics and arts

A CLASSIFICATION OF MATHEMATICAL SCULPTURE

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Abstract: *In this paper, we define the term* Mathematical Sculpture*, a task somehow complex. Also, we present a classification of mathematical sculptures as exhaustive and complete as possible. Our idea consists in establishing general groups for different branches of Mathematics, subdividing these groups according to the main mathematical concepts used in the sculpture design.*

Keywords: Mathematical sculpture

Introduction

There are several studies on the so-called *Mathematical Sculpture*, a concept that we will try to define in the next section.

These studies deal with specific aspects, such as the mathematical study of the works of a particular sculptor or the analysis of specific types of mathematical sculptures. Also, there are general studies. However, as far as we know, there is no work in the scientific literature providing a systematic analysis of the connections between mathematics and sculpture.

Neither is there any study that offers a complete and exhaustive classification of mathematical sculpture. The scarcity and lack of research on this artistic topic led us to choose it as the main topic of the doctoral thesis developed by Ricardo Zalaya, assistant lecturer at the Polytechnic University of Valencia, tutored by Javier Barrallo, professor at the University of the Basque Country, Spain.

In this paper we propose a classification for mathematical sculpture, based on the results of our research in the last years, and on the comments and observations provided by other experts on the topic. This approach was presented at the Alhambra ISAMA–BRIDGES 2003 Meeting [12].

Figure 1: Sculpture located in front of Picasso Tower, José María Cruz Novillo, Madrid, Spain, 1989.

At the meeting, the International Congresses of I.S.A.M.A (The International Society of the Arts, Mathematics and Architecture) and BRIDGES (society for the promotion of "bridges" between mathematics, arts and music) were held simultaneously [1]. These two associations include some experts in the field of mathematics and arts.

Mathematical sculpture works can be found in many places, in addition to museums and exhibition halls. Figure 1 shows a simple sculpture which presents an interesting geometric shape formed by thin cylindrical metal tubes. The design of this sculpture presents different concepts related to geometry and topology: surfaces (cylinders), intersections, symmetries, closed loops, etc. Geometry is the branch of mathematics more widely used in this sculpture.

Below, we have included two works of the well-known mathematical sculptor John Robinson, clearly illustrating the mental process of abstraction and subsequent geometrization (Figure 2). After, Figure 3 shows a more complex example by Bathsheba Grossman. Its design reflects several mathematical concepts: polyhedral geometry, surface topology, isometric transformations.

Most researchers and experts in mathematical sculpture come from United States of America, where this subject has gained great importance. Although in some West European countries, like Great Britain, and in other countries, like Japan, we can also find very good artists and experts on this topic. Among them is Carlo Sequin, professor at Berkeley University and a worldwide well-known expert. His webpage is an excellent reference [11].

Figure 2: Left: *Acrobats*, John Robinson, 1980; Right: *Elation*, John Robinson, 1983.

Figure 3: *Metatrino*, Bathsheba Grossman, 2007.

The Concept of Mathematical Sculpture

Before any attempt to classify the sculptures, we have to define the type of sculpture that we are trying to classify – the so-called *Mathematical Sculpture*. We propose the following definition:

Definition 1. *A* Mathematical Sculpture *is a sculpture that has mathematics as an essential element of conception, design, development or execution.*

In order to include a given artistic work in the set of sculptures that satisfy the definition, some mathematical concept or property must be significantly essential. In this definition we include from the simplest mathematical concept to the most complex mathematical concept (for instance, it may be trivial elementary geometry or sophisticated non-euclidean geometry). The definition is very general and covers a wide spectrum of possibilities, as one can understand by looking at the different artistic works analyzed here.

As an example, we present two sculptures based on the same concept – ruled surfaces, that is, surfaces that can be described as the set of points swept by a moving straight line in space. However, the complexity of both sculptures is

Figure 4: Left: *Ruled surface*, Andréu Alfaro, Valencia, Spain, 1982; Right: *Eclipse*, Charles O. Perry, Hiatt Regency Hotel, San Francisco, USA, 1973.

clearly different. The work shown on the left side of the Figure 4 is very simple whereas the sculpture shown on the right side of the Figure 4 is very complex. The latter sculpture is an extension of the concept of ruled surface, allowing the movement of any curve for the surface generation. It has been made by one of the most complete mathematical sculptors, Charles O. Perry [9, 10].

The mathematical sculptures included in our classification may use concepts related to many branches of mathematics: geometry, differential calculus or vector calculus, algebra, topology, logic, etc. An interesting example is the group of sculptures $\pi r^2 a$, made by Javier Carvajal (see [5], where the Spanish expert Eliseo Borrás summarizes his research on the mathematics used in the design of these Javier Carvajal's works).

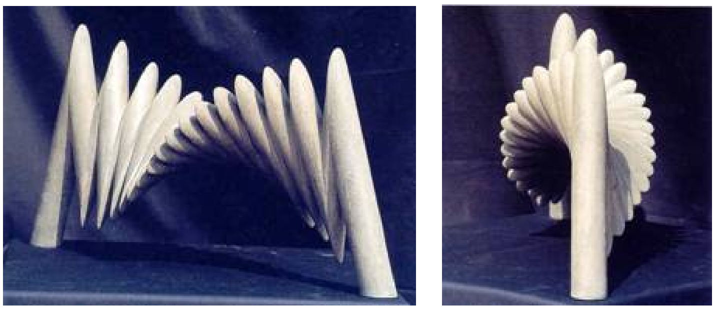

Figure 5: Left: *Parallel slices*, front view, Javier Carvajal; Right: *Parallel Slices*, side view, Javier Carvajal.

An element of this group is the sculpture presented in Figure 5. Another work by Javier Carvajal, from his series *Solomon Columns*, is presented in Figure 6. This example illustrates the difficulty of delimiting the concept of mathematical sculpture and whether a particular work may be considered as a mathematical sculpture.

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Figure 6: *Two columns of inverse rotation*, series *Solominic Columns*, Javier Carvajal, 1991–1994.

Note that some sculptures explicitly show their mathematical nature; an example of that could be a polyhedron; however, in other works, the concepts are present in an implicit or hidden way, such as in the example of the series *Solominic Columns*.

To design the sculptures of the series *Solominic Columns*, the sculptor began by sectioning cylinders in order to obtain some initial objects. Each section is an ellipse and its position is characterized by the angle θ formed by the ellipse's major axis and the central axis of the cylinder (Figure 7, left). The plane determined by these two axes is the main plane of the section (Figure 7, blue plane on the left). Two different sections determine a module for the sculptures (Figure 7, center and right).

In addition to the respective angles, θ_1 and θ_2, the relative position of two sections is determined by ϕ, the angle formed by the two main planes, and by c, the distance between the centers of the ellipses. If c is large enough to prevent the intersection of the ellipses, the module is a "slic" (Figure 7, right). Otherwise, the two modules obtained are "segments" (Figure 7, center).

Each module, obtained using this procedure, is characterized by a 4-uple $(\theta_1, \theta_2, \phi, c)$, where $0° \leqslant \theta_1, \theta_2 \leqslant 90°$, $0° \leqslant \phi \leqslant 180°$, and c depends of the cylinder size. There is an infinite number of different possible modules.

It is possible to place one module beside another module with equal or different radius, rotating the modules by an angle α, and using direct or opposite orientations. Javier Carvajal used that idea to create pieces for his sculptures. Examples are ovoids, spheres, pumpkins, Solominic columns, torus, cones and swirling blades, etc. The translations and rotations of the modules generate shapes that frequently are similar to the geometrical figures found in nature.

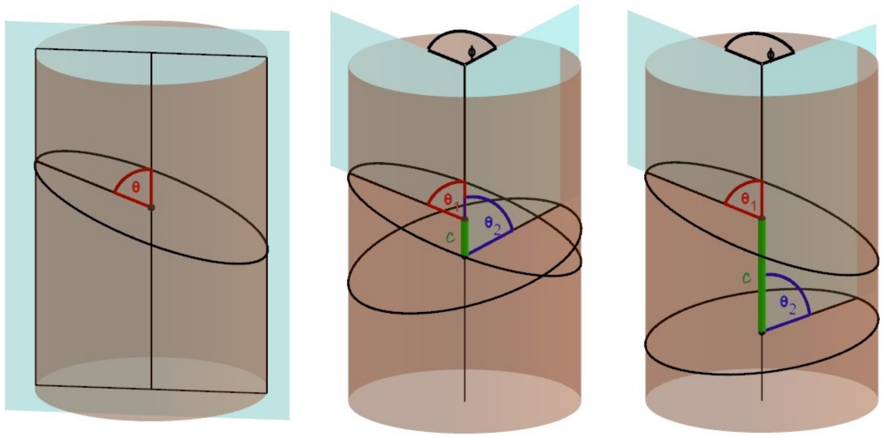

Figure 7: Sections of the pieces obtained from a cylinder and used by Javier Carvajal. Previous series of sculptures.

But not all shapes touch our senses in the same way. Some objects are more attractive than others and correspond to different numerical rhythms. In some works, the sculptor uses "polygonal spirals"; in another type of sculptures he uses "multipolygonal spirals" (Figure 8).

Figure 8: *Multipolygonal spirals*, series $\pi r^2 a$, Javier Carvajal, Spain, 1991–1994.

Since the main goal of this paper is the classification of mathematical sculpture, we don't use the standard mathematical notation. That is replaced by images and photos of sculptures and by grids and drafts used in their conception.

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The use of the computer by many sculptors has allowed the development and evolution of mathematical sculpture. The computer has allowed the precise realization of very sophisticated sculptures. An example of this can be seen in Figure 9 (left), a sculpture by Bathsheba Grossman. Another example is a virtual piece made by the expert in computer science, Javier Barrallo, one of the authors of this paper. That is a case in which interdisciplinarity leads to interesting virtual experiences, such as this virtual sculpture from Barrallo's series *Hypersculpture*. Javier Barrallo uses parametric programming for the artistic design and fractal theory concepts for the work with textures [2, 3].

Figure 9: Left: *Seven spheres*, Bathsheba Grossman, 2005; Right: virtual sculpture of the series *Hypersculpture*, Javier Barrallo, 1994.

Educational purpose of classifying mathematical sculpture

A classification of mathematical sculpture provides a more systematic approach to this field, facilitating its incorporation in secondary or higher education. Courses devoted to connections between mathematics and arts already exist, being included in the contents of artistic and technical syllabi, such as in architecture.

We believe that, without a classification, courses devoted to mathematical sculpture lack structure, focusing only on the enumeration of a number of works or authors, based on particular studies. Examples of studies devoted to a particular sculptor can be consulted in [4, 9] (analyzis of the works of John Robinson and Charles O. Perry, respectively). Other examples can be found in [6, 7], written by the mathematical sculptor George Hart, about two particular types of his sculpture.

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Other approaches for the classification of mathematical sculpture

The only approach to classify mathematical sculpture we know is based on the materials used, since they give the works varied geometrical properties. However, this typology does not permit to include all types of sculptures. The usual materials are the following:

- Wood. This material is used to emphasize curved surfaces (Figure 10, left). Due to its lightness, wood permits to create pieces that would otherwise be unstable (Figure 10, right).

Figure 10: Left: Photo of a Brent Collins workshop with several wood sculptures; Right: *Fire and ice*, George Hart, 1997.

- Welded metal. It is commonly used in polyhedral shapes (Figure 11).

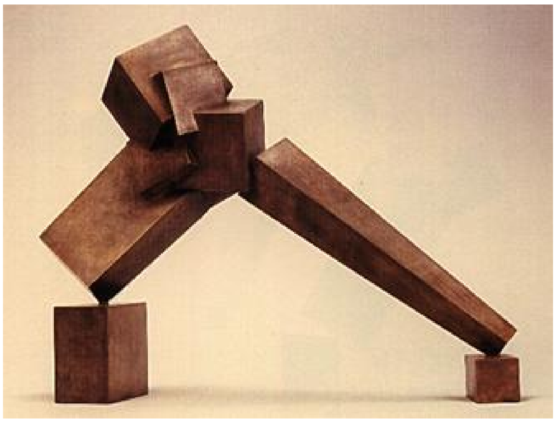

Figure 11: *Intersections II*, bronze, Bruce Beasley, 1991.

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- Concrete. Suitable for architectural works. A good example is the sculpture by Eduardo Chillida, shown in Figure 12.

Figure 12: *Elogio del horizonte*, concrete, Eduardo Chillida, Gijón, Spain, 1990.

- Stone. This material also emphasizes curved surfaces. As a result of the high weight, the works cannot be very complex. A good case is the sculpture by Nathaniel Friedman, shown in Figure 13. Nathaniel Friedman's works, made of stone, are examples of works difficult to classify as mathematical sculptures.

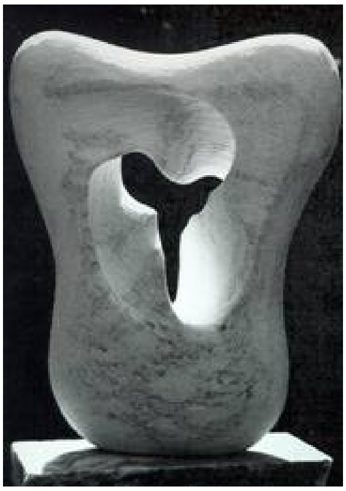

Figure 13: *Grand Canyon*, stone, Nathaniel Friedman, 1995.

The type of typology that we propose is based on mathematical properties. Our first approach was presented in 2003 (July), dividing the artistic works according to mathematical properties or concepts, or by combinations of both. The types considered were the following:

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- Classic and polyhedral geometry
- Nonorientable surfaces
- Topological knots
- Quadric and ruled surfaces
- Modular and symmetric structures
- Boolean operations
- Minimal surfaces
- Transformations
- Others

As can be noted, some of the groups of this classification cover a wide range of sculptures. For example, "the classic and polyhedral geometry" group includes works with different properties. However, other groups included in this taxonomy, for example the "minimal surfaces" group, are more restricted.

A proposal for a classification of mathematical sculpture

Our first approach has been improved and we will present our final proposal in this section. This new classification is based on the different branches of mathematics. The limits between the different groups are not very strict, which is not surprising since the same thing happens regarding the limits of the different branches of mathematics. Our final proposal is the following:

- **Sculpture with geometric characteristics**
  - Polyhedrons
  - Curved mathematical surfaces
    - Quadric and surfaces of revolution
    - Ruled surfaces
    - Nonorientable surfaces
    - Minimal or zero-mean curved surfaces
  - Other surfaces
- **Sculpture with algebraic concepts**
  - Symmetry
  - Transformations and modular sculptures
  - Boolean operations
- **Topological sculpture**
- **Sculpture with varied mathematical concepts**

In some cases, the inclusion of a particular work in one of the groups may be difficult. A clear example of that difficulty is the sculpture by Bathsheba Grossman, shown in Figure 14. That work embraces several topics such as surfaces, topological knots, symmetry, etc. We will classify each particular case by attributing it a *dominating characteristic* that “dominates” its conception. That standard approach is explained in detail in [12].

Figure 14: *Alterknot*, Bathsheba Grossman, 1999.

It is important to note that the numbers of sculptures in each group have different magnitudes. There are groups with large sculpturesque potential, containing a large number of works. Some examples of that are the group type “Minimal or zero-mean curved surfaces” (an example is shown in Figure 15), and the group type “Nonorientable surfaces” (an example is shown in Figure 16).

Figure 15: *Minimal surface costa X*, snow, Helaman Ferguson, 1999.

Figure 16: *Nonorientable surface*, Brent Collins, 1985–1989.

General description and examples

In this section, we present a general description of the different groups mentioned in our classification, giving some examples to illustrate their main characteristics.

Geometrical sculpture

Geometrical sculpture is the widest group in the classification. That happens due to the intrinsic relation between plastic arts, specially sculpture, and geometry. Geometrical sculpture includes most of the mathematical sculpture. To check that fact, it is enough to look at the examples previously exposed, almost all examples included in this category — Figures 1, 2 (right), 4 (left and right), 5, 6, 8, 10 (left), and 14.

There are examples of sculptures for almost all possible types of solids, from the simplest ones like cubes, spheres, cones, cylinders, prisms, etc., to the most complex, like irregular polyhedrons or surfaces defined by highly complex mathematical equations. In addition, in some works, the most relevant element is not a particular type of solid or a combination of solids, but some property or properties.

Geometrical sculpture includes from simple shapes (an example is shown in Figure 17), to much more complex pieces (an example is shown in Figure 18). Also, regarding sizes, sculptures range from very small sculptures (Figure 14, with only 13 cm in height) to huge dimensions (Figure 4, right, with 13 m in height).

Figure 17: *Amaryllis (plant family)*, size 350 x 129 x 350 cm, Tony Smith, Wadswhorh Atheneum, Conneticut, USA, 1965.

Figure 18: *Escher on double torus*, size 15 cm, Helaman Ferguson.

Geometrical sculpture is a type of mathematical sculpture with a lot of tradition, specially in the 20th century. At the beginning of the century, cubist movement produced some works that can be included in this group. With respect to its origins, it is possible to look at [13], reference with analysis of important artistic trends and movements of the last century. Some artists, fans of abstract movements, minimal and conceptual movements, etc., made use of geometry.

Polyhedral sculpture is the first subcategory of Geometrical sculpture. First, we analyze the well-known platonic solids. These solids are some of the figures more widely used by mathematical sculptors and by other artists due to their beauty and simplicity. Although their description is well-known, it is worth mentioning some characteristics of these regular solids. A convex polyhedron is regular if all its faces are equal regular polygons and the same number of faces meet at every vertex. There are only five regular polyhedra, known as platonic (after the Greek philosopher Plato) or cosmic. These five solids are the following: tetrahedron (4 faces); hexahedron or cube (6 faces); octahedron (8 faces); icosahedron (20 faces) and dodecahedron (12 faces).

Like the platonic solids, the truncated polyhedrons[1] have been the inspiration for many mathematical sculptures. The number of possible cases is infinite. Archimedean solids are particular cases. An archimedean solid (or semiregular) is a convex polyhedron that has a similar arrangement of nonintersecting regular convex polygons of two or more different types arranged in the same way about each vertex with all sides the same length. Seven of the 13 Archimedean solids can be obtained by truncation of a platonic solid. These have also been widely used in sculpture.

Another type of figures commonly used by mathematical sculptors are those resulting from transformations of polyhedrons, such as deformation, star-shaping or rotation, or any other geometric transformation that may result in aesthetic effects. Figure 19 (left) shows a work by John Robinson, based on a dodecahedron. The faces have been replaced by 5-point stars. This work also presents other aesthetic values, like its color, or reflections depending on its illumination, etc.


Figure 19: Left: *Star burst*, John Robinson, 1996. Right: *Permutational Sculpture*, Francisco Sobrino, Valencia, Spain, 2000.

Figure 19 (right) shows a work by Francisco Sobrino, consisting of a single module of stainless steel.

[1]Truncation is the removal of portions of solids falling outside a set of symmetrically placed planes.

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Curved mathematical surfaces is the second subcategory of Geometrical sculpture. This subcategory has been subdivided into a few non-excluding types. For example, a surface widely used in both art and architecture is the hyperbolic paraboloid, also called saddle, which is simultaneously a quadric and a ruled surface. The further subcategories are the following.

Quadrics and surfaces of revolution: Quadrics are surfaces defined by a second degree polynomial equation (here, in three variables). Non-degenerated quadrics are: spheres, cones, cylinders, ellipsoids, hyperboloids (one or two sheets) and paraboloids (elliptic and hyperbolic). An example is shown in Figure 20.

Figure 20: *Hyperbolic paraboloid 8*, Jerry Sanders, 2000.

Surfaces of revolution, as the name suggests, are created by rotating a curve (the generatrix) around an axis of rotation. Surfaces of revolution have been used profusely in art and sculpture. An interesting simplification of human figures is shown in Figure 21.

Figure 21: *Couple*, Carmen Grau, Valencia, Spain, 2000.

Ruled surfaces: Ruled surfaces are described as the set of points swept by a moving straight line in space. These surfaces have also inspired many artists and architects. An example is shown in Figure 22.

Figure 22: *Hyperbolic ribbed mace*, Charles O. Perry, Dublin Ohio, USA, 1987.

Nonorientable surfaces: A surface is orientable if a two-dimensional figure cannot be moved around the surface and back to where it started so that it looks like its own mirror image. Otherwise, the surface is nonorientable. The simplest nonorientable surface is the Möebius strip, one of the first objects of that kind that appeared in sculpture. A pioneer in mathematical sculpture, Max Bill, extensively used Möebius strips, obtaining very beautiful works, like the work shown in Figure 23 (see [8], an analysis of Max Bill's works, by Tom Marar).

Figure 23: *Endless surface*, Max Bill, Antverpen, Belgium, 1953–1956.

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Another mathematical sculptor, Brent Collins, has developed many different models of nonorientable surfaces. One of his works can be seen in Figure 16. Also, the Japanese sculptor Keizo Ushio has based some of his works on the Möebius strip, or extensions of that concept, creating very simple, though splendid sculptures. Figure 24 shows a transformation of the double Möebius strip. That piece clearly illustrates the non-orientability of the surface.

Figure 24: *Mihama*, Keizo Ushio, 1990.

Minimal surfaces: Minimal surfaces, with zero-mean curvature, are surfaces of minimal surface area for given boundary conditions. A well-known example are the surfaces created by soap films. The sculptor Helaman Ferguson has created different works based on this concept, like the piece shown in Figure 15[2]. Another example, by Stewart Dickinson, is shown Figure 25.

Figure 25: *Ennepers minimal surface*, Stewart Dickinson.

[2]The Brazilian mathematician Celso Costa formulated its equations.

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Other surfaces: This subcategory includes those surfaces that do not belong to any of the other types mentioned above. In this group, we have from sculptures with geometric objects as simple as planes, to others that can acquire very complex shapes or that combine different types of surfaces. Figure 26 shows a photo of a Richard Serra's exhibition. His works present simple geometrical shapes, planes, ellipses, truncated cones, etc. One of his most famous works, *Snake*, shown in the figure, is based on a third degree polynomial equation.

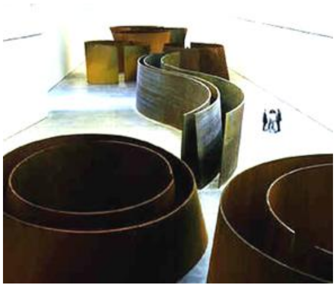

Figure 26: Some works by Richard Serra, exhibited in the Guggenheim Museum, Bilbao.

This subcategory also includes those surfaces given by equations not considered in the other groups. For example, transcendent equations such as trigonometric equations, exponential equations, etc.

Sculpture with algebraic concepts

This category comprises sculptures that make use of some algebraic concept. These works can also adopt geometric shapes like the sculptures of the first category. However, the algebraic properties that characterize them are so determinant for their conception, that we include them in this group.

Symmetry: One of the mathematical concepts with more occurrences in art is symmetry. Figure 27 illustrates a work by Robert Longhurst, showing symmetry with respect to the planes whose angles are multiple of sixty degrees.

Figure 27: *Arabesque XXIX*, Robert Longhurst, 2007.

Transformations and modular sculptures: The already mentioned works by Javier Carvajal fit the idea of transformation. The design of these sculptures is based on cylindric sections that subsequently are joined to create the complete pieces (Figures 5, 6 and 8). Modular sculptures are those sculptures in which a given pattern is repeated. The modules may be combined in many different ways. Brent Collins, a well-known mathematical sculptor, has created some modular sculptures, like those entitled *Modular spirals*. Figure 28 shows a modular sculpture by another artist, Michael Waren.

Figure 28: *Pascua*, Michael Waren, Valencia, Spain, 2000.

Boolean sculpture: In other works, operations with shapes are carried out, using some algebraic structure as, for example, boolean algebra. An example is the work by Bruce Beasley, already shown in Figure 11. The possible results of boolean operations are "true" or "false". This algebra applied to sculpture is used to describe how two solids relate, forming a new volume or emptiness. All logical operations are used: union, intersection, inversion, complement, and exclusion. Figure 29 (left and right) presents two crosses by Eduardo Chillida. The first one (left) can be interpreted as the complementary of the second (right), that is, its "negative".

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Figure 29: Left: cross in Santa Maria Church, Eduardo Chillida, San Sebastian, Spain, 1975; Right: cross in Buen Pastor Church, Eduardo Chillida, San Sebastian, Spain, 1997.

Topological sculpture

Mathematicians have studied “knots” for many centuries. This interesting and fascinating category of topological objects presents a wide range of possibilities to be used in sculpture. Most mathematical sculptors have made use of this concept. The examples in Figures 3, 14, and 18 belong to this group. Figure 30 (left) shows a sculpture by Keizo Ushio, a torus that, when sectioned by positioning a Möebius band, is divided into two topologically nested parts. Figure 30 (right) shows a computer image of the separation.

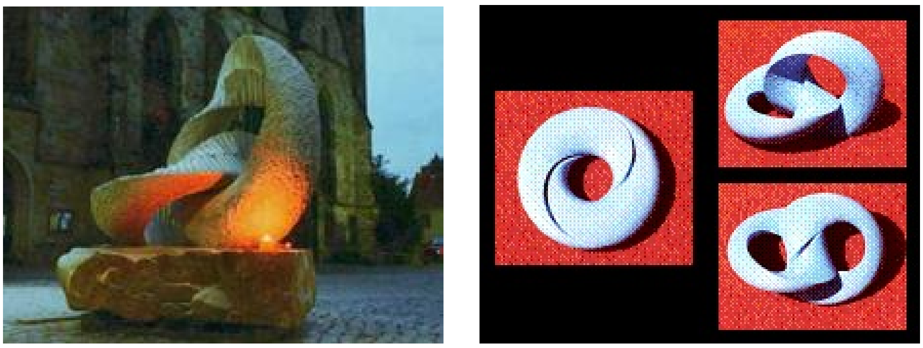

Figure 30: Left: *Oushi-Zokei* (sectioned torus), Keizo Ushio, San Sebastián, Spain, 1999; Right: separation.

Also, the mathematical sculptor John Robinson has made many works that can be included in this category. His works are simple, though very interesting from the topological point of view. Figure 31 shows his series *Trilogy*. These sculptures are inspired by the Borromean rings, three topological circles which are linked and form a Brunnian link (i.e., removing any ring results in two

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unlinked rings). The name of Borromean rings comes from their use in the coat of arms of the aristocratic Borromeo italian family.

Figure 31: Left: *Creation*, *Trilogy* series, John Robinson, 1990; Center: *Intuition*, *Trilogy* series, John Robinson, 1993; Right: *Genesis*, *Trilogy* series, John Robinson, 1995.

Sculpture with varied mathematical concepts

Although we develop and improve our classification, it is very difficult to include all the mathematical sculptures in the proposed categories. Because of that, we have established this last category. For example, the piece illustrated in Figure 31, by the sculptor Ken Herrick, has very little to do with those we have shown previously throughout the article.

Figure 32: *Cloud*, Ken Herrick.

Other interesting examples are some of Helaman Ferguson's sculptures, like the one shown in Figure 33. Although it is a nonorientable surface, we also highlight the texture, whose design required computer help. It is the Hilbert curve, a continuous fractal space-filling curve, first described by the German mathematician David Hilbert in 1891.

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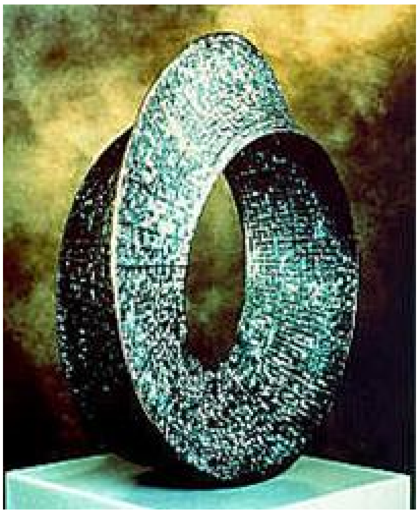

Figure 33: *Umbilic Torus NC*, Helaman Ferguson, 1988.

In addition, Figure 34 shows a cube divided into two complementary fractal parts. We observe that, if we ignore Pauli's exclusion principle, when joining again the two parts, we obtain the original cube.

Figure 34: *The Unit Cube*, Bathsheba Grossman, 2002.

We must mention the possibilities for mathematical sculpture that can be open by the use of non-euclidean, elliptic and hyperbolic geometries. The sculptures motivated by these geometries should be included in this last category. We believe that the use of this type of geometries will occur more often in mathematical sculpture, as it happened in painting, especially after M. C. Escher legacy. An interesting example is shown in Figure 35.

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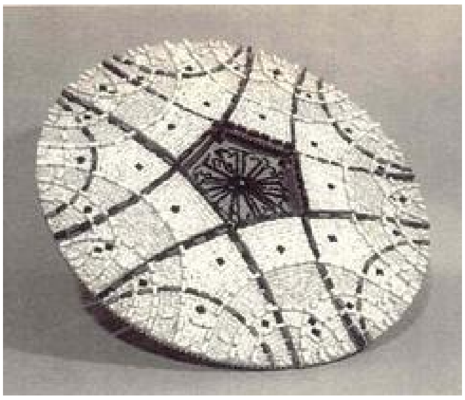

Figure 35: *Hyperbolic Diminution I*, Irene Rousseau, 2005.

Conclusions

- Mathematics relates to sculpture. Moreover, mathematics relates to most artistic manifestations.
- The breakthroughs in mathematics that have taken place in the 20th century have made possible the development of a new type of mathematical art.
- Mathematical sculpture has reached a remarkable status at present. To this has contributed, in addition to the recent advances in mathematics, the development of computer science.
- We believe that mathematical sculpture will expand. This is due to the causes mentioned above, as well as the growing interest of artists and public.
- Courses on mathematics and art, either at secondary level or at the university level, should be encouraged.
- Possibly, the proposed classification can be improved. Other concepts and different mathematical properties may be introduced.

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