ISSN 2182-1976

Recreational Mathematics Magazine

Number 8
December, 2017

DE GRUYTER OPEN
FCT Fundação para a Ciência e a Tecnologia
MINISTÉRIO DA CIÊNCIA, TECNOLOGIA E ENSINO SUPERIOR
CIUHCT
Ludus

Edition
Associação Ludus, Museu Nacional de História Natural e da Ciência,
Rua da Escola Politécnica 56 1250-102 Lisboa, Portugal
email: rmm@ludus-opuscula.org
URL: http://rmm.ludus-opuscula.org

Informations
The *Recreational Mathematics Magazine* is electronic and semiannual.
The issues are published in the exact moments of the equinox.
The magazine has the following sections (not mandatory in all issues):

Articles
Games and Puzzles
Problems
MathMagic
Mathematics and Arts
Math and Fun with Algorithms
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ISSN 2182-1976

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Games and Puzzles

“A difficult case”: Pacioli and Cardano on the Chinese Rings

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Abstract: *The Chinese rings puzzle is one of those recreational mathematical problems known for several centuries in the West as well as in Asia. Its origin is difficult to ascertain but is most likely not Chinese. In this paper we provide an English translation, based on a mathematical analysis of the puzzle, of two sixteenth-century witness accounts. The first is by Luca Pacioli and was previously unpublished. The second is by Girolamo Cardano for which we provide an interpretation considerably different from existing translations. Finally, both treatments of the puzzle are compared, pointing out the presence of an implicit idea of non-numerical recursive algorithms.*

Keywords: Chinese rings, recreational problems, Pacioli, Cardano.

1 Introduction

The Chinese rings feature as a well-known puzzle in many works on recreational mathematics. It is covered in all the classic works such as [22, 163–186], [3, 80–85], [8, Problem 417] and [12, 15–17]. In French nineteenth-century works ([22], [13]), the puzzle is also known by the name ‘Baguenaudier’. As the earliest source in Europe one often quotes Girolamo Cardano’s *De subtilitate* [6, 294f], which gave it its alternative name ‘Cardan’s rings’. The aim of this paper is not to provide a complete overview on the history of the Chinese rings but to present for the first time an English translation of the earliest known European source, namely *De viribus quantitatis* by Pacioli [25, Capitolo CVII], which gives a reasonably detailed explanation of the puzzle as it was known at the beginning of the sixteenth century (Appendix 1). We will also discuss Cardano’s treatment of the puzzle, which has become available recently in an English translation

[10]. However, we are providing here an alternative translation based on a mathematical interpretation of Cardano's description (Appendix 2). While the puzzle has always fascinated mathematicians, such as Pacioli, Cardano and John Wallis in his *De algebra tractatus* [31, 472–478], it attracted the attention of modern mathematicians only since the booklet [13] became known through [22]. The mathematical explanation that we present here is necessary for the interpretation of the texts by Pacioli and Cardano and our choice of material is focussed on this interpretation and the translations.

As for the Chinese sources, there remains a lot of confusion. References to rings, linked rings or interlocked rings date back to Sun Tzu's (544–496 BCE) *The Art of War* and the logic school of Hui Shih (380–305 BCE). However, the terse text fragments do not allow us to identify these as references to the puzzle as we know it. As an additional complication, there also exist a conjuring trick by the same name, with solid metal rings that appear to link and unlink and pass through each other, as well as wire puzzles with several pairs of interlocked irregular shaped rings that need to be separated. The tangram puzzle was also commonly referred to as the 'Chinese puzzle' during the nineteenth century. Joseph Needham briefly discusses the Chinese rings in his volume on mathematics [23, 111] and shows a drawing of a purchased specimen from the beginning of the nineteenth century (see Figure 1).

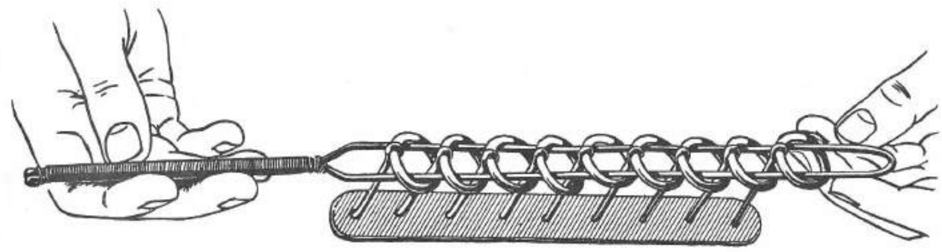

Figure 1: The puzzle as discussed by Joseph Needham.

Needham mentions that the puzzle is known by the name 'Ring of linked rings' but he remains cautious about its source, admitting that "its origin is quite obscure". The earliest depiction of the Chinese rings in a Chinese work appears to be in a painting by Yu Ji (1738–1823) of a lady holding a version of the puzzle with nine rings. Illustrations of the puzzle in books or material evidence are all of later dates. Curiously, the puzzle is treated in books on Japanese mathematics, or *wasan*, that precede these earliest Chinese sources, most prominently in the *Shūki Sanpō* by Yoriyuki Arima, [2] (see Figure 2); cf. [7].

DOI: https://doi.org/10.1515/rmm-2017-0017

Figure 2: A solution to the puzzle from the *Shūki Sanpō* (National Diet Library Digital Collections).

2 The name of the puzzle and its parts

That this popular puzzle is termed in the West as 'Chinese rings' reveals the awareness of a cross-cultural influence. Remarkably however, none of our early sources mention a Chinese origin. The name may have come in use because of the popularity of exotic specimens of wonder from the Far East during the nineteenth century.

The rings, supported by individual rods, are arranged on a kind of loop as in Figure 1. In the chapter title, Pacioli uses the term 'strenghetta' for this loop. *Queen Anna's New World of Words*, a seventeenth-century Italian—English dictionary translates 'strengata' as 'a yark, or strip with a point'. The verb 'to yark' refers to the shoemaker's practice of sewing leather. In German, the most likely translation for 'strenghetta' would be 'Schlaufe', which translates into a heft or loop. Cardano refers to the puzzle as 'instrumentum ludicrum', or a playful tool. According to the Oxford English Dictionary, the puzzle was known as Tarriers or Tarrying irons in England at the beginning of the seventeenth century [28]. In *De algebra tractatus*, Wallis spends six pages on *De complicatis annulis* or connected rings, and mentions Cardano as his main source. However, his version of the puzzle (see Figure 3) does not follow the description by Cardano but instead uses connected metal lamella, which suggests that he had such a specimen at his disposal. For the loop, Wallis does not adopt the term 'naviculum' from Cardano, but uses 'acum' (hair pin) or 'orbiculum' (revolving figure). Intriguingly, his analysis is based on nine rings, which is the more common version in Chinese sources. The configuration with nine rings is

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confirmed by the Chinese name for the puzzle which is *jiulianhuan* (nine linked rings)[1].

PROBLEMA.

His ita paratis; Requiritur, Ut ſic imponatur Acus, ut per omnes tranſeat Annulos, omnes intra ſe Clavos complectens: Atque ut (ſic implicata) inde demum Expediatur.

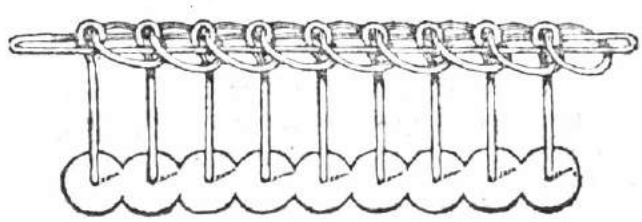

Figure 3: The puzzle as discussed by Wallis.

In a four-volume encyclopedic work on recreational problems by Jacques Ozanam, first published in two volumes as [24], the puzzle is shown as an illustration only (1725, IV, plate 16), (see Figure 4). However, Ozanam does not discuss it anywhere in the text. His version shows seven rings and the rods holding the rings are secured in a wooden or leather slat.

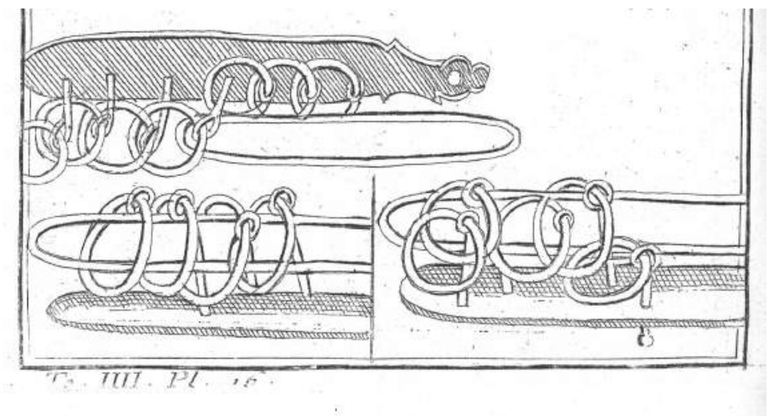

Figure 4: An illustration of the puzzle from [24].

None of the sources of the early nineteenth century mention a Chinese origin of the puzzle. Even Louis Gros, in his dedicated essay [13] on the *baguenodier* [sic!],

[1]However, the terms used in other languages do not refer to Chinese rings. In Korean, it is addressed as *yu gaek ju* (delay guest instrument). In German, it is called *Zankeisen* (quarrel iron) or *Nürnberger Tand*. Italians call it *anelli di Cardano* (Cardano's rings). In Russian the puzzle is known as *meleda*. In Swedish: *Sinclairs bojor* (Sinclair's shackles), and Finnish: *Vangin lukko* (Prisoner's lock) or *Siperian lukko* (Siberian lock).

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which contains an etymological introduction, does not mention the Chinese. Édouard Lucas, who discussed the puzzle first in the *Revue Scientifique* (2) 19(1880), 36–42, and later in his book [22], also uses the French term without mentioning a Chinese origin. However, in England in the 1860s, the term 'Chinese rings' starts to appear and grows more and more common. The earliest reference to the puzzle by this term we could find is a question proposed by G. S. Carr in *Mathematical Questions and Solutions from The Educational Times*, Vol. XVIII (1873, 31). The puzzle then appeared by this name in sales catalogues on games and pastimes as in the one by Peck and Snyder from New York (reprinted in [26]) or as a Japanese rings puzzle in Mr. Bland's *Illustrated Catalogue of Extraordinary and Superior Conjuring Tricks* (1889). Earlier English books describing the puzzle, such as *The Magician's own book* (1857, 280–283) do not refer to a Chinese origin. H. J. Purkiss calls it, in 1865, 'the common ring-puzzle' [27, 66]. So, it seems that the term 'Chinese rings' has been introduced only late in the nineteenth century.

3 Origin and transmission

The lack of any historical Chinese text describing the puzzle before Pacioli and Cardano, and the late adoption of the term 'Chinese rings' makes us circumspect about the possible Chinese origin of the puzzle. While it is difficult to determine a single origin of the contraption, it was probably invented only once. The rather complicated arrangement of rings, rods and bars makes it very unlikely that it was designed by several people independently or that it originated in different cultures simultaneously. Moreover, all extant versions are very similar, only differing in the number of rings, being mostly 7 or 9. Also, whoever has invented the puzzle must have had an idea of how to solve it. The great majority of extant Chinese rings puzzles have an *odd* number of rings. This implies that, when starting with all rings on the bar, the best first move is by ring 1, the one which is always free to be moved; for an even number of rings it would be ring 2, the next in succession. There are several references to its use as a lock. Cardano describes it as a useless subtlety, "though it can be applied to the artful locks made for chests". The Finnish names refer to it as a lock and [22, 165] writes that it was used as a lock in Norway. The Japanese scholar of Dutch language and science, Genpaku Sugita [30] narrates how Gennai Hiraga managed to unlock a bag with the Chinese rings belonging to the Dutch chief Jan Crans at the Dutch trading company at Deshima, c. 1769 [28]. As we do not know the inventor of the Chinese rings, his or her intention may have been to design an intricate lock, but we believe it served foremost a recreational purpose.

DOI: https://doi.org/10.1515/rmm-2017-0017

Figure 5: The *mamakodate* problem as it appears in the *Jinkōki* of 1627.

The Chinese rings puzzle is not the only recreational problem whose cultural origin is difficult to establish. While the use of the term 'Chinese rings' suggests a Chinese origin, the direction of influence might actually be from the West to the East. We may compare the question of its origin with the so-called Josephus problem, also treated by Pacioli in the *De viribus quantitatis* (problems 56–60) and by Cardano in his *Practica Arithmetice* ([5], chap. 61, section 18, ff. $\mathrm{T}^{\mathrm{iiii}}$r–$\mathrm{T}^{\mathrm{v}}$r). The problem concerns counting out every kth person standing in a circle of n people, so that depending on the version of the problem, a given number of people remain or only the last person remains. The name 'Josephus problem' (*Ludus Josephi*) is given by Cardano who relates it to a passage in Josephus' *De Bello Judaico* (c. 80, Book III, chap. 8, sect. 7). The most common version in the Latin world is about 15 Christians and 15 Turks on a boat that is about to sink, unless 15 people are thrown overboard. If all 30 stand in a circle and every 9th person is thrown overboard the following sequence guarantees that only Turks are thrown overboard: 4, 5, 2, 1, 3, 1, 1, 2, 2, 3, 1, 2, 2, 1, meaning first 4 Christians, then 5 Turks, then 2 Christians, etc. These solutions are memorised by means of a vowel mnemonic. [29] was the first Western source to document that the Josephus problem was well known in Japan at least since the *Jinkōki* of 1627, one of the first books on *wasan*, traditional Japanese mathematics. The story here is of a wealthy farmer having thirty children, 15 from his first wife and 15 from the second, leaving his heritage to his single favorite son. Western, but also Japanese historians of mathematics have entertained the hypothesis that Jesuits residing in Japan from 1549 to the 1630s may have influenced the development of *wasan*[2]. Furthermore, there is the case of Hartsingus, a 'Japanese student' of mathematics at Leyden in 1654. However, none of these supposed influences can pass closer scrutiny [16]. A further complication is that the Japanese version of the problem, known as *mamakodate*, also features in literary works of a much older date. Though Josephus predates the earliest Japanese sources, it is nowhere evident to determine the direction of influence. It could have originated in Japan as well as in the Latin world.

Recreational problems such as the Josephus problem leave evidence of their existence as stories (Josephus), figures (as in Figure 5) or solidified solution recipes as the vowel mnemonics which allow us to trace the problem through historical sources. Such problems travel easily across cultural boundaries along the trade routes while embedding mathematical practices which become adapted to a new cultural context [15]. With the Chinese rings puzzle, we also have the material culture of crafting, selling, transporting and using the contrivance. Material remains could shed some light on the origin of the puzzle, but extant examples are of recent dates. The puzzle shown by Needham (Figure 1) is a nineteenth-century copy. Unless a specimen would be excavated in a tomb dating before the sixteenth century, there is no solid evidence of its Chinese origin.

Pacioli's text is the earliest source of a mathematically inspired treatment of the Chinese rings puzzle. The *De Viribus Quantitatis* is an extensive compilation of puzzles, conjuring tricks and recreational problems. According to a dedicatory letter it was written from 1496 to 1508. However, in problem 129 there is a reference to a date in 1509, which indicates that the manuscript was still being edited during that year. The book was never published but survived as a manuscript owned by an eighteenth-century book collector, Giovanni Giacomo Amadei. It was later acquired by the University Library of Bologna [25]. The dedication of the book and a request for a printing privilege was published by Boncompagni in his journal [4, 430-432]. These texts provide sufficient evidence for the attribution to Pacioli and the dating of the manuscript. A transcription of the text was published only in 1997 [11]. In 2009 a luxurious facsimile edition was published by the Aboca Museum with some commentaries [19].

4 Mathematical interpretation

The Chinese rings puzzle with $n \in \mathbb{N}_0$ rings can be modelled mathematically by a graph R^n whose vertex set is B^n, $B := \{0, 1\}$, and where a pair $\{s, t\} \in \binom{B^n}{2}$ belongs to the edge set, if the states represented by s and t differ by the move

[2] [29], 135: "could the mathematics of the West, or any intimation of what was being accomplished by its devotees, have reached Japan in Seki's time? These questions are more easily asked than answered, but it is by no means improbable that the answers will come in due time. We have only recently had the problem stated, and the search for the solution has little more than just begun."

DOI: https://doi.org/10.1515/rmm-2017-0017

of one ring[3]. We write $s = s_n \dots s_1$, where $s_r = 0$, if ring $r \in \{1, \dots, n\}$ is *off* the loop and $s_r = 1$, if it is *on* the loop; here ring 1 is the outermost ring and ring n is the one closest to the handle. So s is a *word* over the *alphabet* B or a *binary string* (of *length* n). Figure 6 shows the example of graph R^3.

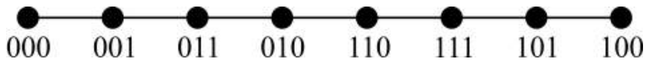

Figure 6: The graph R^3.

An easy theory shows that R^n is isomorphic to P_{2^n}, the path graph on 2^n vertices, whence the shortest path between any two vertices is always unique. The pendant vertices of R^n are $\alpha^{(n)} := 0^n = 0 \dots 0$ and $\omega^{(n)} := 10^{n-1} = 10 \dots 0$ (with $\alpha^{(0)} = \emptyset = \omega^{(0)}$)[4]. More on the mathematical theory of the Chinese rings can be found in [18, Chapter 1].

We will designate by ℓ_n (the length of) the path from 0^n to 1^n in R^n and by $\overline{\ell}_n$ the $1^n, 0^n$-path (with the same length). The ℓ stands for Georg Christoph Lichtenberg (1742–1799), who in 1769 published the first (known so far) account of the lengths of these paths for general n; see [17]. We then have[5]

$$\ell_n + \ell_{n-1} = M_n \,, \tag{1}$$

the *Mersenne number* $M_n = 2^n - 1$ being the length of the whole $0^n, 10^{n-1}$-path with $1^n = 11^{n-1}$ lying on that path. Lichtenberg did not write out (1) explicitly, but instead derived the equivalent recurrence

$$\ell_0 = 0,\ \ell_{n+1} = \ell_n + 2\ell_{n-1} + 1 \,. \tag{2}$$

4.1 *De viribus quantitatis*

Pacioli, who, of course, does *not* refer to a graph, describes the minimal solution for the task to get from 0^7 to 1^7, but does *not* account for the number of moves on that path. The first 8 moves, i.e. from 0000000 to 0001100, are given explicitly in great detail. Then, for the first time, he alludes to a sequence of moves already done, namely $\overline{\ell}_2$, and tells us to do these moves in reverse, i.e. ℓ_2, to arrive at 0001111. The next instruction corresponds to an $\overline{\ell}_3$ leading to 0001000 and allowing ring 5 to go up: 0011000. Again, Pacioli evokes an ℓ_3 which would result in 0011111, but he says "and you will have 6 on the loop". So he forgot or skipped voluntarily a 0011111, 0111111-path, which amounts to $\overline{\ell}_4$ (4 rings down) plus 1 move upwards of ring 6 and then ℓ_4 to reposition the first four rings on the loop. But then he resumes his strategy by saying that to move ring 7 up, "it is first necessary to pull down the first 5 ones", i.e. $\overline{\ell}_5$. To do this he can just run the procedure which led from 0000000 to 0011111 in reverse order, thereby switching ℓ and $\overline{\ell}$, because the presence of ring 6 on the loop does not change the mobility of the first 5 rings. Again he explains this explicitly,

[3]We have chosen a graph for a mathematical model of the Chinese rings specifically to connect with the explanations provided by Pacioli and Cardano. Readers unfamiliar with basics from graph theory are referred to, e.g., [32].

[4]Inside a word, l^k means a k-fold repetition of the *letter* l.

[5]By convention $\ell_{-1} = 0$.

i.e. $\overline{\ell}_3$, ring 5 down, ℓ_3, $\overline{\ell}_2$, ring 4 down, ℓ_2, $\overline{\ell}_1$, ring 3 down. At this point, at state 0100010, a somewhat cryptic instruction follows, which can only mean ℓ_1, ring 2 down, ring 1 down. since it must lead to 0100000, when ring 7 can go up. Arriving at 1100000, he spares the reader an ℓ_5 to the final state 1111111 or either another $\overline{\ell}_6$ to arrive at 1000000 which would allow an 8th ring to move up and so on. This second interpretation is compatible with Pacioli's words "And in this way, successively you can place the others, one after the other".

Obviously, Pacioli had made the following observations:

1. Ring 1 is free, i.e. it can move at any time.

2. Ring $r+1$ can only be moved if ring r is on the loop and all rings $< r$ are off: $0\omega^{(r)} \leftrightarrow 1\omega^{(r)}$. We may omit equal prefixes of any length, because rings $> r+1$ do not affect such a move.

3. After an up move of ring $r+1$, i.e. at $1\omega^{(r)}$, one has to move up rings 1 to $r-1$ to arrive at 1^{1+r} and thereafter r rings down to get to ω^{1+r}. (This takes all in all $\ell_{r-1} + \overline{\ell}_r = \ell_{r-1} + \ell_r = M_r$ moves.)

4. Equivalently, starting in 01^r, taking $r-1$ rings off leads to $0\omega^{(r)}$, such that ring $r+1$ can be lifted up to $1\omega^{(r)}$, and again lifting $r-1$ rings ends in 1^{1+r}. (Here $\overline{\ell}_{r-1} + 1 + \ell_{r-1} = 2\ell_{r-1} + 1 = \ell_{r+1} - \ell_r$ moves have been performed; cf. (2).)

Pacioli, after the inevitable first lift of ring 1 according to observation 1, follows point 4 for r from 1 to 4, then skips $r=5$ and resuming with $r=6$, he repeats details for the performance of $\overline{\ell}_5$ as the inverse of the ℓ_5 that led him from 00000 to 11111 at the beginning. The final ℓ_5, which would lead from 1100000 to 1111111, or any further goal he then "spare[s] the reader".

The last sentence of Pacioli's Chapter 107 is very remarkable: if our interpretation, which is supported by the title of the chapter, is correct, he describes a two-player game in which one of the players starts in 1^n, the other in 0^n and the winner is who first reaches the opposite state of the instrument, i.e. who can take off all n rings from the loop, in other words pull the loop from the rings, or put them all up, i.e. place the loop into them, respectively. There is no other example in literature where such a competition based on the Chinese rings has been proposed!

4.2 *De subtilitate*

In Book XV, *On uncertain-type or useless subtleties*, Cardano first describes the material which is obviously the same as Pacioli is referring to; the number of rings is fixed to 7. Cardano uses the word *navicula* for what Pacioli named by the obscure word *strenghetta*. While we translated the latter with *loop*, we will employ *shuttle*, deriving from the weaver's tool, for Cardano's text.

This material, in addition to Pacioli's observations 1 and 2, allows for the first two rings to move simultaneously—up, if both are off, and down, if both are on the shuttle. This is what Lucas later called the *marche accélérée* (accelerated run) [22, 183f]. The graph $\widetilde{R}^n$ modelling this version, obviously for $n \geq 2$, is obtained from R^n by adding edges between vertices ending in 00 and 11 and deleting all vertices (and their incident edges) ending in 01. It is again a path graph, but whose length is shortened to $3 \cdot 2^{n-2} - 1$. In this setting, in contrast to all the other rings, ring 2 never walks alone, but always together with ring 1. The example of $\widetilde{R}^3$ is drawn in Figure 7.

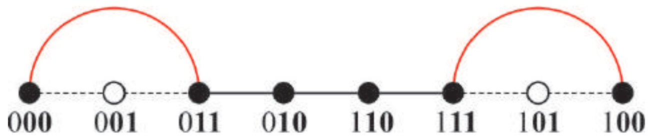

Figure 7: The graph $\widetilde{R}^3$.

Accordingly, Cardano starts, from state 0000000, by pulling up rings 1 and 2 to get 0000011, from where he can drop down ring 1 to arrive at 0000010. So he used two "exchanges" instead of the three moves of individual rings. Now tbe third ring can be lifted to 0000110 and moving up ring 1 results in three rings on the shuttle 0000111. Then rings 1 and 2 can be dropped simultaneously to reach state 0000100 from which ring 4 can go up to 0001100.

Somewhat surprisingly, Cardano stops here with his stepwise description of the solution, because he believes that the procedure is already clear from that short introduction and that it follows three recipes:

1. "that the ring to be pulled up or dropped down should have only one in front of it enclosing the shuttle"

   This is the same as Pacioli's observation 2. (Cardano mentions the first ring only later as the one which is free; cf. Pacioli's observation 1.)

2. "when you are dropping down, you should always let down the first two together after pulling up the first, or before dropping the first down you pull up the first two"

   This is the statement that rings 1 and 2 should move simultaneously, if possible. The wording is not particularly clear, but it can only mean that the exchanges involving the first two rings always come in two, either two rings up, one down, i.e. $00 \to 11 \to 10$, or one ring up and two down, i.e. $10 \to 11 \to 00$.

3. "whichever is pulled up or dropped, all those in front must be pulled up, and again be dropped"

   Again the wording is a bit ambiguous, but it probably refers to $00^r \to 01^r \to 010^{r-1} \to 110^{r-1} \to 11^r \to 10^r$ and $10^r \to 11^r \to 110^{r-1} \to 010^{r-1} \to 01^r \to 00^r$ for ring $r+1$ to move up or down, respectively, for $r \in \{2, \ldots, 6\}$. This can be compared with Pacioli's observation 4.

DOI: https://doi.org/10.1515/rmm-2017-0017

After stating his three instructions for the solution, Cardano emphasises the independence of the first two rings and claims, without proof, that state 1111111 is reached (from 0000000) in 64 exchanges and that another 31 lead to 1000000. He then adds these numbers to arrive at 95 for the whole path $\widetilde{R}^7$ and even considers the way back with the same number of exchanges to summarize the round trip $\alpha^{(7)}(\to 1^7) \to \omega^{(7)}(\to 01^6) \to \alpha^{(7)}$, completed in 190 exchanges.

These four numbers are correct, i.e. they represent the values p_7, p_6, $3 \cdot 2^5 - 1$ and $3 \cdot 2^6 - 2$, where $p_n = 3\ell_{n-2} + 1$ is the length of the path from 0^n to 1^n, named for Henri John Purkiss (cf. [17]), and the other two numbers stand for the length of the path $\widetilde{R}^7$ as seen before and its double. Note that, similarly as in (1), we have, for $n \geq 2$,

$$p_n + p_{n-1} = 3 \cdot 2^{n-2} - 1\,. \tag{3}$$

4.3 Comparison

Given that the optimal, i.e. shortest, solution for the Chinese rings puzzle is unique, it comes as no surprise that the descriptions of Pacioli and Cardano are very similar. The main difference is that Cardano uses the accelerated rule to move ring 2 always together with ring 1 in one exchange. This could have led him, already after the first 6 exchanges, to the recipe of employing his instruction 2 alternately with a move of ring 3 in between and the only possible move of one ring different from 1, 2 and 3 thereafter: 00 $\to$ 10, one move of ring 3, 10 $\to$ 00, one move of some ring $r > 3$. In that case, his instruction 3 is only needed to deduce the numbers of moves of rings $r \in \{4, \ldots, 7\}$, namely 2^{7-r}, on the path from $\alpha^{(7)}$ to $\omega^{(7)}$. Then, after $6 \cdot \sum_{r=4}^{7} 2^{7-r} = 6 \cdot 15 = 90$ exchanges we are in state 1000100, so that after 5 more we arrive at the terminal state. Similarly, if we interrupt the sequence of moves of the larger 4 rings after the 10th already, i.e. after 60 exchanges, we are in state 1111000 and 4 more exchanges lead to 1^7. This is a possible explanation for the four numbers occurring in Cardano's text.

Let us finally mention that none of the authors addressed the question of optimality of their solutions although minimality of the path length is usually assumed to be part of the problem in comparable puzzles like, e.g., the *wolf, goat and cabbage* problem. Note that describing a solution, as Pacioli and Cardano did, only gives an upper bound for the optimal length. Therefore, an additional argument is needed for the lower bound as, e.g., provided above by looking at the corresponding graph.

5 Conclusion

It is remarkable that the Chinese rings puzzle caught the attention of mathematicians such as Pacioli, Cardano, Wallis, Lichtenberg, and Arima long before a full theory was established by L. Gros. They all showed an interest in the mathematical aspects of the puzzle and all arrived at interesting results. They treat the Chinese rings not as a mere curiosity or pastime but recognize the combinatorial and recursive aspects of the mathematics behind the puzzle.

Recursive thinking is present in both Pacioli's and Cardano's accounts. The former claims that once you know "the method", namely to get to 1100000, you can put up the other rings, the number of which he declares to be arbitrary from the beginning. The figure Pacioli refers to would obviously decide the question whether he is just describing a *generalizable example*, but the figure is unfortunately missing in the extant manuscript. Cardano restricts his treatment to seven rings (Wallis has nine) and definitely considers a finite path, namely from 0000000 to 1000000 and back again. But also here, some recursive thinking is present, because he stops his stepwise description already at 1100, which not coincidentally is the prototype of Pacioli's solution.

The first evidence for a general n is due to Lichtenberg [20]. [2] again has 9 rings, but something like recursion is mentioned (*jiyaku-jutsu*). Also he gives a general rule for how to get the next entry in his table(s) from the previous one. By the time of Gros and Lucas recurrence is, of course, well-established.

While Pacioli's description of the solution is more lucid, he does not deal with move numbers at all. Interestingly, both authors employ a certain form of non-numerical recursion in their solution procedure. As with mathematical induction, an implicit idea of recursive procedures seem to have been present at the beginning of the sixteenth century[6]. Although they both restrict themselves to $n = 7$ rings, Pacioli admits that n could be any number (larger than 2) and his as well as Cardano's algorithm could easily be extended to any number of rings $n \geq 2$.

Apart from the rather specific recursive character of the *Arithmetical triangle*, an often cited example of an early use of recursion is Fibonacci's rabbit problem [9, 404–5] leading to the first definition of an integer sequence by recurrence; cf. [21, p. 131]. However, it was not really noticed before the seventeenth century and we believe the case of Pacioli and Cardano here presented to be much closer to the modern concept of recursion than that of Fibonacci, or the inventors of the Arithmetical triangle for that matter, by applying it to a "method", thereby anticipating a non-numerical recursive algorithm.

Acknowledgements

The authors wish to thank Alexandra Wieser-Seibt (Krailling) and Giuseppe Primiero (London) for their help in translating the Italian texts. Albrecht Heeffer was supported by the Research Foundation Flanders (FWO Vlaanderen), project nr. 3G002713. Andreas M. Hinz was supported by the Slovenian Research Agency (ARRS) under the grant J1–7110.

[6]For a discussion on the use of mathematical induction in the abbaco period (before 1500), see [14].

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Appendix 1: Pacioli

The transcription is made from [25], reproduced by Honsell and Bagni [19]. Ligatures and abbreviations have been expanded, the letters 'u' and 'v' have been differentiated and punctuation has been added, but otherwise the original spelling has been maintained, even where it is used inconsistently. The English translation closely follows the original text. For convenience to the reader, the states of the rings on the loop are shown with binary strings s as explained in Section 4 put between square brackets: $[s]$.

Capitolo CVII: Do cavare et mettere una strenghetta salda in al quanti anelli saldi difficil caso

Molti hanno certa quantita de annelli saldi messi in certi gambi, quali asettano in una steccha piatta de legno o altro metallo, la quali gambi sonno commo chiuodi o vero aguti ognuno ficto nel suo foro alla fila in ditta stecha, in modo chel capo loro tenga, ch'non posino uscire, et la ponta de ognuno revoltata aluncino ch' tenga ognuna uno anello et ognuno delli anelli ha la ponta de laguto dentro.
Et poi, in ditta ponto fermato la nello el chiuodo non po ne avanze per lo capo ne adrieto per la nello ch' sta in la ponta revoltata dentro laltro anello. Et questi anelli possano essere piu de tre quanti te piaci. Ma manco non per chel giuoco non seria bello, et sonno situati uno in laltro commo vedi qui in [f. 212r] figura, salvo chel primo diloro non ha niuno dentro.
Da poi hanno una strengheta facta commo vedi, salda da ogni capo et in quella con bellissimo modo et in gegno in filzano tutti ditti anelli commenzando dal primo. Cioe da quello ch'é libero et metese in questo modo. Videlicet: Prima metti quello. 1$^{\text{o}}$ p$^{\text{o}}$ solo in traversandolo per taglio, ch' possa intrare, et gli altri tutti stanno a giacere insu ditta steccha.
Poi prendi el 2$^{\text{o}}$ annello tirando lo su pel gambo del primo et tirandolo inanze al capo de detta strengheta. Lo mettera et arane' doi gia. Poi gettarai giu il primo insula steccha a giacere' et su pel gamdo del 2$^{\text{o}}$ tirerai su el 3$^{\text{o}}$, pingendolo inanze ch' nentri la stringhetta, comme prima festi per lo 2$^{\text{o}}$, et messo ch' tu arai questo 3$^{\text{o}}$, torneravi poi el primo ch' giu gettasti, quale dase e' libero, et tirato ch' larai su, et tu giu pel suo gambo getterai giu el secundo et ancho ditto primo in mo ch' in stangheta non resta se non el 3$^{\text{o}}$.
Poi su pel gambo de questo tertio tirerai su el quarto con ducendolo in capo della strenghetta. Commo prima festi aglialtri. Poi tornarai superordine quelli doi primi ch' getasti gui et arane 4 in stanghetta. Poi ordinamente gettarai giu p$^{\text{o}}$ 2$^{\text{o}}$ 3$^{\text{o}}$ et solo el 4$^{\text{o}}$ resti in stanghetta il quale a modo ditto tirera su el 5$^{\text{o}}$. Poi pertira- [f. 212v] re su el 6$^{\text{o}}$ te bisogna retornare su tutti li 3 primi a modo ditto a uno auno et arane 6 in stanghetta et per tirare el 7$^{\text{o}}$ bisogna ut prius gettar giu li primi 5. Cioe primo, secundo, tertio, quarto, quinto.

El 5^o non sepi gettare se non quando sia gettati giu prima, p^o, 2^o, 3^o, poi per lo gambo del quatro se getta giu el 5^o. Poi se retorna, p^o, 2^o, 3^o per buttar giu el q^o, qual non se po gettare se non per lo gambo del 3^o, per li quale medesimamente fo messo su, et pero se getta li primi doi. Cioe p^o, 2^o, et poi per lo gambo de ditto tertio se getta el quarto, poi se getta giu el tercio ma prima se tornano su li ditti doi, cioe primo, secundo e getta se giu el primo. Poi per lo gambo del 2^o se getta el 3^o commo sali su. E cosi urevitere li arai cavati tutti 5 é sira solo restato in stanghetta el 6^o anello per lo cui gambo farai andar su el 7^o e' cosi sucessive de mano in mano metterai li altri et parcas lector.
Per ch' non solo ascrivere el modo ma actu mostrandolo con fatiga el giovinelo a prende'. Ma son certo chel tuo ligiero in gegno alla mia diminuta scriptura suplire in quo plurimum confido ideo. Et di questi anche sene propone doi, uno ligato et laltro sciolto, achi prima asetta luno commo laltro venca.

Chapter 107: Where a solid loop is pulled from and placed into any number of solid rings, a difficult case

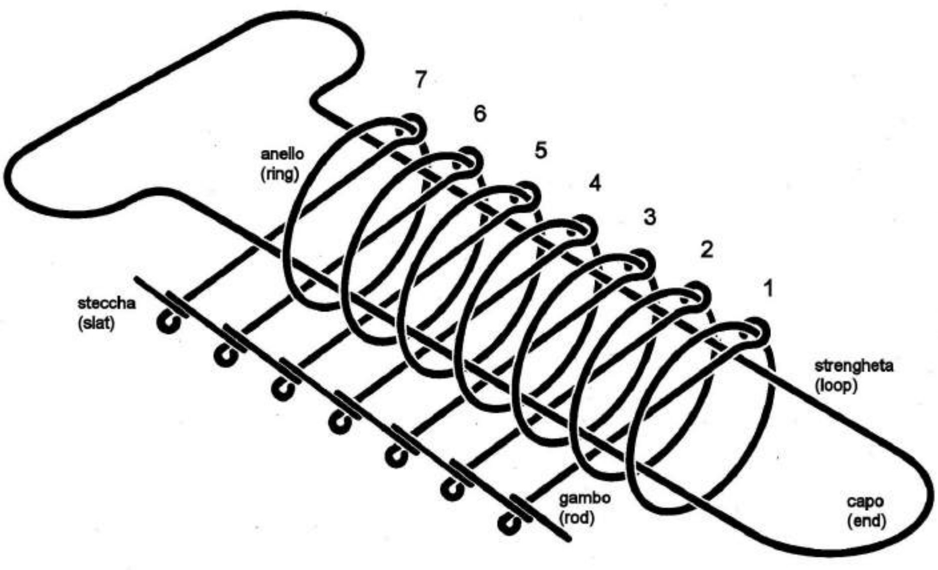

Figure 8: A reconstruction of the Chinese rings (state 1^7) from Pacioli's description.

Many have a certain quantity of solid rings held by some rods which come together in a slat made of wood or else of metal, which rods are as locks, or each fixed together at their hole in rank at that slat, in such a way that its end holds it so that none can move out and each of their tips is coiled so that each holds one of the rings and each of the rings has its tip inside.

And then at these closed tips the lock of the ring cannot move at the end, nor at the other side as the ring rests by the coiled tip inside the other ring. And these rings may be more than three as many as you please. But with less the game would not be as nice. And they are placed one to the other as can be seen in the figure [missing in the manuscript, cf. Figure 8 for our tentative reconstruction] except that the first of them does not have one inside.

Then, since we have made a loop, as you can see, closed at each end and such in a most beautiful way and skill are lined up all these rings starting from the first [0000000]. Thus the one which is free is placed in such a way: first, place this first one alone by passing it through the cut so that it can enter and all the others are remaining along that slat [0000001].

Next, take the second ring by pulling it up along the rod of the first and pulling it forward to the end of that loop. Leave it so that we have two already [0000011]. Then throw the first down to the slat to remain there [0000010]. And by the rod of the second you will pull up the third, pushing it forward so that the loop enters it, as you did before for the second [0000110]. When you have put up this third, next return to the first which you had thrown down, which is free by itself. And pull it up [0000111] and along the rod [of the first ring], you will throw down the second [0000101], and also that first [0000100], such that the only one remaining on the loop is the third [0000100].

Then, along the rod of this third you pull the fourth leading it to the end of the loop as you did before for the others [0001100]. Then reverse the order [of the moves] of the first two which you had thrown down and we have four on the loop [0001111]. Then, in the right order pull off the first, second and third, and only the fourth remains on the loop [0001000]. In the same way pull the fifth one up [0011000] and for the sixth it is necessary to return to all the first three in the same way, one by one [0011111]. And you will have six on the loop [0111111]. And to pull on the seventh it is first necessary to pull down the five first ones. Thus the first, second, third, fourth, fifth.

The fifth cannot be thrown when not having thrown the first, second, third [0111000]. Then, along the rod of the fourth one can throw down the fifth [0101000]. Then we return, the first, second, third [0101111], to drop down the fourth, which cannot be thrown if not along the rod of the third, which itself has to be placed up. And therefore we have to drop the first two. Thus, first, second [0101100], and by the rod of the said third we can throw down the fourth [0100100]. Then we throw down the third, but first we return to these two. Thus, the first, second [0100111] and we throw down the first [0100110]. Then, by the rod of the second the third is thrown down [0100010] as it climbed up. And so to shorten, you have all five off, and there only remains the sixth ring on the loop [0100000] by whose rod you can let go [up] the seventh [1100000]. And in this way, successively you can place the others, one after the other, and we spare the reader.

By which I not only described the method but actually demonstrated to exhaustion the youthful to learn. But I am very confident that your swift intelligence will fill the gaps of my terse writing in which many therefore rely on. And if of these [instruments] two are offered, one interlinked and the other loose, who first arranges one like the other wins.

DOI: https://doi.org/10.1515/rmm-2017-0017

Appendix 2: Cardano

The transcription is based on the *Opera Omnia* and changes are tracked with the three previous editions of the *Subtilitate*: 1) Paris 1550, 2) Basel 1554 and 3) Basel 1560 (other editions are reprints of any of these). The changes are very minimal. The English translation is adapted from [10] as well as the French sixteenth century translation. As before, the states of the rings on the loop are shown with binary strings s as explained in Section 4 put between square brackets: [s].

Liber XV, De incerti generis aut inutilibus subtilitatibus

Verùm nullius vsus est instrumentum ex septem annulis. Bractea ferrea digitum lata, palmi longitudine, tenuis, in qua septem foramina, rotunda, angusta, æquísque spatiis, secundum longitudinem disposita, septem excipiunt virgulas tenues, altitudine vnciæ fermè, mobiles in imo, & in suprema parte circumflexas, vt annulos digiti magnitudine inclusos retineant, ipse verò virgæ à sequenti annulo infra flexuram continentur. Ob idque omnes annuli præter primum, ab antecedente, ne exiliant liberè extra anteriorem virgam prohibentur: ferrea omnia & ferrea etiam nauicula, cuius speciem ad vnguem in margine reddidimus, longa latáque pro magnitudine subiectæ laminæ. Hoc instrumento ludus excogitatus miræ subtilitatis. Primus, secundúsque annulus per inane A, spatium immittitur, inde nauicula per eosdem annulos, pòst illorum primus per inane A, demittitur, post quem tertius annulus per nauiculæ vacuam mediam partem, vt primi duo sursum trahitur, illique nauicula intruditur: tum etiam primò sursum deducto, iam tres circumambiunt nauiculam ipsam: demittes igitur duos primos exempta prius nauicula, ita illa soli tertio inclusa manebit, inde quartum superinducere licebit, vt omnis hæc industria tribus præceptis contineatur. Primùm, quòd annulus sursum trahendus, dimittendúsve, vnum tantùm habeat ante se, cui nauicula includatur. Secundùm, vt dùm demittis, vnà semper primos duos demittat[7], & vnum trahat[8], vel vnum demittendo duos primos trahat[9]. Tertiùm, vt quocunque sursum tracto, vel demisso, omnes qui antè sunt sursum trahere necesse est, ac rursus demittere. Primi itaque duo à nullo alio impediuntur, ne intercurrant: primum voco eum annulum, qui liber est. In sexagintaquatuor vicibus (si sine errore agatur) nauicula in omnibus includitur annulis, virgásque omnes inclusas continet in triginta vna, aliis vt nonaginta quinque[10] sint ab absolutione ad primi transitum seu vltimi, redeat verò totidem. Igitur circulus perficietur totus in centum nonaginta vicibus. Inutile est hoc per se, sed tamen ad seras artificiosas arcarum transferri potest.

Book XV, On uncertain-type or useless subtleties

A tool made from seven rings is [a] real useless [subtlety]. A slat of iron a finger wide, a palm long, thin and with seven circular holes in it, delicately placed at equal distances along its length, of which come seven thin rods nearly an inch high, movable at the bottom, and at the top coiled around so that they hold

[7] 1550, 1554, 1560: demittas

[8] 1550, 1554, 1560: trahas

[9] 1550, 1554, 1560: trahas

[10] 1550: [aliis vt nonaginta quinque] alii in vltimam virgam navicula redigitur. Totidem verò vicibus redit, ita ut LXXXXV

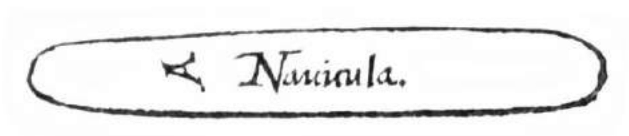

Figure 9: Cardano's illustration in the *Subtilitate* only shows the loop or shuttle (in 1550 without the word 'Nauicula')

enclosed rings a finger in size; the rods themselves are held beneath the coiled tip by the subsequent ring. Consequently, all the rings, apart from the first one, are restrained by the preceding ring from falling off by the rod in front. It is all of iron, and there is a small iron shuttle (which we presented with an accurate picture in the margin [cf. Figure 9]) that is long and wide according to the size of the underlying slat. With this tool a game of marvelous subtlety has been devised. The first and second ring are inserted through the empty space A and then the shuttle through these rings [0000011]. Afterwards, the first of these is passed through the empty space A [0000010], after which the third ring is taken up through the empty middle part of the shuttle, like the first two, and the shuttle is pushed into it [0000110]. Then when the first is pulled up, there are already three enclosing the shuttle [0000111]; you will pull out the first two that were freed from the shuttle first; so it rests enclosed by the third only [0000100]; next one may pull the fourth over [0001100], so that all this hard work is covered by three instructions. First, that the ring to be pulled up or dropped down should have only one in front of it enclosing the shuttle. Second, when you are dropping down, you should always let down the first two together after pulling up the first, or before dropping the first down you pull up the first two. Thirdly, whichever is pulled up or dropped, all those in front must be pulled up, and again be dropped. Thus the first two are not blocked by any other and neither do they interact [with each other]; the one I call the first ring is the one that is free[.][11] In sixty-four exchanges, if no error is made, the shuttle is enclosed by all the rings, and encloses all the rods [1111111], after thirty-one others the shuttle is held only by the first rod [1000000]. So that there are ninety-five from the release [of the shuttle from the rings] to the transition of the first or the last, and equally many returning. So the circle will be completed in a hundred and ninety exchanges. This [subtlety] is of no use in itself, though it can be applied to the artful locks made for chests.

[11]This full stop is missing in the printed editions. This important omission drastically changes the interpretation of the text. Without the stop, the text is translated by [10, 754] as "the first ring is the one that is free in sixty four exchanges". In our interpretation as two separate sentences, the text now makes sense within the mathematical context discussed above.

Games and Puzzles

INDEPENDENCE AND DOMINATION ON SHOGIBOARD GRAPHS

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Abstract: *Given a (symmetrically-moving) piece from a chesslike game, such as shogi, and an $n \times n$ board, we can form a graph with a vertex for each square and an edge between two vertices if the piece can move from one vertex to the other. We consider two pieces from shogi: the dragon king, which moves like a rook and king from chess, and the dragon horse, which moves like a bishop and rook from chess. We show that the independence number for the dragon kings graph equals the independence number for the queens graph. We show that the (independent) domination number of the dragon kings graph is $n-2$ for $4 \leqslant n \leqslant 6$ and $n-3$ for $n \geqslant 7$. For the dragon horses graph, we show that the independence number is $2n-3$ for $n \geqslant 5$, the domination number is at most $n-1$ for $n \geqslant 4$, and the independent domination number is at most n for $n \geqslant 5$.*

Keywords: shogi, n-queens problem, combinatorics.

Introduction

Hundreds of papers have been written on problems involving the placement of chess pieces on a board so that the placement satisfies given constraints [1, 6, 10]. One famous example is the *n-queens problem* of placing the maximum number of queens on an $n \times n$ chessboard so that no queen "attacks" any other queen (i.e., no queen can reach the position of another queen in one move). Another problem, the *queens domination problem*, calls for placing the minimum number of queens on an $n \times n$ board so that each empty square is attacked by at least one queen. A third example, the *queens independent domination problem*, calls for the placement of the minimum number of queens necessary on an $n \times n$ board so that no two queens attack each other and every empty square is attacked by at least one queen.

Among other approaches, these placement problems have been framed and studied as problems in graph theory. Suppose M is a piece in a chesslike game played on an $n \times n$ board and suppose its set of legal moves form a symmetric relation on the set of positions; i.e., if M can move from position a to position b it can also move from b to a (we exclude the case where M is a chess pawn, which only moves in forward directions). Then we define the *pieces graph* M_n to have vertex set $V(M_n) = \{(i,j)|i,j \in \{0,\dots,n-1\}\}$, with (i,j) representing the square in column i and row j, and edge set $E(M_n) = \{\{(a,b),(i,j)\}|M$ can move from (a,b) to $(i,j)\}$. For example, since the rook attacks all squares in its row and column, the rooks graph R_n has edge set $E(R_n) = \{\{(a,b),(i,j)\}|a = i$ or $b = j\}$. Also, since the bishop attacks all squares in its "falling diagonal" (i.e., squares for which the sum of row and column coordinates is a constant) and its "rising diagonal" (i.e., squares for which the row coordinate minus the column coordinate is a constant), the bishops graph B_n has edge set $E(B_n) = \{\{(a,b),(i,j)\}|a + b = i + j$ or $a - b = i - j\}$. Finally, since the queen combines the powers of rook and bishop, the edge set of the queens graph is $E(Q_n) = E(R_n) \cup E(B_n)$.

Recall from [4] that in any graph G, two vertices are *adjacent* if they share an edge, a set of vertices of G is *independent* if no two of the vertices in that set are adjacent, and a set of vertices is a *dominating set* if each vertex of G is either in the set or adjacent to an element of the set. For any graph G we define the following domination parameters:

- the *independence number* $\beta(G)$ is the maximum cardinality of an independent set of vertices of G
- the *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of vertices of G
- the *independent domination number* $i(G)$ is the minimum cardinality of an independent dominating set of vertices of G.

These parameters are related by an inequality chain: for any graph G, we have $\gamma(G) \leqslant i(G) \leqslant \beta(G)$ [4, Corollary 3.7].

The values of these parameters for chessboard graphs are known for, among other pieces, the rook and bishop: $\beta(R_n) = i(R_n) = \gamma(R_n) = n$, $\beta(B_n) = 2n-2$ for $n \geqslant 2$, and $i(B_n) = \gamma(B_n) = n$. Also, $\beta(Q_n) = n$ for $n \geqslant 4$ ($\beta(Q_n) = 1$ for $n = 1$ or 2, and 2 for $n = 3$). However, the values of $i(Q_n)$ and $\gamma(Q_n)$ are known for only finitely many n [5, Chapter 6].

In this paper we consider two pieces from shogi, a Japanese relative of chess: the dragon king, which can move one square diagonally or any number of squares vertically or horizontally, and the dragon horse, which can move any number of squares diagonally or one square vertically or horizontally. (For more information on shogi, we refer the interested reader to [2].) Let D_n be the dragon kings graph on the $n \times n$ board — so $E(D_n) = \{\{(a,b),(c,d)\}|a = c$ or $b = d$ or $\max(|a-c|,|b-d|) = 1\}$. Also, let H_n be the dragon horses graph on the $n \times n$ board – so $E(H_n) = \{\{(a,b),(c,d)\}|a + c = b + d$ or $a - c = b - d$ or $\max(|a-c|,|b-d|) = 1\}$. We note that $E(D_n) \cup E(H_n) = E(Q_n)$, so exploring domination parameters for the dragon kings graph and the dragons horses graph may provide insight into the problems of queens domination and queens independent domination.

In Section 1 we determine $\beta(D_n)$, $\gamma(D_n)$, and $i(D_n)$. In Section 2 we determine $\beta(H_n)$ and find upper bounds for $\gamma(H_n)$ and $i(H_n)$. We also discuss computer calculations for $\gamma(H_n)$ and $i(H_n)$. In Section 3 we discuss open problems and avenues for further study.

1 Dragon kings

First we show that the dragon kings independence number equals the queens independence number.

Proposition 1. *For $2 \leqslant n \leqslant 3$, $\beta(D_n) = \beta(Q_n) = n - 1$. For all other values of n, $\beta(D_n) = \beta(Q_n) = n$.*

Proof: It is easy to check that $\beta(D_2) = 1 = \beta(Q_2)$ and $\beta(D_3) = 2 = \beta(Q_3)$, so suppose $n \neq 2$ and $n \neq 3$. Since $E(R_n) \subseteq E(D_n) \subseteq E(Q_n)$, we have $\beta(R_n) \geqslant \beta(D_n) \geqslant \beta(Q_n)$. So $n = \beta(R_n) \geqslant \beta(D_n) \geqslant \beta(Q_n) = n$, and therefore $\beta(D_n) = \beta(Q_n) = n$ for $n = 1$ and $n \geqslant 4$. ■

The reader might wonder what the maximum independent sets of D_n are. For $n \geqslant 4$, we obtain a maximum independent set of dragon kings on the squares $(i, \sigma(i)), i = 0, \ldots n-1$, where σ is any permutation of $\{0, \ldots, n-1\}$ such that $\sigma(i+1) \neq \sigma(i) \pm 1$ for $i = 0, \ldots, n-2$. Such permutations exist; for example, consider $\{(i, (2i+1) \mod n) | i = 0, \ldots, n-1\}$ for $n \geqslant 4$ (as pictured in Figure 1). These permutations have been studied in many settings, and the sequence $\{a(n)\}_{n=1}^{\infty}$, where $a(n)$ is the number of such permutations of $\{0, \ldots, n-1\}$, is the sequence A002464 of the OEIS [8].

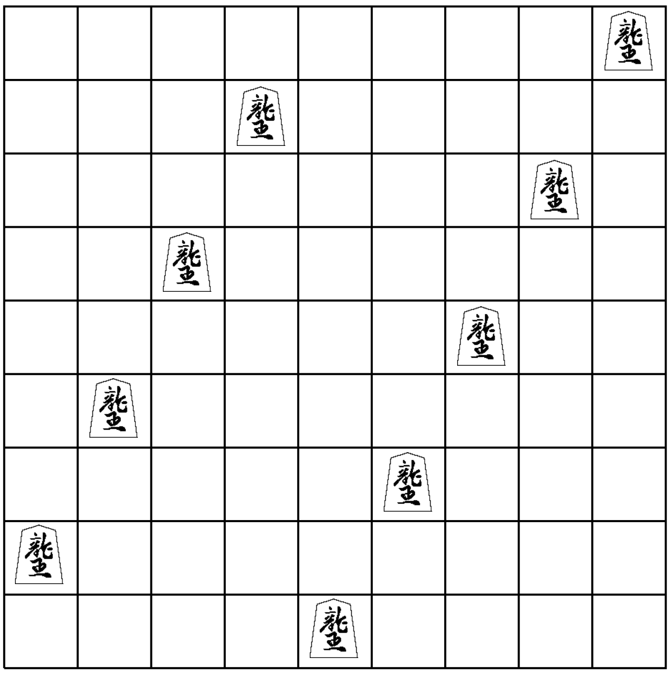

Figure 1: A 9×9 board with 9 independent dragon kings.

DOI: https://doi.org/10.1515/rmm-2017-0018

Next we consider the domination and independent domination numbers. For $1 \leqslant n \leqslant 3$, $i(D_n) = \gamma(D_n) = 1$. In the next few theorems we show that the domination number and the independent domination number of the dragon kings graph are equal for all n.

Lemma 2. *Suppose we have an $n \times n$ board dominated by k independent dragon kings. Then we can dominate an $(n+1) \times (n+1)$ board by $k+1$ independent dragon kings.*

Proof: We note that at least one of the four corner squares is empty, since otherwise the dragon kings are not all independent. Without loss of generality, say that square is $(n-1, n-1)$. (If not, we can rotate until it is.) Add row n and column n and place a dragon king on square (n, n). The new dragon king dominates the newly added squares and attacks no other dragon king. ■

Proposition 3. *For $3 \leqslant n \leqslant 6$, $\gamma(D_n) = i(D_n) = n - 2$.*

Proof: First we show $\gamma(D_n) \leqslant i(D_n) \leqslant n-2$. For a 3×3 board, place a dragon king in the center square and note that all other squares are attacked. We can now prove our claim by induction on n using Lemma 2.

To complete the proof it remains to show $n - 2 \leqslant \gamma(D_n)$ for $3 \leqslant n \leqslant 6$. This is obviously true for $n = 3$, so let $n \geqslant 4$ and suppose we have a dominating set A for D_n of size $n - 3$. We have at least three empty columns (that is, columns containing no elements of A) and at least three empty rows.

Let $c_1 < c_2 < c_3$ indicate the numbers of three empty columns and $r_1 < r_2 < r_3$ indicate the numbers of three empty rows. Consider squares $(c_1, r_1), (c_1, r_3)$, (c_3, r_1), and (c_3, r_3). Each of those squares must be attacked diagonally by members of A. For any i, the squares in column c_i (respectively, row r_i) can only be diagonally attacked by pieces in column $c_i - 1$ or $c_i + 1$. (respectively, row $r_i - 1$ or $r_i + 1$.) We show that no single piece covers any pair of the squares under consideration. If a dragon king attacked both (c_1, r_1) and (c_1, r_3) diagonally, that piece would be in row $r_1 + 1 = r_3 - 1$. But that row would then also be row r_2, which is empty. So the dragon king attacking (c_1, r_1) must be distinct from the dragon king attacking (c_1, r_3). Similar arguments work for any other pair of the considered squares. Hence our dominating set must have at least 4 elements. But for $4 \leqslant n \leqslant 6$, $n - 3 < 4$, a contradiction. We must have $n - 2 \leqslant \gamma(D_n)$ for $n = 4, 5, 6$. ■

An anonymous reviewer of a previous draft of this paper kindly provided the following proof:

Proposition 4. *(Anonymous) For $n \geqslant 7$, $n - 3 \leqslant \gamma(D_n) \leqslant i(D_n)$.*

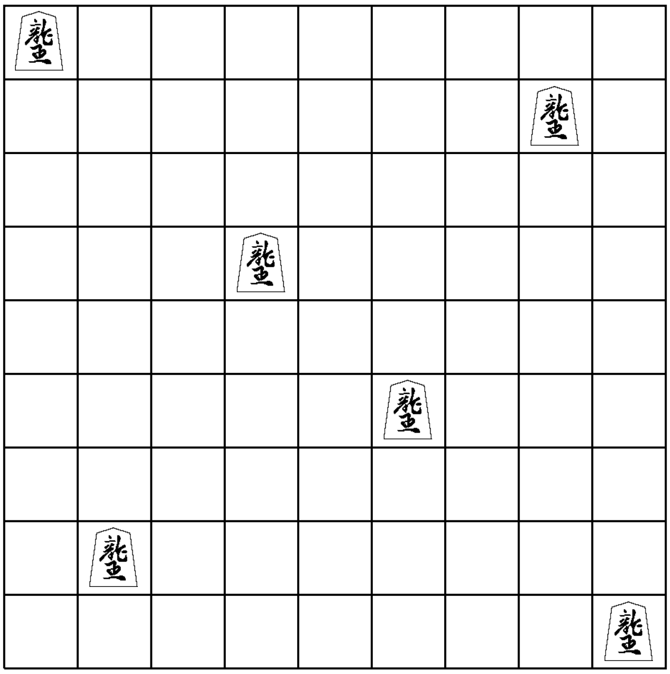

Figure 2: A 9×9 board dominated by 6 independent dragon kings.

Proof: Suppose that for some integer $n \geqslant 7$ there exists a dominating set A of size $n-4$ for D_n. Let e denote the number of empty rows and let f denote the number of empty columns. By rotating the board if necessary, we may assume that $e \leqslant f$.

For $i = 0, 1, 2$ let $A_i = \{s \in A :$ square s is in a row adjacent to exactly i empty rows$\}$ and let $a_i = |A_i|$. Then

$$a_0 + a_1 + a_2 = n - 4. \tag{1}$$

For $i = 0, 1, 2$, let d_i be the number of rows having a square in A_i. Then

$$d_0 + d_1 + d_2 = n - e. \tag{2}$$

The definitions imply $d_i \leqslant a_i$ for each i, so (1) and (2) give $e - 4$ as a sum of nonnegative integers:

$$(a_0 - d_0) + (a_1 - d_1) + (a_2 - d_2) = e - 4. \tag{3}$$

Since at most $2e$ rows are adjacent to empty rows,

$$d_1 + 2d_2 \leqslant 2e. \tag{4}$$

Let U be the set of board squares not in the same row or column as any square of A; these squares must be covered diagonally. There are ef squares in U.

DOI: https://doi.org/10.1515/rmm-2017-0018

For $i = 1, 2$, a square of A_i covers at most $2i$ squares of U, and squares of A_0 cover no squares of U. Thus $2a_1 + 4a_2 \geqslant ef$. We rewrite this as

$$2[d_1 + (a_1 - d_1)] + 4[d_2 + (a_2 - d_2)] \geqslant ef. \tag{5}$$

By (4), $4e \geqslant 2d_1 + 4d_2$, so (5) implies

$$4e + 2(a_1 - d_1) + 4(a_2 - d_2) \geqslant ef.$$

From (3), $4(e - 4) \geqslant 4(a_1 - d_1) + 4(a_2 - d_2) \geqslant 2(a_1 - d_1) + 4(a_2 - d_2)$, so $4e + 4(e - 4) \geqslant ef$. This gives

$$0 \geqslant e(f - e) + (e - 4)^2. \tag{6}$$

Since each of e, $f - e$, and $(e - 4)^2$ is nonnegative, (6) implies $e = f = 4$. Then (3) implies $a_i = d_i$ for $i = 0, 1, 2$: every nonempty row contains just one square of A. Since $e = f$, we may similarly conclude that every nonempty column contains just one square of A.

Let $c_1 < c_2 < c_3 < c_4$ be the numbers of the empty columns, and $r_1 < r_2 < r_3 < r_4$ be the numbers of the empty rows. The four squares of U in row r_1 must be diagonally covered by squares of A in rows $r_1 + 1$ and $r_1 - 1$. Then these rows must contain exactly one square each of A, and these must be in columns $c_1 + 1$ and $c_3 + 1$, with $c_2 = c_1 + 2$ and $c_4 = c_3 + 2$. However, the four squares of U in row r_4 will similarly need to be covered by squares of A in columns $c_1 + 1$ and $c_3 + 1$, and these squares cannot be the same as those covering the squares of U in row r_1. This means A contains two squares in some columns, a contradiction. Thus no dominating set of size $n - 4$ exists. ■

Corollary 5. *For $n \geqslant 7$, $\gamma(D_n) = i(D_n) = n - 3$.*

Proof: By Proposition 4, it suffices to show an independent dominating set of D_n for each $n \geqslant 7$. For $n = 7$, take an 7×7 board and place dragon kings on squares $(0, 0), (2, 4), (4, 2)$ and $(6, 6)$. We can check that this set is an independent dominating set. Then we can prove the statement by induction on n using Lemma 2 (an example of the construction is shown in Figure 2). ■

2 Dragon horses

We start the section by determining the independence number of the dragon horses graph.

Proposition 6. *The independence number of the dragon horses graph, $\beta(H_n)$, is 1 for $n = 1$ and $n = 2$, n for $n = 3$ and $n = 4$, and $2n - 3$ for $n \geqslant 5$.*

Proof: On a 1×1 or a 2×2 board, a single dragon horse placed on any square leaves all other empty squares attacked, so $\beta(H_1) = \beta(H_2) = 1$.

On a 3×3 board, we can place dragon horses on $(0, 1), (2, 0)$ and $(2, 2)$ to show $\beta(H_3) \geqslant 3$. To see $\beta(H_3) \leqslant 3$, suppose 4 pieces are on the board. If they are not on the four corners, then at least two are on physically touching squares, and if the pieces are on the four corners, then we have pieces on the same diagonal.

DOI: https://doi.org/10.1515/rmm-2017-0018

To see that $\beta(H_4) \geqslant 4$, place dragon horses on squares $(0,0), (0,2), (2,1)$, and $(2,3)$. To see that $\beta(H_4) \leqslant 4$, partition the board into a 2×2 array of 2×2 blocks and note that each block can only hold one dragon horse of an independent set.

For $n \geqslant 5$, we show $\beta(H_n) \geqslant 2n-3$. If n is odd, place dragon horses on $(\frac{n-1}{2}, \frac{n-1}{2})$, $(1,0)$, $(1, n-1)$, and, for each $i = 1, \ldots, \frac{n-3}{2}$, $(0, 2i)$, $(n-1, 2i), (2i+1, 0)$, and $(2i+1, n-1)$. If n is even, place dragon horses on $(0,0)$,$(n-2,0)$,$(1,n-1)$,$(n-1,n-2)$, $(\frac{n}{2}-1, \frac{n}{2})$, and, for each $i = 1, \ldots, \frac{n}{2}-2$, $(2i, 0)$, $(2i+1, n-1)$, $(0, 2i+1)$, and $(n-1, 2i)$. Examples of these constructions are pictured in Figures 3 and 4. We can check that none of these positions are on the same diagonal and that for any pair of positions, either the row coordinates or the column coordinates differ by at least 2.

To conclude the proof, we show $\beta(H_n) \leqslant 2n-3$ for $n \geqslant 5$. Partition the board into $2n-3$ "slices", where the first slice consists of squares $(0,0), (0,1)$, and $(1,0)$, the $(2n-3)^{rd}$ slice consists of squares $(n-1, n-2), (n-2, n-1)$, and $(n-1, n-1)$, and the i^{th} slice for $i = 2, \ldots 2n-2$ consists of the squares in the i^{th} falling diagonal. Each slice can have at most one independent dragon horse, so the board can have at most $2n-3$ independent dragon horses. ■

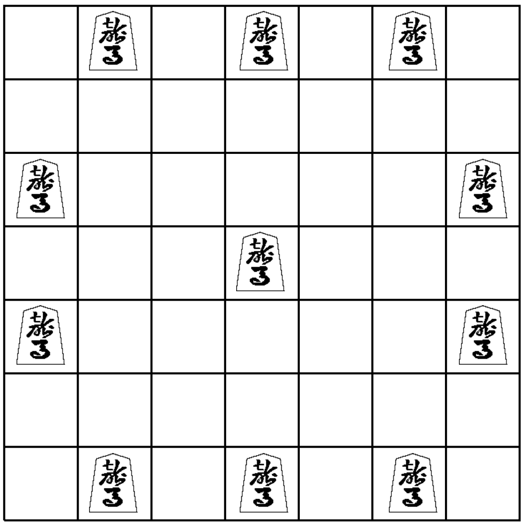

Figure 3: A 7×7 board with 11 independent dragon horses.

Next we consider the domination and independent domination numbers. We note that for $1 \leqslant n \leqslant 3$, $i(H_n) = \gamma(H_n) = 1$. We obtain an upper bound for the domination number of the dragon horses graph.

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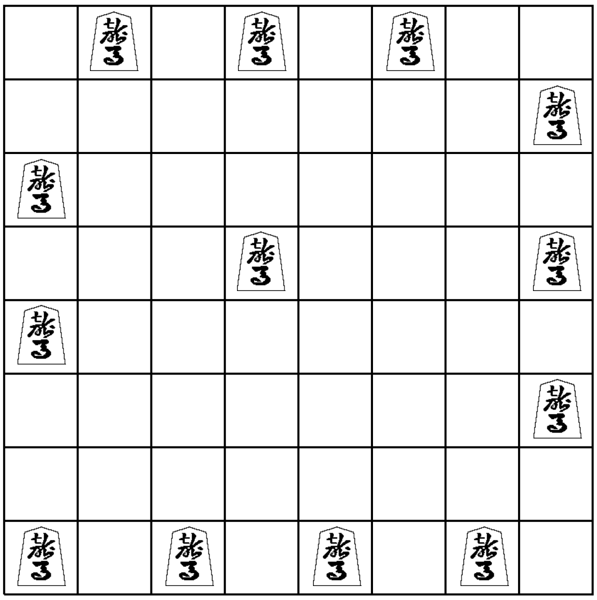

Figure 4: An 8×8 board with 13 independent dragon horses.

Proposition 7. *For $n \geqslant 4$, $\gamma(H_n) \leqslant n-1$.*

Proof: If n is odd, we place dragon horses on all but the bottom row of the central column, as shown in Figure 5, and check that the empty squares are dominated. If $n = 2k$ is even, we place dragon horses on all but the first and last rows of column k and square $(n-1, k)$, as shown in Figure 6, and check that the empty squares are dominated. ■

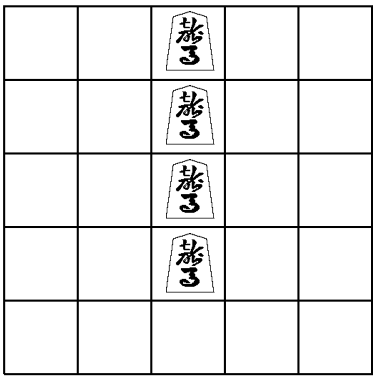

Figure 5: A 5×5 board dominated by 4 dragon horses.

We next obtain an upper bound for the independent domination number of the dragon horses graph.

DOI: https://doi.org/10.1515/rmm-2017-0018

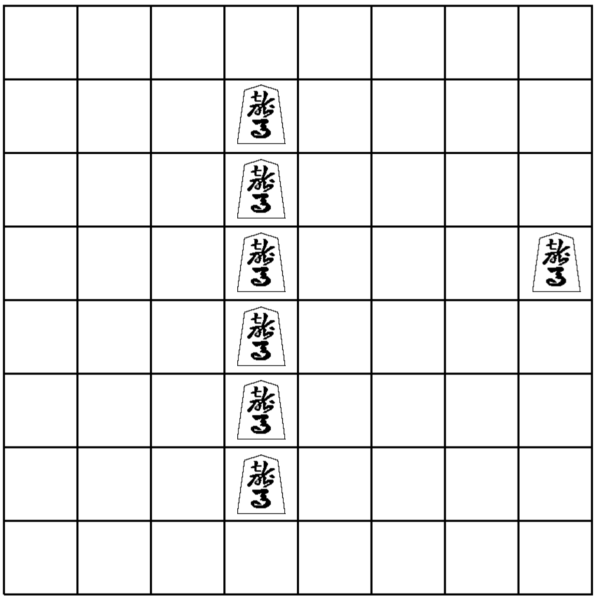

Figure 6: A 8×8 board dominated by 7 dragon horses.

Proposition 8. *For $n \geqslant 5$, $i(H_n) \leqslant n$.*

Proof: If n is odd, place dragon horses on squares $(1, 2i+1)$ and $(n-2, 2i)$ for each i such that $0 \leqslant i \leqslant \frac{n-1}{2} - 1$ and square $(n-2, n-1)$, as shown in Figure 7. If n is even, place dragon horses on squares $(1, 2i)$ and $(n-2, 2i)$ for each i such that $0 \leqslant i \leqslant \frac{n}{2} - 1$, as shown in Figure 8. In each case, we can check that each empty square is dominated and that none of the dragon horses attack each other. ■

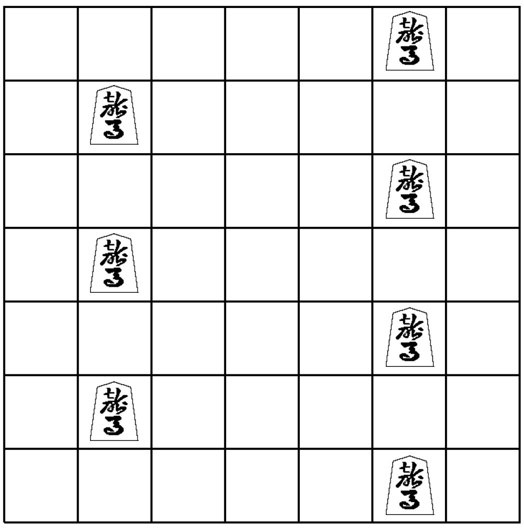

Figure 7: A 7×7 board dominated by 7 independent dragon horses.

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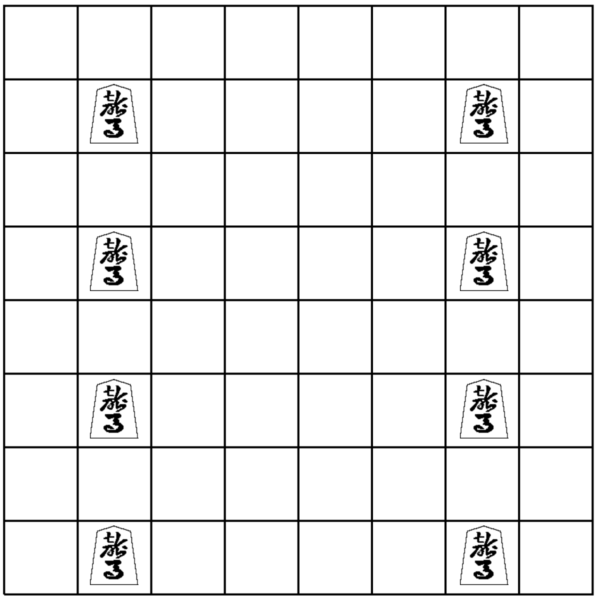

Figure 8: A 8×8 board dominated by 8 independent dragon horses.

```
include "globals.mzn";

int: n;

array[0..n-1,0..n-1] of var 0..1: horse;
array[1..2] of var string: piece;

piece=[".","h"];

% Each space is either occupied or attacked by a horse
constraint forall(i in 0..n-1, j in 0..n-1)((sum(h in 0..n-1,k in
     0..n-1 where h+k==i+j\/h-k==i-j\/(abs(h-i)<=1/\abs(j-k)<=1))(
     horse[h,k])>=1));

% No "sum diagonal" (i.e. squares (i,j) where i+j=s for some
     constant s) has two horses
constraint forall(s in 0..2*n-2)(sum(j in 0..n-1 where 0<=s-j/\s-j
     <=n-1)(horse[s-j,j])<=1);

% No "difference diagonal" (i.e. squares (i,j) where i-j=d for some
      constant d) has two horses
constraint forall(d in 1-n..n-1)(sum(j in 0..n-1 where 0<=d+j/\d+j
     <=n-1)(horse[d+j,j])<=1);

% No 2x2 block has more than one horse
constraint forall(i in 0..n-2,j in 0..n-2)(horse[i,j]+horse[i+1,j]+
     horse[i,j+1]+horse[i+1,j+1]<=1);

solve minimize sum(i in 0..n-1,j in 0..n-1)(horse[i,j]);

% optimal solution displayed as a 2-D diagram
output[show(piece[horse[i,j]+1]) ++ if j==n-1 then "\n" else ""
     endif|i in 0..n-1,j in 0..n-1];
```

Figure 9: MiniZinc model for determining $i(H_n)$.

The problem of finding the (independent) domination number of a graph can be expressed as an integer programming problem. (See [4, Section 11.1] and [5, Chapter 1].) We set up models in MiniZinc [7, 9] for the domination and independent domination numbers of H_n and applied the G12 MIP solver to those models. The model for $i(H_n)$ is shown in Figure 9. To get a model for $\gamma(H_n)$, take Figure 9 and remove lines 10-11. The results indicate that for $4 \leqslant n \leqslant 18$, $\gamma(H_n) = i(H_n) = n-1$, except that $\gamma(H_6) = 4$, not 5. (To see $\gamma(H_6) \leqslant 4$, place dragon horses on squares $(1,1), (1,4), (4,1)$ and $(4,4)$.) However, we lack a general proof that $\gamma(H_n) = i(H_n) = n-1$.

3 Open problems

The results in this paper provoke many questions, including the following.

1. It is known that there are 2^n arrangements of $2n-2$ independent bishops on an $n \times n$ board where $n \geqslant 2$ [10, Theorem 10.2]. How many arrangements are there of $2n-3$ independent dragon horses on an $n \times n$ board (where $n \geqslant 5$)?

2. Is it true that $\gamma(H_n) = i(H_n) = n-1$ for $n > 6$?

3. For each $0 < k \leqslant n$, we define a *k-step rook* on a $n \times n$ board to be a piece that can move up to k squares in a vertical or horizontal direction and a *k-step bishop* as a piece that can move up to k squares in a diagonal direction. Further, we define a *k-step dragon king* to be the combination of the rook and a k-step bishop and a *k-step dragon horse* to be the combination of the bishop and a k-step rook. What happens to the domination parameters of the k-step dragon kings graph $D_{k,n}$ and k-step dragon horses graph $H_{k,n}$ as k increases? How quickly do those parameters reach the corresponding parameters of the queens graph?

   We note that since $\beta(D_n) = \beta(Q_n)$ and $E(D_n) \subseteq E(D_{k,n}) \subseteq E(Q_n)$, we have $\beta(D_{k,n}) = \beta(Q_n)$ for all k. On the other hand, for $n \geqslant 5$, $n = \beta(Q_n) < \beta(H_n) = 2n-3 < \beta(B_n) = 2n-2$, so we cannot yet determine $\beta(H_{k,n})$ for all k and n.

4. What can be said about the domination parameters of other shogi pieces? The shogi king, bishop, and rook move like their chess counterparts (ignoring rules about castling and the king avoiding check), and their graphs have been studied [5, Chapter 6]. The other shogi pieces have nonsymmetrical moves. Suppose the initial position of a player's king is in row 0 and that the opponent's king starts in row $n-1$. Then for the pieces on the player's side:

   - The *lance* moves any number of squares forward; i.e., from (i,j) to $(i, j+k)$ for $k \geqslant 1$.
   - The *shogi pawn* moves one square forward; i.e., from (i,j) to $(i, j+1)$.
   - The *shogi knight* leaps to positions that are two squares vertically forward plus one square to the left or right; i.e., from (i,j) to $(i-1, j+2)$ or $(i+1, j+2)$.

DOI: https://doi.org/10.1515/rmm-2017-0018

- The *silver general* moves one square diagonally or one square vertically forward; i.e., from (i, j) to $(i \pm 1, j \pm 1)$ or $(i, j + 1)$
- The *gold general* moves one square vertically, one square horizontally, or one square diagonally forward; i.e., from (i, j) to $(i, j \pm 1)$ or $(i \pm 1, j)$ or $(i \pm 1, j + 1)$.

So, if we examine the appropriate directed graphs, what are the domination parameters for these pieces?

5. Since chess queens do not move through other pieces, placing pawns on a board may increase the maximum number of independent queens we can put on that board. In [3] it is noted that the maximum number of mutually nonattacking queens that can be placed on an $n \times n$ board with pawns is $\frac{n^2}{4}$ if n is even and $\frac{(n+1)^2}{4}$ if n is odd. The argument divides the board into 2×2 blocks and notes that each block can take at most one queen, regardless of how many pawns are on the board. That argument also works for dragon kings and dragon horses. So, as [3] asks for queens, how many pawns are needed to allow the maximum number of dragon kings or dragon horses on the board?

Acknowledgment: The author wishes to thank Yoshiyuki Kotani of the Tokyo University of A & T for suggesting the study of the dragon king and dragon horse.

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Articles

A generalization of Trenkler's magic cubes formula

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Abstract: *A Magic Cube of order p is a $p \times p \times p$ cubical array with non-repeated entries from the set $\{1, 2, \dots, p^3\}$ such that all rows, columns, pillars and space diagonals have the same sum. In this paper, we show that a formula introduced in* The Mathematical Gazette 84(2000)*, by M. Trenkler, for generating odd order magic cubes is a special case of a more general class of formulas. We derive sufficient conditions for the formulas in the new class to generate magic cubes, and we refer to the resulting class as regular magic cubes. We illustrate these ideas by deriving three new formulas that generate magic cubes of odd order that differ from each other and from the magic cubes generated with Trenkler's rule.*

Keywords: Magic cube, regular magic cube, magic cube formula, Trenkler's formula.

A Magic Cube of order p is a $p \times p \times p$ cubical array with non-repeated entries from the set $\{1, 2, \dots, p^3\}$, such that all rows, columns, pillars and space diagonals have the same sum. This is a natural extension of the concept of a magic square of order p, defined as a square array of order p consisting of non-repeated entries from the set $\{1, 2, \dots, p^2\}$ whose rows, columns and diagonals add up to the same sum. The sum of the elements in a magic cube of order p is $1 + 2 + \cdots + p^3 = p^3(p^3 + 1)/2$. Since these numbers are divided into p^2 rows each of which has the same sum, that sum (the magic constant) must be $p(p^3 + 1)/2$. The columns, pillars and space diagonals will also have the same sum. A magic cube is usually considered to be identical with the 47 magic cubes obtainable from it by performing rotations and/or reflections.

The following is an example of a magic cube:

$$
\begin{array}{ccc}
M_{:\,:\,1} & M_{:\,:\,2} & M_{:\,:\,3} \\
\begin{array}{ccc} 20 & 6 & 16 \\ 18 & 19 & 5 \\ 4 & 17 & 21 \end{array} &
\begin{array}{ccc} 15 & 25 & 2 \\ 1 & 14 & 27 \\ 26 & 3 & 13 \end{array} &
\begin{array}{ccc} 7 & 11 & 24 \\ 23 & 9 & 10 \\ 12 & 22 & 8, \end{array}
\end{array}
$$

where $M_{:\,:\,k} \stackrel{\text{def}}{=} \{m_{ijk} : 1 \leqslant i, j \leqslant p\}$.

For centuries, magic squares and magic cubes have been sources of mathematical amusements and challenging open problems. They are classic examples of recreational mathematics topics which typically have a large number of enthusiasts, the majority of who are not professional mathematicians.

Different types of magic cubes, some methods for constructing them, and their history, can be found in the books by Andrews [1], Ball and Coxeter [2], Benson and Jacoby [4], the writings of Martin Gardner (cf. [6, 7]), the 1888 paper [3] by Barnard and the more recent papers [8, 9, 10, 11, 12].

In the sequel, the symbols $\mathbb{N}_p$ and $\mathbb{Z}_p$ will always designate, respectively, the sets $\mathbb{N}_p = \{1, 2, \ldots, p\}$ and $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$. Given integers a and b, a mod b and (a, b) will designate, respectively, the remainder when b divides a, and the greatest common factor of a and b.

Trenkler proved in [12] that the formula

$$\begin{aligned} m_{ijk} &= 1 + [(i-j+k-1) \bmod p] + p[(i-j-k) \bmod p] \\ &\quad + p^2[(i+j+k-2) \bmod p],\ i, j, k \in \mathbb{N}_p, \end{aligned} \tag{1}$$

yields a magic cube for all odd orders $p \geqslant 3$. In this paper, we show that a larger class of odd order magic cubes can be obtained from the formula

$$\begin{aligned} m_{ijk} &= 1 + [(a_1 i + b_1 j + c_1 k + d_1) \bmod p] \\ &+ p[(a_2 i + b_2 j + c_2 k + d_2) \bmod p] \\ &+ p^2[(a_3 i + b_3 j + c_3 k + d_3) \bmod p],\ i, j, k \in \mathbb{N}_p, \end{aligned} \tag{2}$$

where $p \geqslant 3$ is an odd number and the coefficients a_r, b_r, c_r, d_r are elements of $\mathbb{Z}_p$ for $r = 1$, 2, 3. The main purpose of the paper is to derive sufficient conditions that the coefficients must satisfy to yield a magic cube. The following results will be used to derive these conditions.

Lemma 1 (cf. [5]). *Let $\theta \in \mathbb{Z}_p$. Then the Diophantine equation*

$$\theta z \equiv b \pmod p$$

has one unique solution $z \in \mathbb{Z}_p$ associated with each integer b if and only if $(p, \theta) = 1$.

Proposition 2 ([13]). *Let p be a positive odd number, let q and z be any integers and let $\alpha = (p, q)$. Then the equation*

$$\sum_{i=1}^{p} [(qi + z) \bmod p] = p(p-1)/2$$

holds if and only if $z \bmod \alpha = (\alpha - 1)/2$.

Lemma 3. *Let* $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ *and let* $\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$ *where a_i, b_i, $c_i \in \mathbb{Z}_p$ and $f_i \in \mathbb{Z}$ for $i = 1$, 2, 3. Let $\Delta = \det(A)$ and suppose that $(\Delta, p) = 1$. Then the system $A\mathbf{x} \equiv \mathbf{f} \pmod p$ has a unique solution $\mathbf{x} \in \mathbb{Z}_p^3$.*

DOI: https://doi.org/10.1515/rmm-2017-0019

Proof. Let $\hat{A}$ be the adjugate matrix of A, defined as the transpose of the matrix of cofactors of A. It is well known that $A\hat{A} = \hat{A}A = \Delta I_3$, where I_3 is the identity matrix of order 3. An application of Lemma 1 shows that, since $(\Delta, p) = 1$, there exists a unique $\Delta' \in \mathbb{Z}_p$ such that $\Delta\Delta' \equiv 1 \pmod{p}$. It follows that $\Delta'\hat{A}$ is the inverse matrix of A, modulo p, and hence that the vector $\mathbf{x} = \Delta'\hat{A}\mathbf{f} \bmod p$ is the unique solution of the given system in $\mathbb{Z}_p^3$. □

We now give sufficient conditions for the coefficients in (2) to yield magic cubes.

Theorem 4. *Let* $[a_r, b_r, c_r, d_r] \in \mathbb{Z}_p^4$, *and let* $\alpha_r = (p, a_r + b_r + c_r)$, $\beta_r = (p, a_r - b_r + c_r)$, $\delta_r = (p, -a_r + b_r + c_r)$ *and* $\theta_r = (p, a_r + b_r - c_r)$, *for* $r = 1, 2, 3$. *Suppose that*

$$(\Delta, p) = 1 \tag{3}$$

and

$$(p, a_r) = (p, b_r) = (p, c_r) = 1, \tag{4}$$
$$d_r \bmod \alpha_r = (\alpha_r - 1)/2, \tag{5}$$
$$(d_r + b_r) \bmod \beta_r = (\beta_r - 1)/2, \tag{6}$$
$$(d_r + a_r) \bmod \delta_r = (\delta_r - 1)/2, \tag{7}$$
$$(d_r + c_r) \bmod \theta_r = (\theta_r - 1)/2, \tag{8}$$

for $r = 1, 2, 3$. *Then the cubical array* $M = (m_{ijk})$ *defined in equation* (2) *is a magic cube.*

Proof. Since $(p, a_r) = 1$, the equation

$$(b_r j + c_r k + d_r) \mod (p, a_r) = ((p, a_r) - 1)/2$$

holds trivially for all $j, k \in \mathbb{N}_p$. Therefore it follows from Proposition 2 that the following equation holds for $r = 1, 2, 3$ and $j, k \in \mathbb{N}_p$:

$$\sum_{i=1}^{p} (a_r i + b_r j + c_r k + d_r) \bmod p = p(p-1)/2.$$

Similarly, since $(p, b_r) = 1$, the equation

$$(a_r j + c_r k + d_r) \mod (p, b_r) = ((p, b_r) - 1)/2$$

holds trivially, and, since $(p, c_r) = 1$, the equation

$$(a_r j + b_r k + d_r) \mod (p, c_r) = ((p, c_r) - 1)/2$$

holds trivially, for all $j, k \in \mathbb{N}_p$. Therefore it follows from Proposition 2 that the following equation holds for $r = 1, 2, 3$ and $j, k \in \mathbb{N}_p$:

$$\sum_{i=1}^{p} (a_r j + b_r i + c_r k + d_r) \bmod p = p(p-1)/2,$$
$$\sum_{i=1}^{p} (a_r j + b_r k + c_r i + d_r) \bmod p = p(p-1)/2.$$

DOI: https://doi.org/10.1515/rmm-2017-0019

It follows from Proposition 2, the definitions of α_r, β_r, δ_r , θ_r and the fact that conditions (5), (6), (7), (8) hold, that the following equations hold, respectively, for $r = 1, 2, 3$:

$$\sum_{i=1}^{p}[(a_r + b_r + c_r)i + d_r] \bmod p = p(p-1)/2,$$
$$\sum_{i=1}^{p}[(a_r - b_r + c_r)i + d_r + b_r] \bmod p = p(p-1)/2,$$
$$\sum_{i=1}^{p}[(-a_r + b_r + c_r)i + d_r + a_r] \bmod p = p(p-1)/2,$$
$$\sum_{i=1}^{p}[(a_r + b_r - c_r)i + d_r + c_r] \bmod p = p(p-1)/2.$$

By substituting these equations in (2) we see that (m_{ijk}) satisfies the $3p^2 + 4$ defining equations for a magic cube, namely:

$$\text{Rows}: \sum_{i=1}^{p} m_{ijk} = p(p^3+1)/2, \quad \forall k, j \in \mathbb{N}_p,$$

$$\text{Columns}: \sum_{i=1}^{p} m_{jik} = p(p^3+1)/2, \quad \forall k, j \in \ N_p,$$

$$\text{Pillars}: \sum_{i=1}^{p} m_{jki} = p(p^3+1)/2, \quad \forall k, j \in \mathbb{N}_p,$$

$$\text{Space Diagonals}: \sum_{i=1}^{p} m_{iii} = \sum_{i=1}^{p} m_{i\bar{\imath}i} = \sum_{i=1}^{p} m_{ii\bar{\imath}} = \sum_{i=1}^{p} m_{\bar{\imath}ii} = p(p^3+1)/2,$$

where $\bar{\imath} \equiv p + 1 - i$.

To complete the proof we have to show that $M = \{m_{ijk} \colon i, j, k \in \mathbb{N}_p\}$ and $P = \{1, 2, \ldots, p^3\}$ coincide. It follows from the definition of the cubical array (m_{ijk}) in (2) that $m_{ijk} \in \mathbb{Z}$ and $1 \leqslant m_{ijk} \leqslant 1+(p-1)+p(p-1)+p^2(p-1) = p^3$ for all i, j, $k \in \mathbb{N}_p$. Therefore the inclusion $M \subseteq P$ holds. To prove the opposite inclusion, we let $z \in P$ be arbitrary. Then it follows from the remainder theorem that $z - 1$ can be written uniquely in the form $z - 1 = r + p^2 w$ where $0 \leqslant r \leqslant p^2 - 1$ and $0 \leqslant w \leqslant p - 1$. Since r can also be written uniquely in the form $r = u + pv$, it follows that z can be written uniquely in the form $z = 1 + u + pv + p^2 w$ where u, v, $w \in \mathbb{Z}_p$.

Since (3) holds, an application of Lemma 3 shows that the Diophantine system:

$$\begin{aligned}
a_1(i-1) + b_1(j-1) + c_1(k-1) &\equiv (u - d_1 - a_1 - b_1 - c_1) \pmod p \\
a_2(i-1) + b_2(j-1) + c_2(k-1) &\equiv (v - d_2 - a_2 - b_2 - c_2) \pmod p \\
a_3(i-1) + b_3(j-1) + c_3(k-1) &\equiv (w - d_3 - a_3 - b_3 - c_3) \pmod p
\end{aligned}$$

DOI: https://doi.org/10.1515/rmm-2017-0019

has a unique solution $[i,j,k] \in \mathbb{N}_p^3$. Therefore, the equivalent linear system

$$\begin{aligned} a_1 i + b_1 j + c_1 k + d_1 &\equiv u \pmod p \\ a_2 i + b_2 j + c_2 k + d_2 &\equiv v \pmod p \\ a_3 i + b_3 j + c_3 k + d_3 &\equiv w \pmod p \end{aligned}$$

also has a unique solution $[i,j,k] \in \mathbb{N}_p^3$. It follows from equation (2) that $m_{ijk} = 1 + [(a_1 i + b_1 j + c_1 k + d_1) \bmod p] + p[(a_2 i + b_2 j + c_2 k + d_2) \bmod p] + p^2[(a_3 i + b_3 j + c_3 k + d_3) \bmod p] = 1 + u + pv + p^2 w = z \in M$. We conclude that $\{m_{ijk} : i,j,k \in \mathbb{N}_p\} = \{1,2,\ldots,p^3\}$ and therefore (m_{ijk}) is a magic cube. $\square$

As an application of this theorem, we can easily deduce the following result which was the subject of the paper [12].

Theorem 5 (Trenkler). *Equation* (1) *yields a magic cube for all odd values of* $p \geqslant 3$.

Proof. On rewriting the formula (1) with positive coefficients in the form

$$\begin{aligned} m_{ijk} \quad = \quad & 1 + [i + (p-1)j + k + p - 1) \bmod p] \\ & + p[(i + (p-1)j + (p-1)k \bmod p] \\ & + p^2[(i + j + k + p - 2) \bmod p], \end{aligned}$$

we see that it corresponds to the regular magic cube rule (2) with $[a_1, b_1, c_1, d_1] = [1, p-1, 1, p-1]$, $[a_2, b_2, c_2, d_2] = [1, p-1, p-1, 0]$ and $[a_3, b_3, c_3, d_3] = [1, 1, 1, p-2]$.

It is easy to verify, using the notation of Theorem 4, that $(\Delta, p) = (p^2 - 2p + 2, p) = (2, p) = 1$ and that $(p, a_r) = (p, b_r) = (p, c_r) = 1$ for $r = 1, 2, 3$. Therefore conditions (3) and (4) of Theorem 4 hold. The evaluations $\alpha_1 = (p, p+1) = 1$ and $\alpha_2 = (p, 2p-1) = 1$ show that condition (5) holds trivially for $r = 1$ and $r = 2$. Similarly, it follows from the evaluations $\beta_1 = (p, 1-p) = 1$, $\beta_2 = \beta_3 = (p, 1) = 1$ $\delta_1 = (p, p-1) = 1$, $\delta_2 = \delta_3 = (p, 1) = 1$, $\theta_1 = (p, p-1) = 1$, $\theta_2 = \theta_3 = (p, 1) = 1$, that conditions (6) and (8) holds trivially for $r = 1, 2, 3$. The evaluations $\delta_1 = (p, p-1) = 1$ and $\delta_2 = \delta_3 = (p, 1) = 1$ show that condition (7) holds trivially for $r = 1$ and $r = 2$. Finally, since $\delta_2 = (p, 2p-3) = (p, 3) = \alpha_3$, and

$$(p,3) = \begin{cases} 3 \text{ if } p \mid 3 \\ 1 \text{ otherwise,} \end{cases}$$

it is not hard to verify, by considering these two cases, that condition (5) holds when $r = 3$ and condition (7) holds when $r = 2$. We conclude that Trenkler's cubical array (1) satisfies the hypotheses of Theorem 4 and is therefore a magic cube. $\square$

Other magic cube formulas that obtainable from equation (2) and verifiable in

DOI: https://doi.org/10.1515/rmm-2017-0019

a similar way with Theorem 4 include the equations

$$\begin{aligned} m_{ijk} &= 1 + [(i + j + k + 1) \bmod p] + p[(-i - j + k) \bmod p] \\ &\quad + p^2[(i - j - k) \bmod p], \\ m_{ijk} &= 1 + [(-2i + j - 2k + 1) \bmod p] + p[(i - j + k - 1) \bmod p] \\ &\quad + p^2[(i - j - k) \bmod p], \\ m_{ijk} &= 1 + [(i + j + k - 2) \bmod p] + p[(i - j - k) \bmod p] \\ &\quad + p^2[(i - j + k - 1) \bmod p]. \end{aligned}$$

These equations yield magic cubes that are different from each other and from magic cubes that are obtainable with Trenkler's formula (1) for all odd values of $p \geqslant 3$. Such examples illustrate how the regular magic cube rule (2) extends Trenkler's equation (1) in a nontrivial manner.

In the sequel, given $\mathbf{a} = [a_1, a_2, a_3]$, $\mathbf{b} = [b_1, b_2, b_3]$, $\mathbf{c} = [c_1, c_2, c_3]$ and $\mathbf{d} = [d_1, d_2, d_3]$, $M_p(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ will designate the magic cube generated with equation (2). The next result shows that the magic cube $M_p(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ is uniquely defined by its coefficients $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$.

Theorem 6. *If $M_p(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = M_p(\mathbf{a}', \mathbf{b}', \mathbf{c}', \mathbf{d}')$, and both are magic cubes, then $\mathbf{a} = \mathbf{a}'$, $\mathbf{b} = \mathbf{b}'$, $\mathbf{c} = \mathbf{c}'$ and $\mathbf{d} = \mathbf{d}'$,*

Proof. For all i, j, $k = 1, 2, \ldots, p$, we have

$$\begin{gathered} [(a_1'i + b_1'j + c_1'k + d_1') \bmod p] + p[(a_2'i + b_2'j + c_2'k + d_2') \bmod p] \\ + p^2[(a_3'i + b_3'j + c_3'k + d_3') \bmod p] = [(a_1 i + b_1 j + c_1 k + d_1) \bmod p] \\ + p[(a_2 i + b_2 j + c_2 k + d_2) \bmod p] + p^2[(a_3 i + b_3 j + c_3 k + d_3) \bmod p]. \end{gathered}$$

Hence

$$\begin{aligned} [(a_1' - a_1)i + (b_1' - b_1)j + (c_1' - c_1)k + (d_1' - d_1)] &\equiv 0 \pmod p \\ [(a_2' - a_2)i + (b_2' - b_2)j + (c_2' - c_2)k + (d_2' - d_2)] &\equiv 0 \pmod p \\ [(a_3' - a_3)i + (b_3' - b_3)j + (c_3' - c_3)k + (d_3' - d_3)] &\equiv 0 \pmod p. \end{aligned}$$

On setting $i = j = k = p$, we obtain the relations $|d_1' - d_1| \equiv 0 \pmod p$, $|d_2' - d_2| \equiv 0 \pmod p$ and $|d_3' - d_3| \equiv 0 \pmod p$, which imply (since $|d_1' - d_1|$, $|d_2' - d_2|$ and and $|d_3' - d_3|$ are elements of $\mathbb{Z}_p$) that $d_1' = d_1$, $d_2' = d_2$ and $d_3' = d_3$. On taking $i = 1$, $j = p$ and $k = p$, we obtain the relations $|a_1' - a_1| \equiv 0 \pmod p$, $|a_2' - a_2| \equiv 0 \pmod p$ and $|a_3' - a_3| \equiv 0 \pmod p$, which imply that $a_1' = a_1$, $a_2' = a_2$ and $a_3' = a_3$.

On setting $i = p$, $j = 1$ and $k = p$, we obtain the relations $|b_1' - b_1| \equiv 0 \pmod p$, $|b_2' - b_2| \equiv 0 \pmod p$ and $|b_3' - b_3| \equiv 0 \pmod p$, which imply that $b_1' = b_1$, $b_2' = b_2$ and $b_3' = b_3$.

On setting $i = p$, $j = p$ and $k = 1$, we obtain the relations $|c_1' - c_1| \equiv 0 \pmod p$, $|c_2' - c_2| \equiv 0 \pmod p$ and $|c_3' - c_3| \equiv 0 \pmod p$, which imply $c_1' = c_1$, $c_2' = c_2$ and $c_3' = c_3$. ☐

Acknowledgment

The authors are grateful to the anonymous referee for suggestions that improved the presentation of the paper.

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Articles

Rules for folding polyminoes from one level to two levels

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Dedicated to Lunch Clubbers Mark Elmer, Scott Preston, Amy Hannahan, and Max Robertson

Abstract: *Polyominoes have been the focus of many recreational and research investigations. In this article, the authors investigate whether a paper cutout of a polyomino can be folded to produce a second polyomino in the same shape as the original, but now with two layers of paper. For the folding, only "corner folds" and "half edge cuts" are allowed, unless the polyomino forms a closed loop, in which case one is allowed to completely cut two squares in the polyomino apart. With this set of allowable moves, the authors present algorithms for folding different types of polyominoes and prove that certain polyominoes can successfully be folded to two layers. The authors also establish that other polyominoes cannot be folded to two layers if only these moves are allowed.*

Keywords: Folding polyominoes.

A *polyomino* is a geometric figure formed by joining one or more squares of equal side length together, edge-to-edge. The most familiar polyominoes are dominoes, formed by joining two squares together, and the tetrominoes used in the popular game Tetris. Researchers (including those engaging in mathematical recreation!) have enjoyed studying polyominoes in relation to tiling the plane [5, 6], game play in Tetris [1, 2], dissecting geometric figures [3], and a whole host of other fascinating problems.

This current investigation addresses the question of how to fold a paper cut-out of a polyomino, using prescribed allowable folds, so that the resulting shape is exactly the same as the original but now with exactly two layers ("levels") of paper over the entirety of the polyomino. If one can succeed in this endeavor with a particular polyomino, it is said that the polyomino can be *folded from one level to two levels*, or *folded* for short. This question was originally brought up in Frederickson's "Folding Polyominoes from One Level to Two" [4] from 2011. Frederickson showed several solutions and introduced readers to the terminology of the field; here the authors give explicit algorithms for how to fold certain polyominoes and establish some of the theory dictating which polyominoes can be folded from one level to two levels.

In the language introduced by Frederickson, a *HV-square* is a single square in a polyomino that is attached to at least one square along a vertical edge and to at least one square along a horizontal edge. A *well-formed polyomino* has no adjacent HV-squares. By contrast, a *non-well-formed polyomino* has two or more adjacent HV-squares and a polyomino with no HV-squares at all is called a *chain polyomino*. A chain of squares attached to a HV-square is called an *appendage*. These concepts are illustrated in Figure 1. The number of squares in a chain polyomino is referred to as its *length*.

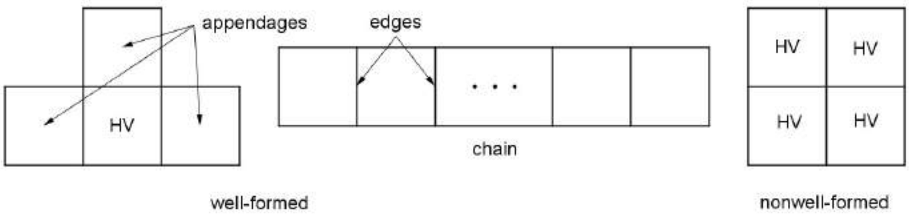

Figure 1: Well-formed and non-well-formed polyominoes.

The word *genus* refers to the number of holes in a polyomino, following the usage by topologists. All of the polyominoes shown in Figure 1 are of genus 0 and the folding of such well-formed polyominoes is addressed in Section 1. Polyominoes of larger genus are addressed in Section 2.

Frederickson and others were particularly interested in the figures that arise from "dissecting" a polyomino at the fold creases. These figures are called *dissections*. In this paper, we refer specifically to four main types of dissections: corner triangle, middle square with adjacent corner triangle, skewed parallelogram, and right triangle. Figure 2 illustrates these dissections.

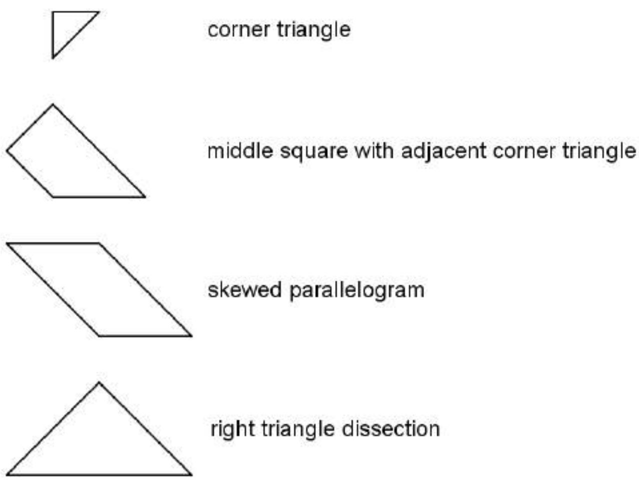

Figure 2: Four main types of dissections.

DOI: https://doi.org/10.1515/rmm-2017-0020

From the dissections shown in Figure 2, one can deduce the type of fold that we employ. A *corner fold* can be performed on any corner of a square. Mark the midpoints of two consecutive sides and fold the corner of the square inwards, towards the center of the square. Corner folds can be extended across an edge to a second adjacent square. The reader may find it easiest to follow along with paper if after every fold, the polyomino is reoriented so that the next fold is created by *folding away*.

Half edge cuts are also allowed when working with well-formed polyominoes, and in fact necessary when not dealing with chain polyominoes. In this move, one cuts halfway along one edge of a square in the polyomino. Remember that this is literally folding paper cutouts of polyominoes so performing half edge cuts in the middle of a block of HV-squares is not reasonable. Additionally, the only time we cut an entire edge is when folding a polyomino of genus greater than 0.

1 Well-Formed Polyominoes of Genus 0

The easiest polyomino to fold is, without contest, a chain polyomino of any length.

Proposition 1.1. *Let n be an integer, at least 2. A chain polyomino of length n can be folded from one level to two. Moreover, the dissections will be 4 corner triangles, 2 middle squares with adjacent corner triangle, and $n-2$ skewed parallelograms.*

Proof. We want to show that chain polyominoes can be folded using three of the shape dissections of well-formed polyominoes seen in Figure 2. Let a chain polyomino of length n be given, as in Figure 3, with the squares labeled from 1 to n.

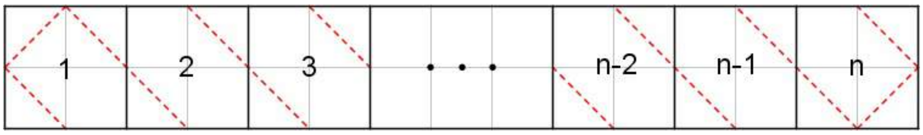

Figure 3: Chain of length n.

Then in square 1, make two corner folds. Make a fold across the edge between square 1 and square 2 that is parallel to one of the corner folds. Repeat that fold between square i and square $(i+1)$ until the folds cross into square n. Finish with the last two corner folds in square n. Therefore, by using four corner triangle dissections, two middle square with adjacent corner triangle dissections, and $n-2$ skewed parallelogram dissections, a chain polyomino of length n can be folded from one level to two levels. ∎

Note that there are two distinct ways to fold a chain polyomino because the "folder" makes a choice after the first step - there are two possible ways to make create a fold across square 1 and square 2 that is parallel to one of the corner folds in square 1.

DOI: https://doi.org/10.1515/rmm-2017-0020

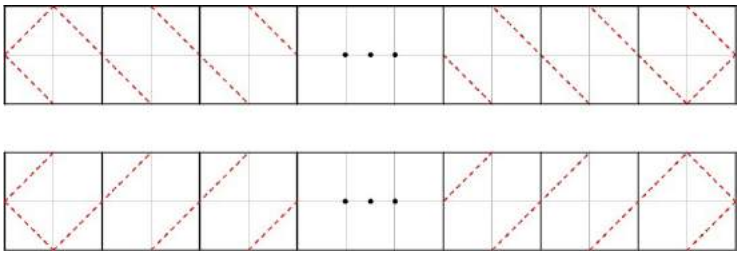

Figure 4: Alternative folds for chain.

Proposition 1.2. *Let n be an integer, at least 2. For a chain of length n, two perpendicular folds in any one of squares 2 through $(n-1)$ (i.e., a right triangle dissection) is an obstruction to folding the polyomino from one level to two.*

For example, suppose a chain of length n is folded with two perpendicular folds in one of the squares besides square 1 and n, as in Figure 5.

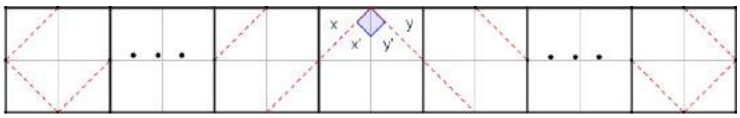

Figure 5: Two perpendicular folds in a middle square.

The two-level polyomino is not a chain because the resulting square overlaying the right triangle dissection will be an HV-square. Hence, two perpendicular folds in any one of squares 2 through $(n-1)$ is an obstruction to folding a chain polyomino from one level to two levels.

Corollary 1.3. *To introduce an HV-square with two appendages while folding a chain from one level to two, make two perpendicular corner folds across the edges of a square (i.e., make a right triangle dissection) that is not at the end of a chain.*

Because there are no other types of folds allowed, the only possible ways to fold squares in the middle of a chain are using parallel folds across the edges (giving skew parallelogram dissections) or making two perpendicular corner folds (giving a right triangle dissection). Thus Proposition 1.2 tells us that the only way a chain can be folded is the ways described by Proposition 1.1, yielding the first theorem of this section:

Theorem 1.4. *There are exactly two ways to fold a chain polyomino from one level to two levels.*

In a well-formed polyomino, there are three possible configurations for a HV-square: two appendages, three appendages, or four appendages. These configurations are referred to as an L shape, a T shape, and an X shape, respectively. We now provide algorithms to successfully fold a polyomino with one of these configurations from one level to two. Before starting, it's important to notice that by rotating a polyomino, with the HV-square as the center of rotation, there is essentially one orientation for each configuration.

DOI: https://doi.org/10.1515/rmm-2017-0020

Algorithm 1.5 (*L* **shape**). *Turn the polyomino so there is a right horizontal appendage and an upward vertical appendage. Cut halfway along the edge of the HV-square and the vertical appendage from the right. Make a corner fold in the HV-square. Make a fold across the edge between the HV-square and the horizontal appendage that is perpendicular to the corner fold in the HV-square; from there, fold as needed down the appendage. Fold a corner fold across the square in the vertical appendage directly next to the HV-square. This corner fold should be parallel to the corner fold in the HV-square. From there, fold as needed down the appendage.*

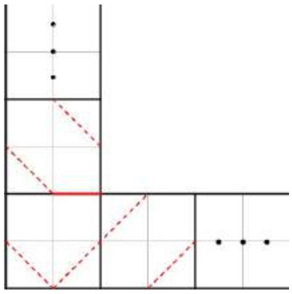

Figure 6: Cuts and folds for an L shape.

Algorithm 1.6 (*T* **shape**). *Orient as in Figure 7. Make two cuts: first cut halfway along the edge of the HV-square and the upper vertical appendage from the right, then make another cut halfway along the edge between the HV-square and the right horizontal appendage from below. Fold across the edge between the HV-square and the square below, proceed to fold down the appendage as needed. As with the L shape, make a corner fold on the horizontal appendage square that, if extended into the HV-square, would be perpendicular to the HV-square corner fold. Fold the upper vertical appendage as in L shape algorithm.*

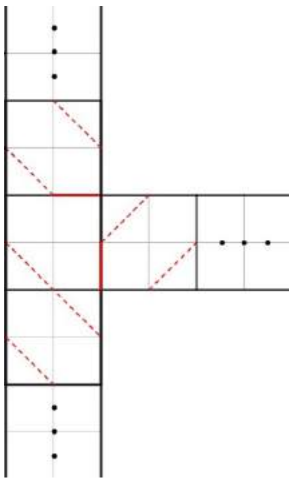

Figure 7: Cuts and folds for a T shape.

Algorithm 1.7 (X **shape**)**.** *Cut halfway along the edge between the HV-square and the upward vertical appendage from the right, turn the polyomino 90° and repeat the cut; do this twice more until there is a cut between the HV-square and all appendages. Then make a corner fold in the square directly next to the HV-square in the vertical appendage, but do not fold into the HV-square. Proceed down the appendage as needed. Rotate the polyomino 90° and repeat the same fold three more times in each appendage and fold down each appendage as needed.*

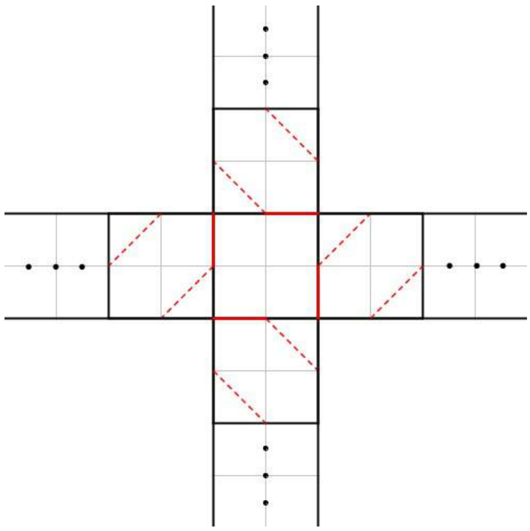

Figure 8: Cuts and folds for an X shape.

An X shape with the appendages not continuing beyond a single square has also been referred to as a *Greek Cross* (see [4, p. 266]).

These three algorithms provide one way to fold well-formed polyominoes with genus 0 and one HV-square. However each algorithm could be performed from the perspective of a "mirror image". For example, one could cut and fold the L shape in Figure 6 according to a diagram in Figure 9, which is simply the original algorithm but reflected across the 45° diagonal of the HV-square.

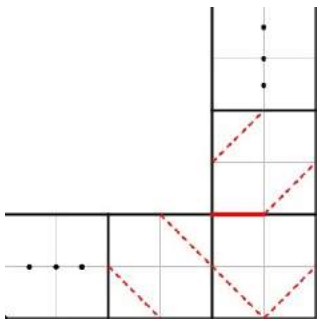

Figure 9: Alternative cuts and folds for an L shape.

DOI: https://doi.org/10.1515/rmm-2017-0020

When the polyomino is well-formed, there is at least one square between each pair of consecutive HV-squares and this provides enough space for the algorithms to be performed consecutively. The question is whether an appendage extending to the left will continue to left after folding, or if it will now extend to the right of its HV-square. The algorithms for HV-squares all preserve the orientation of the appendages relative to one another, and do not produce a "mirror image" of the original HV-square. However, the folds and cuts for the HV-square are not the only orientation-changing maneuvers.

The squares between a pair of consecutive HV-squares are essentially a chain and are folded using the method of Proposition 1.1. Each corner fold that extends across adjacent squares flips the appendage 180°. If there is an odd number of squares between consecutive HV-squares then there will be an even number of such folds, for no net change in orientation. If there is an even number of squares between consecutive HV-squares then there is an odd number of folds across adjacent squares, and the orientation at the second HV-square will be incorrect. Since the algorithms presented here for folding X shapes, L shapes, and T shapes *all* preserve orientation, there's no way to correctly fold a polyomino with an even number of squares between a pair of consecutive HV-squares using only corner folds and half edge cuts.

Other algorithms may exist to fold the three configurations for HV-squares in well-formed polyominoes, leading to a solution to the problem of an even number of squares between a pair of consecutive HV-squares. However, these algorithms do lead to the conclusion:

Theorem 1.8. *Any well-formed polyomino of genus 0 with an odd number of squares between each pair of consecutive HV-squares can be folded from one level to two levels with only corner folds and half edge cuts. A well-formed polyomino of genus 0 with an even number of squares between any pair of consecutive HV-squares cannot be folded from one level to two with only corner folds and half edge cuts.*

2 Well-Formed Polyominoes of Genus > 0

2.1 Genus At Least 1

The most basic configuration for a well-formed polyomino with genus at least 1 is a rectangular "loop" of squares with no appendages. The algorithms developed earlier, with the modification of allowing for one complete side cut, can be successfully applied to fold some of these polyominoes from one layer to two.

Proposition 2.1. *Let a closed polyomnio have no appendages and genus 1 with a rectangular hole be given. If each side has odd length then the polyomino can be folded to 2 levels.*

Proof. First, cut the edge between square $(0,0)$ and square $(0,1)$ in Figure 10. Make two corner folds on square $(0,0)$. Fold across into square (1,0) parallel to the corner fold adjacent to square $(0,1)$. Fold across square $(1,0)$ through square $(2m-1,0)$ using the chain algorithm. Notice that when you fold as a chain, at

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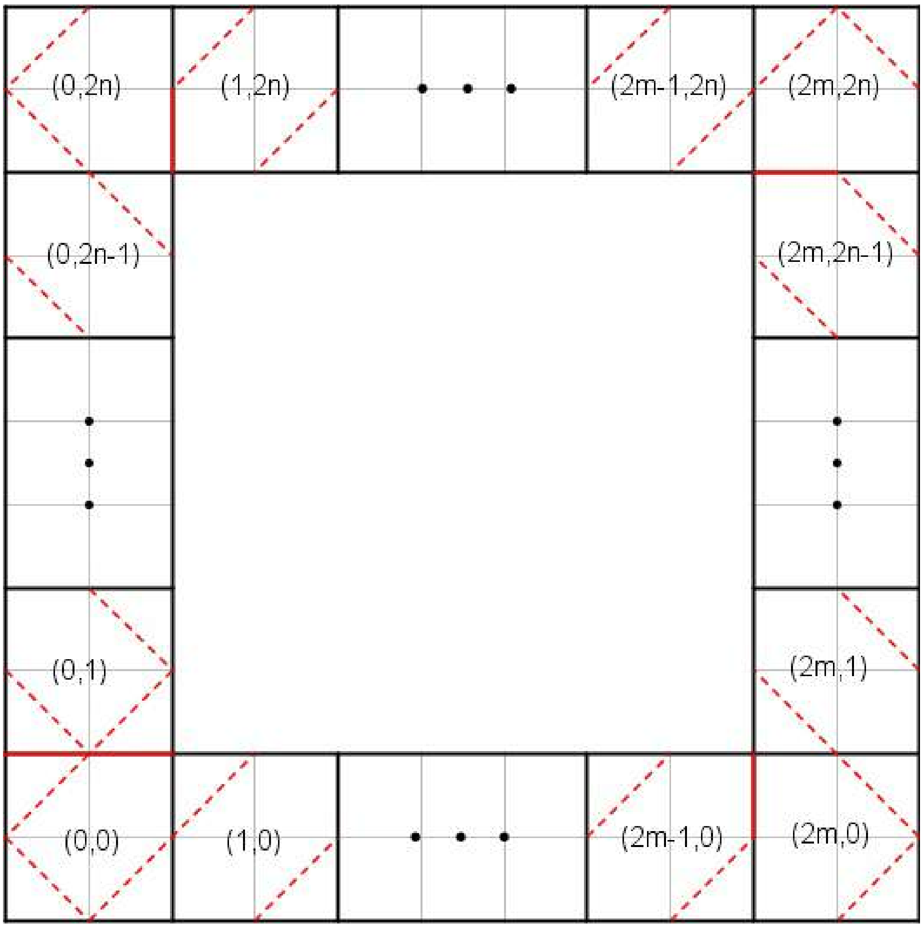

Figure 10: Rectangular loop.

the end of the appendage you are left with a half-square and a HV-square in the perfect position for the L-shape algorithm to be applied. Use the L-shape algorithm to turn the corner. Repeat this procedure until the polyomino is completely folded.

Why is it that the 2 ends of the folded polyomino meet? It has to do with the folding on the squares $(2m, y)$ where $0 \leq y \leq 2n$ and squares $(x, 2n)$ where $1 \leq x \leq 2m$. The squares $(2m, x)$ where $1 \leq x \leq 2m$ form a chain of length $2n + 1$ with both square $(2m, 2n)$ and square $(2m, 0)$ being HV-squares. There will be $2n - 1$ complete folds across adjacent squares in the chain; in square $(2m, 2n - 1)$ we have a corner fold that also flips the appendage 180°. So there are a total of $2n$ folds, each flipping the appendage 180° for no net change in orientation. Then the two appendages on the HV-squares will be parallel but in the same direction. The same will happen for squares $(x, 2n)$ since this is also a chain of odd length. Then the two end cuts will come back together to create a polyomino of genus 1 and 2 levels. □

Corollary 2.2. *Since the sides of a $k \times j$ polyomino of genus 1 will have $k-1$ folds and $j-1$ folds on respective sides, the only way orientation can be preserved is if k and j are both odd.*

The folding algorithm does not require that the cut be made where specified; that designation was made only for consistency. However, the folding algorithm truly *does* require the polyomino have an odd number of squares between each consecutive pair of HV-squares. Consider a well-formed polyomino of genus 1 where the sides are 4×3, as in Figure 11.

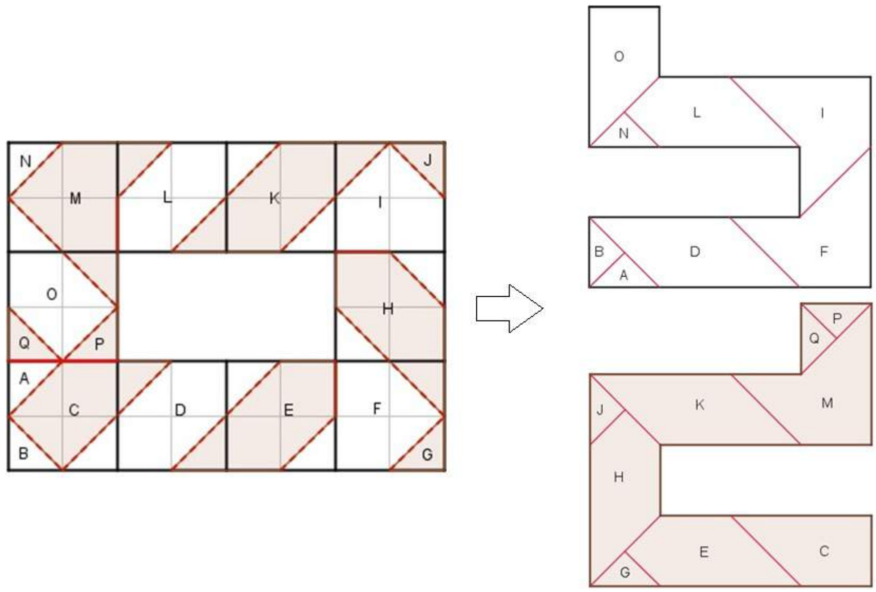

Figure 11: Folding a well-formed 4×3 polyomino.

If one orients the polyomino to begin by folding along a side with three squares, the folding seems to work – it's only when one "turns the corner" and starts to work on a side with four squares that trouble strikes. In Figure 11 we show how the folding will result if the algorithm starts along a side with four squares.

Having observed that the algorithms created completely fail when there is an even number of squares between pairs of consecutive HV-squares, one might conclude the following: *A closed polyomino with no appendages, genus 1, and rectangular hole can be folded to 2 levels if and only if each side has odd length.*

In fact, the algorithm described in Proposition 2.1 is even broader; no part of the proof required that the well-formed polyomino be of genus 1, other than simply dictating a single cut. In a closed polyomino of genus $g > 0$, one needs to make g cuts, one to connect each "hole" to the "outside" and then the algorithms for folding do the rest of the work – of course, this only applies if the number of squares between pairs of consecutive HV-squares is always odd.

Theorem 2.3. *A well-formed closed polyomino with no appendages and an odd number of squares between each pair of consecutive HV-squares can be folded from one level to two.*

DOI: `https://doi.org/10.1515/rmm-2017-0020`

2.2 Another Look at Genus 1

While only allowing corner folds and half-edge cuts, the authors stumbled upon an alternative method for folding certain closed polyominoes of genus 1 with no appendages from one level to two levels. This began with the following observation:

Proposition 2.4. *A 1-level L shape polyomino can be folded into a 2-level chain polyomino.*

Proof. Orient the L shape polyomino as in the L shape algorithm and fold the bottom left corner of the HV-square using a corner fold as in Figure 12. Fold the bottom left corner in the square above the HV-square; follow the fold into the HV-square top right corner and the square adjacent to the right bottom left corner. Make two more parallel folds on each appendage, then finish the appendages by following the chain algorithm. □

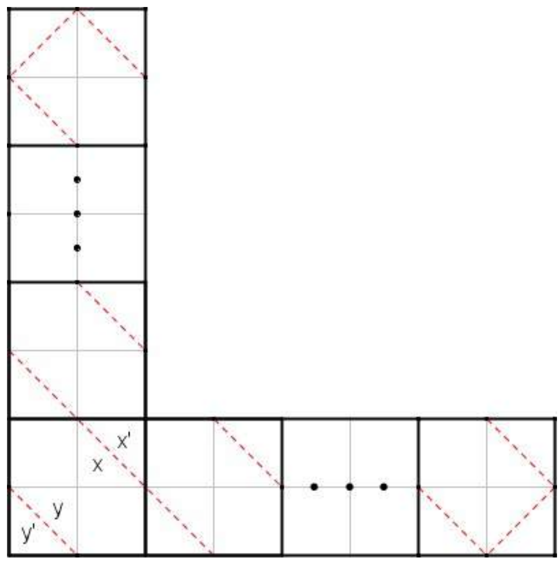

Figure 12: Folding a 1-level L shape polyomino.

In the situation of a well-formed polyomino of genus 0, the result of Proposition 2.4 is not particularly useful. However, when considering well-formed polyominoes of genus 1 with a rectangular hole, the procedure described in Proposition 2.4 suddenly opens a door for folding without the use of any cuts.

Proposition 2.5. *There is an alternative way of folding a $(2n+1) \times (2m+1)$ well-formed polyomino of genus 1 without needing any cuts.*

Proof. Begin on the middle square of all sides of the polyomino and make perpendicular folds as shown in Proposition 1.2; this results in a perpendicular fold in the center of squares $(m+1, 0)$, $(0, n+1)$, $(2m, n+1)$, and $(m+1, 2n)$. Ensure the vertex of the corner is on the outside edge of each square. Taking a closer look at the polyomino squares between $(0, n+1)$ and $(m+1, 0)$, half of

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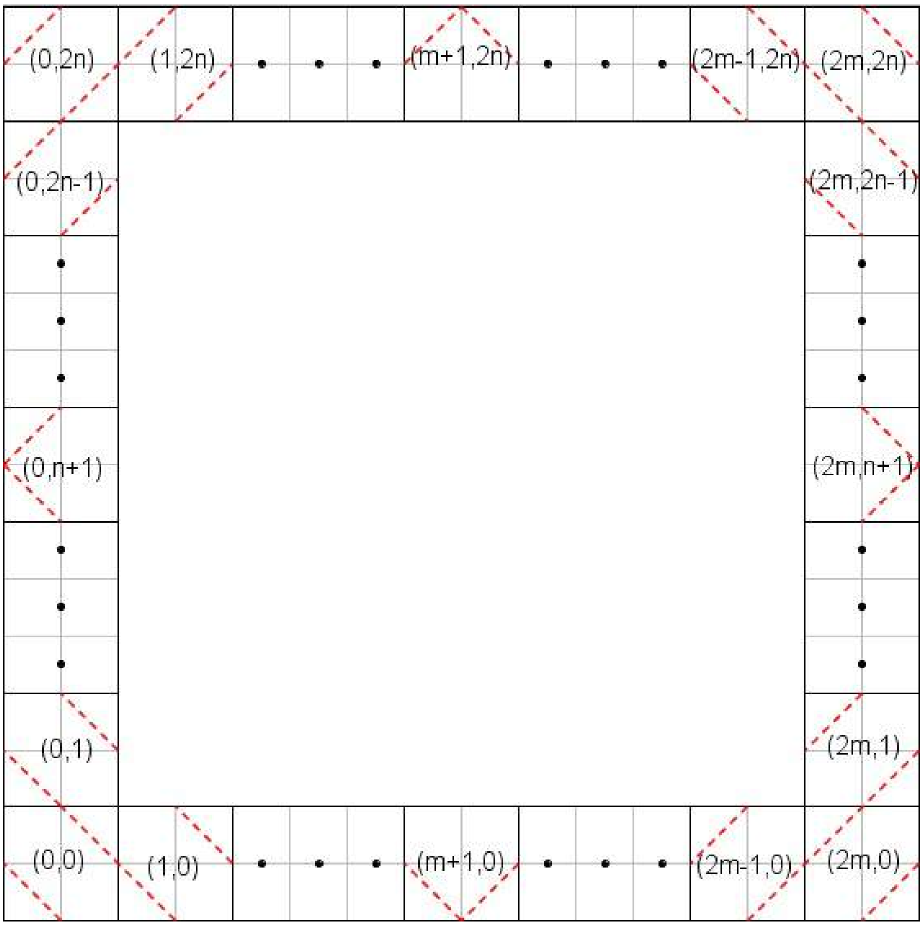

Figure 13: Folding a $(2n+1) \times (2m+1)$ well-formed polyomino of genus 1.

$(m, 0)$ and $(0, n)$ is available for a corner fold into the adjacent square closer to $(0, 0)$. Make these two folds and repeat, moving closer to $(0, 0)$. At last, half of cells $(0, 1)$ and $(1, 0)$ will be left; fold the corner as we folded the corner in the proof of Proposition 2.4. Repeat for the other 3 corners. □

Notice that this alternative folding does not work for $2n \times 2m$ well-formed polyominoes of genus 1. The key component to success in the odd-sided case is that a right triangle dissection created using the folds discussed in Proposition 1.2 must have its right-angle vertex at the midpoint on the outside edge of the center square. This ensures that the sides of the resulting two-layer polyomino are exactly the same number of squares as in the one-layer polyomino. In a $2n \times 2m$ polyomino of genus 1, there is no center square on each side and thus the right-angle vertex of the right triangle dissection will be placed off-center, creating a two-layer polyomino with odd side-lengths. One can fold such a polyomino, but it will be not be a *successful fold* in the manner stipulated.

Conclusion

This paper provides clear algorithms to fold certain well-formed polyominoes from one level to two, including some very complicated and beautiful polyominoes. Frederickson claimed that *all* well-formed polyominoes can be folded from one level to two [4, p. 270], but the systematic method for folding well-formed polyominoes of genus 0 appears to require more than corner folds and half edge cuts, and even allowing a side cut for those of genus greater than 0 is insufficient. For instance, the authors do not have a way to fold a basic "U shape" polyomino as seen in Figure 14 from one level to two.

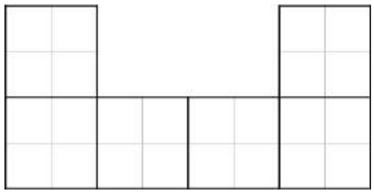

Figure 14: Basic "U shape" polyomino.

Furthermore, the matter of non-well-formed polyominoes is an open question. Frederickson [4] explored certain families of non-well-formed polyominoes, using corner cuts (some that extend across adjacent squares) in addition to corner folds and half edge cuts. It's straightforward, when n and k are positive integers, to see that a $(2n+1)\times(2n+1)$ square polyomino (of genus 0) can be folded from one level to two using only corner folds, but a $2n \times 2n$ square polyomino requires diagonal folds to be folded and a $(2n+1)\times(2k+1)$ rectangular polyomino cannot be folded using only corner folds. Beyond those observations, it is not clear which types of folds or cuts may be sufficient or necessary to employ in order to fold a non-well-formed polyomino. All in all, there is still much work left if we are to understand how to fold polyominoes from one level to two.

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