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Articles
Games and Puzzles
Problems
MathMagic
Mathematics and Arts
Math and Fun with Algorithms
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News

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Articles

CONSTRUCTION AND ENUMERATION OF CIRCUITS CAPABLE OF GUIDING A MINIATURE VEHICLE

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Abstract: *In contrast to traditional toy tracks, a patented system allows the creation of a large number of tracks with a minimal number of pieces, and whose loops always close properly. These circuits strongly resemble traditional self-avoiding polygons (whose explicit enumeration has not yet been resolved for an arbitrary number of squares) yet there are numerous differences, notably the fact that the geometric constraints are different than those of self-avoiding polygons. We present the methodology allowing the construction and enumeration of all of the possible tracks containing a given number of pieces. For small numbers of pieces, the exact enumeration will be treated. For greater numbers of pieces, only an estimation will be offered. In the latter case, a randomly construction of circuits is also given. We will give some routes for generalizations for similar problems.*

Keywords: closed paths, toy tracks, combinatorics, exact and asymptotic enumeration.

Introduction

Children's tracks have existed for a long time, and allow the transportation of small wooden trains, as well as cars or model trains. Today there are tracks formed of a very large number of different pieces, i.e. larger than ten. This large number of pieces is interesting, as it allows the production of different circuits from the same set of pieces. On the other hand, due to this large number of different pieces, there are also numerous situations where it is not possible to simply connect the two extremities so as to close the circuit. In certain cases this is simply not possible. Often, the circuits offered demonstrate some play, to a greater or lesser extent, which allows the creation of large circuits. This play, of a geometric origin, is taken into account in the pieces constituting these circuits. For example, the (mortise and tenon) connecting parts of *Brio* ® track pieces allow them to move very slightly with respect to each other. Model train tracks can be slightly deformed in order to close the circuit. The accumulation of this play indeed allows the closure of the constructed circuit, however the play often renders it difficult to close the circuit and, if it does close, it is also possible that the resulting discontinuity derails the miniature vehicles which use these circuits.

The patented system *Easyloop* aims to overcome these drawbacks by offering a circuit or a set of guiding pieces which

- allows the realization of a large number of closed circuits from the same set of pieces,
- uses a minimum of different guide pieces,
- guarantees that it is always possible to simply close the circuit.

The play between the pieces of this system will be strictly zero, in contrast with traditional systems, allowing a perfect fit of circuit loops. The manufacturing will nonetheless provide a very small play, allowing the pieces to be connected to each other by mortise and tenon joints. Typically, this system concerns the domain of train tracks for children, though it may equally concern that of circuits for small cars and the like.

The construction of the pieces of the circuits, which has led to a patent [1, 2], is not the subject of this article, though it is briefly recalled in Section 2. We note however that, from a pedagogical and didactic point of view, the construction of these circuits has been the subject of various presentations to different audiences, from the general public, to high school students, to a informal seminar for final-year undergraduates to doctoral students (see [3, 4]).

These circuits employ various notions of geometry, spanning the curricula of middle and high school up to undergraduate studies (Pythagoras' theorem, tangents, circles, parabolas, Bézier curves, radii of curvature, tessellation and enumerative combinatorics), which may be brought up in a distinguished way, and adapted to the relevant public. A collaboration is planned with Nicolas Pelay from the Plaisir Maths association and we will try together to promote the game on it's pedagogical and didactic aspect.

The initial motivation of the work presented here was a question asked by a manufacturer: "Is it possible to tally all of the circuits which can be realized from a given number of pieces?" The objective of this article is to attempt to respond to this question. We will present in Section 3 the methodology allowing the construction and enumeration of all of the possible circuits containing a given number of pieces. For greater numbers of pieces, only an estimation will be offered (Section 4). In the latter case, a randomly construction of circuits is also given (Section 5). We will give some routes for generalizations in Section 6.

All of the algorithms presented in this article have been implemented computationally and have allowed the determination of the different circuits presented, as well as their to-scale depiction. Four executables (distributed for Windows only) and a documentation in French allowing the installation of graphical libraries, the creation of circuits in a manual or random manner, and the drawing of the circuits are available on the internet at given in Appendix 8.

Two catalogues have been created in a totally automatic manner; see Appendix 8.

Principles of the patented system

Construction of the basic curves

The principle of this system is to define a path Γ in $\mathbb{R}^2$, and of class $\mathcal{C}^1$, which ensures continuity between two successive pieces of the circuit, as well as their good fit. Let N be any non-zero natural number. Two fundamental ideas are used:

- We consider a set of squares $\mathcal{C}_i$, $1 \leqslant i \leqslant N$ each belonging to a square tiling of the plane. The side of each square is defined by

$$L_0 = 1. \tag{1}$$

  We will then assume, without loss of generality, that the coordinates of the centers of the squares $\mathcal{C}_i$ are integers. Each square contains a part of the path Γ, and the intersection of a square $\mathcal{C}_i$ with Γ is denoted Γ_i.

- For each of the squares $\mathcal{C}_i$, the curve Γ_i must satisfy the following constraints:
  - it is contained within the square $\mathcal{C}_i$,
  - it begins on one vertex of the square, or in the middle of one side of the square, at a point A_i, and ends on another vertex of the square, or in the middle of another side, at a point B_i,
  - it is tangent at A_i and at B_i to the straight lines connecting respectively the center of the square to the points A_i and B_i.

Thus, the path Γ will be defined as the union of the curves $(\Gamma_i)_{1\leqslant i\leqslant N}$. For $1 \leqslant i \leqslant N-1$, each of the squares $\mathcal{C}_i$ must have a unique vertex or side in common with the neighboring square $\mathcal{C}_{i+1}$. If $i = N$, then the same rule applies for the squares $\mathcal{C}_1$ and $\mathcal{C}_N$. One may hence define the path Γ, from the centers $(c_i)_{1\leqslant i\leqslant N}$ of the squares $\mathcal{C}_i$ with integer coordinates. This problem is therefore very similar to the research into self-avoiding walks, described in [15, 17], in the planar case, and also in the particular case where the origin and the end are identical, i.e. the case of the self-avoiding polygons, described in [5, 7, 8, 10, 13, 17]. Five essential differences distinguish the game's circuits from self-avoiding polygons. On the one hand, in [5, 7, 8, 10, 13, 17], while the squares must necessarily be distinct, the *Easyloop* system allows two non-successive squares to be confounded; we will return to this point in Section 3.4. On the other hand, in [10, 17], two successive squares may only have one side in common, in contrast with the *Easyloop* system. Furthermore, some additional constraints due to the number of available pieces are to be considered in the *Easyloop* system. It will only be necessary to keep circuits which are different up to an isometry. See Section 3.3. Finally, the number of pieces used in self-avoiding polygons is necessarily even; in the case of an odd number of pieces, no polygon exists, which is not the case for the circuits.

It now remains to define the geometry of each of the curves Γ_i. Let us fix $i \in \{1, ..., N\}$. We call $\mathcal{H}_i$, the set of eight points formed by the four middles and the four vertices of the square $\mathcal{C}_i$. To have a high number of circuits, we seek all of the possible curves corresponding to all of the possible choices of pairs of distinct points A_i and B_i in $\mathcal{H}_i$, which represents, *a priori*, $C_8^2 = 28$ cases. However, the square possesses a group of isometries $\mathcal{S}$ leaving it invariant, of cardinal 8, which reduces the number of possible curves to 6. We define 6 types of curve in the following way:

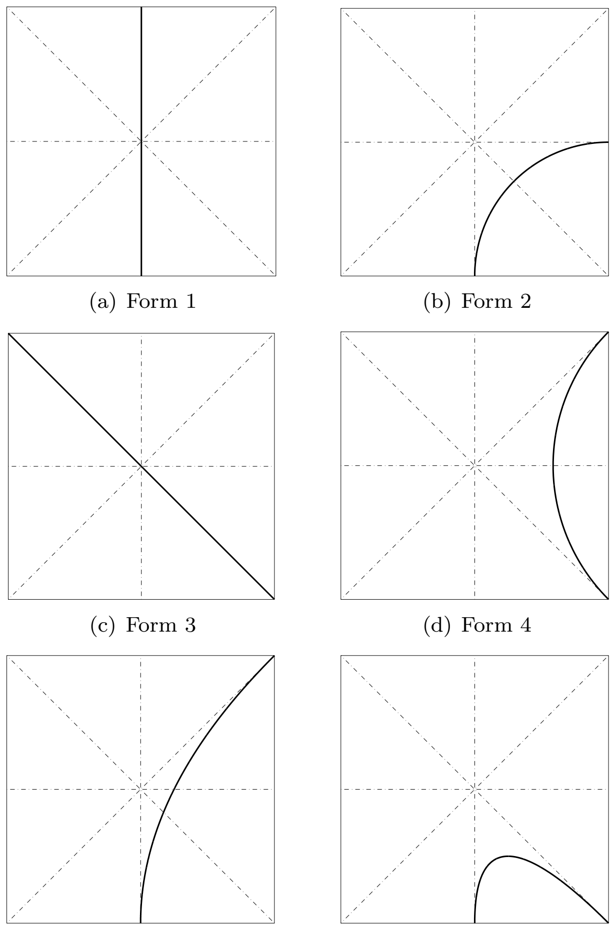
- a first type, grouping together only the curves for which the points A_i and B_i are the middles of two opposite sides of the square,
- a second type, grouping together only the curves for which the points A_i and B_i are the middles of two adjacent sides of the square,
- a third type, grouping together only the curves for which the points A_i and B_i are two diagonally opposite vertices of the square,
- a fourth type, grouping together only the curves for which the points A_i and B_i are two immediately consecutive vertices of the square,
- a fifth type, grouping together only the curves for which the points A_i and B_i are the middle of one side and a vertex of the opposite side of the square,
- a sixth type, grouping together only the curves for which the points A_i and B_i are the middle of one side and a vertex of the same side of the square.

Such constraints still do not totally define the curves, but we now seek them in the set of line segments or circular arcs, as in the world of the toy.

In Figure 1, the six types of curves are presented. The first and third types contain only line segments of respective lengths 1 and $\sqrt{2}$ (Figures 1(a) and 1(c)). The second and fourth types contain only quarter-circles, with respective radii $1/2$ and $\sqrt{2}/2$ (Figures 1(b) and 1(d)). Finally, for the last two types, no circular arcs exist. We therefore seek a solution, for example, in the form of a parabola defined by two points A_i and B_i and the two associated tangents.

On may either determine the unique parabola thus defined, or equivalently, determine the unique Bézier curve of order two, which is then defined by the following control points: the point A_i, the center of the square c_i and the point B_i (Figures 1(e) and 1(f)).

One may refer to [4, 6, 9, 14, 16].



(a) Form 1 (b) Form 2

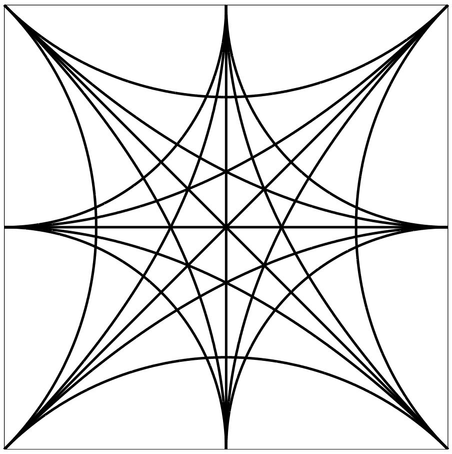
(c) Form 3 (d) Form 4

(e) Form 5 (f) Form 5'

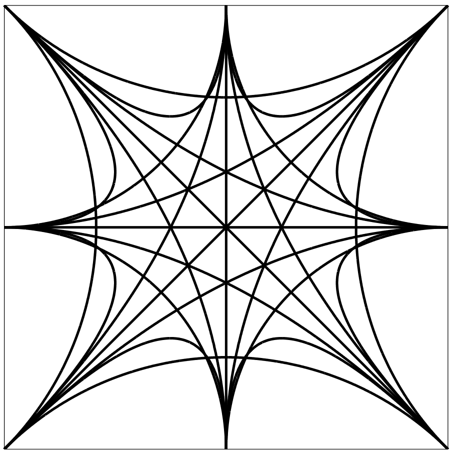
Figure 1: The six basic forms.

Acting on the 6 curves in Figure 1 with the group of 8 isometries $\mathcal{S}$, one indeed obtains the 28 possible curves of Figure 2(b). The sixth type in Figure 1(f) will be eliminated in the following, since the corresponding piece has a radius of curvature which is too small for the miniature vehicles to be able to ride there (see Section 2.2), which reduces the number of possible paths to 20 (see Figure 2(a)).

In this case, the rule "every curve linking any two distinct points in $\mathcal{H}_i$" is to be replaced by "every curve linking any two distinct non-neighboring points in $\mathcal{H}_i$", which, as we will see in the following, nevertheless offers a large number of circuits.



(a) with 5 basic pieces



(b) with 6 basic pieces

Figure 2: The set of possible paths.

The curve Γ is of class $\mathcal{C}^1$; indeed, each of the curves Γ_i is of class $\mathcal{C}^\infty$. Furthermore, the union of all of these curves will be of class $\mathcal{C}^1$. By construction, indeed, at the connecting points, which can only be vertices or middles of the sides of squares, the curves are continuous (since they pass by the same start and end points) and have a continuous derivative, since the tangents coincide.

The curves Γ obtained are of class $\mathcal{C}^1$, but not of class $\mathcal{C}^2$, due to the discontinuity of the radius of curvature, in contrast with real rail and road networks. We note however that this discontinuity is also present in the existing traditional systems, constituted of straight-line and circular pieces of different radii of curvature. On a mechanical level, this generates, for the miniature vehicle which takes the circuit, a discontinuity of the normal acceleration (at constant velocity) and of the steering angle of the wheels. These constraints, significant for real vehicles, directly affect the comfort of the passenger and the wear induced on the mechanical parts, but are not taken into account in the domain of games.

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Indeed, the masses and the velocities of the vehicles are very low, and therefore the shocks due to the discontinuity of the normal acceleration are negligible. Moreover, the notion of the comfort of the passenger has no meaning here. Finally, the wheels of the vehicles may be subjected to a discontinuity of the steering angle since they exhibit a slight play with respect to the chassis.

Construction of the pieces

Once the path Γ has been constructed, it remains to define the different types of tracks constituting the circuit. Each of the types of track will be defined from one of the five types of curves defined above.

- These curves constitute the midline of each of the types of piece.
- The wheel passages are defined as two curves at a constant distance from this middle curve, i.e: each point from one of these two curves is found on a straight line perpendicular to the tangent of the midline at constant distance from the middle curve. The edges of the tracks are defined in the same way.
- The cross section of the piece is defined in a conventional way.
- Note that in [1], we specified that the half-width $e/2$ of the rail must be less than the radius of curvature of the midline in order to avoid that, at the considered point, the curve constructed at equal distance from the midline does not feature a stationary point with a change in the direction of the unit tangent vector. For the parabola in Figure 1(e), the minimal radius is $R_{\min} = 1$, while for the parabola in Figure 1(f), the minimal radius is $R_{\min} = \frac{\sqrt{5}}{25}$. Thus, if we consider the curves in Figures1(a) to 1(e), the minimal radius of curvature is then
$$R_{\min} = \frac{1}{2},$$
whereas if we consider the curves in Figures 1(a) to 1(f), the minimal radius of curvature is then
$$R_{\min} = \frac{\sqrt{5}}{25}.$$
In the first case, the width e of the rail is therefore strictly less than e_0, given by
$$e_0 = 1, \tag{2}$$
while in the second case, e_0 is given by
$$e_0 = \frac{2\sqrt{5}}{25} \approx 0.17889. \tag{3}$$
The choice of a standard cross-section, compatible with Brio®-type vehicles, corresponds to
$$e = 0.18349. \tag{4}$$
Thus, for the curves in Figures 1(a) to 1(e), this choice of width allows us to have no stationary points. On the other hand, for the curves in Figures 1(a) to 1(f), this choice of width presents a stationary point, as exhibited by pieces 7 and 8 in Figure 4.

- Finally, the connectors are mortise and tenon joints, designed such that each track possesses one mortise and one tenon.

Piece-types 1 to 4 are symmetric: they possess a symmetry plane perpendicular to the middle curve, and since the cross section is itself symmetric, it is therefore sufficient to construct a single type of piece for each of these four types. On the other hand, the type 5 piece isn't symmetric, and the two extremities therefore do not play the same role. In order to realize the circuit, it was therefore necessary to construct two different pieces where the male and female connectors are inverted.

Figure 3: The two pieces corresponding to the type 5 midline.

See Figure 3. The extremities of the pieces may be either middles of side or vertices, and it was necessary to mark on the rails the extremities corresponding to vertices, which was done using a yellow dot, visible in Figure 3. This dot also appears on all of the track designs which will be presented in this document. The child which plays at assembling the pieces will therefore have this single rule to obey: "only put pieces together if the two extremities of two contiguous pieces have the same nature (simultaneous absence or presence of dots)". This rule is the only one to be obeyed in order to be able to make circuits which loop properly!

Finally some prototypes were manufactured, of the type of those in Figure 3. The theoretical squares shown with side of length given by (1) have all been multiplied by a reference length given by

$$L = 21.8\,\text{cm}, \tag{5}$$

which is equal to the side of the basic square constituting the real tiling. The pieces are henceforth numbered as indicated in Figure 4.

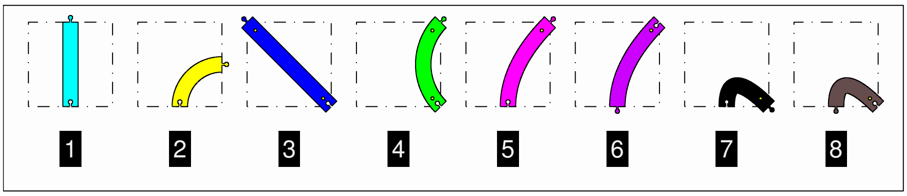

Figure 4: The numbering of the parts.

Recall that pieces 7 and 8 present in theory in the circuit, were not produced in practice, since they are too curved. The calculations presented in the following can certainly take into account these two types of piece, but for simplicity we will henceforth assume that only the first six types of piece are used. However, the programs and algorithms described can also provide for the presence of these two pieces. We give an actually manufactured circuit as an example in Figures 5 and 6.

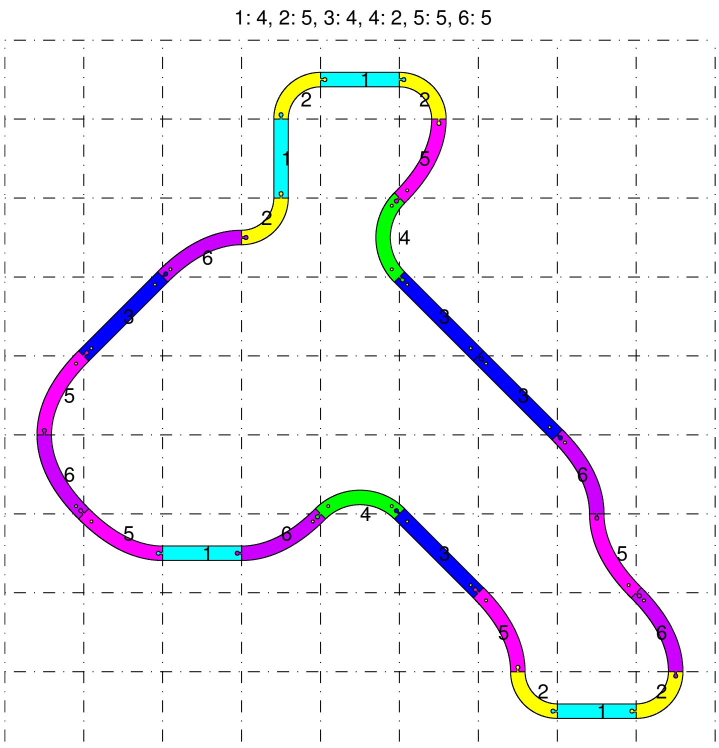

Figure 5: An example track design.

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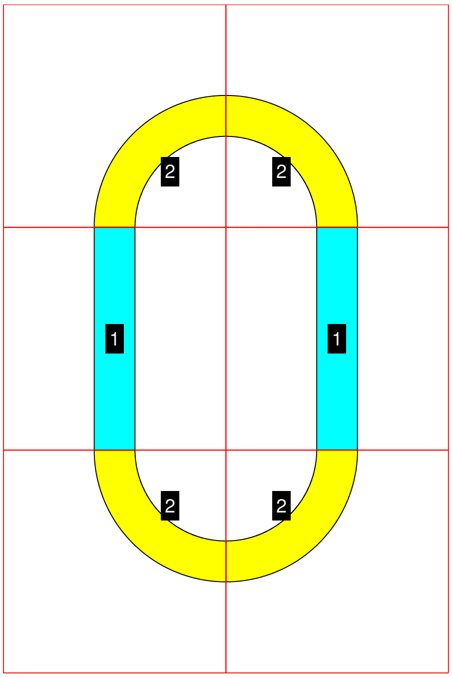
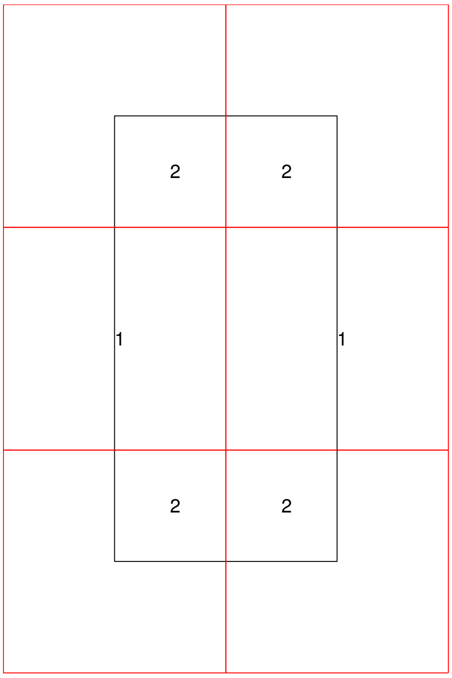
Figure 6: An example track created corresponding to the design in Figure 5.



(a) Circuit map representation

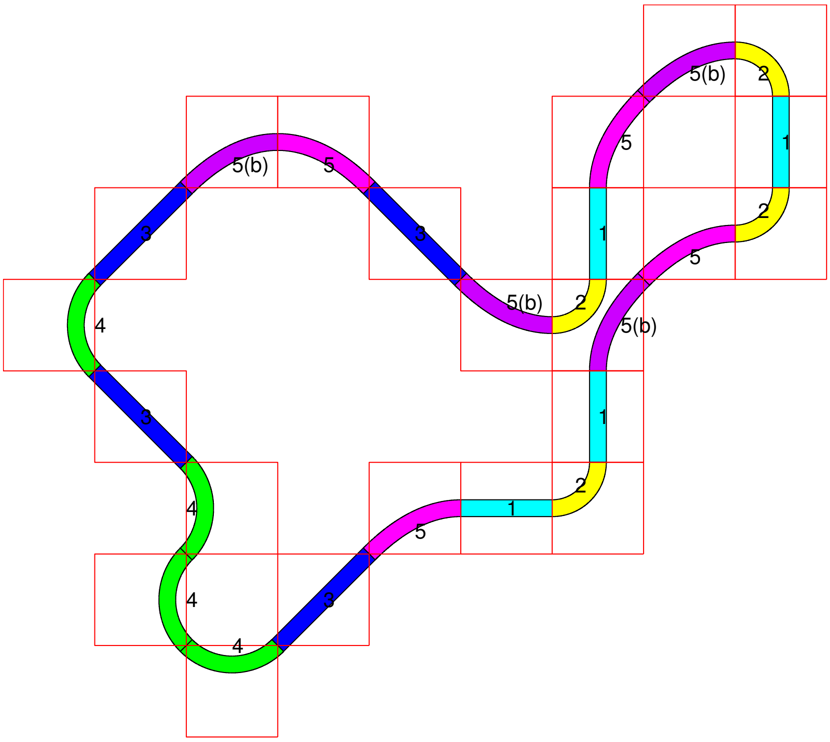


(b) Self-avoiding polygon

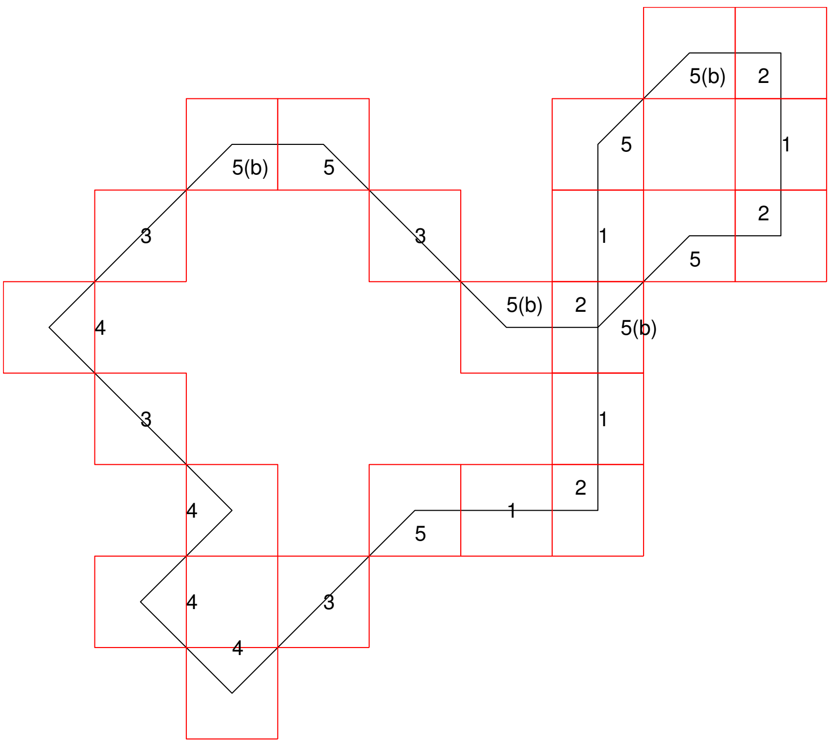
Figure 7: An example circuit.

In Figure 7, we have represented a track design, seen as a game map (Figure 7(a)) or as a self-avoiding polygon (Figure 7(b)), obtained by by joining the centers c_i. In this case, the circuit is a self-avoiding polygon in the classical sense of the term.

In Figure 8, we have represented a track design seen as a game map (Figure 8(a)) or as a self-avoiding polygon (Figure 8(b)). The latter figure highlights the essential differences which exist between our self-avoiding polygons and those in the literature; ours allow successive squares to have a vertex in common, and two different squares may be occupied by two different trajectory parts.



(a) Circuit map representation



(b) Self-avoiding polygon

Figure 8: An example circuit.

Construction and enumeration of the circuits

Posing the problem

Recall the question which is of interest to the manufacturer: "Is it possible to tally all of the circuits which can be realized from a given number of pieces?" For $1 \leqslant j \leqslant 6$, we denote by $N_j \in \mathbb{N} \cup \{+\infty\}$ the number of pieces 1 to 6 available, and by N the total number of pieces used, and we seek all of the circuits which close containing exactly N pieces in all, and such that, for each type of piece, the number of pieces used in less than N_j. We necessarily have

$$N \leqslant \sum_{j=1}^{6} N_j. \tag{6}$$

The case $N_j = +\infty$ corresponds to the case where the type of pieces concerned is not *a priori* limited. However, the number of pieces of this type is necessarily less than N.

A circuit is totally determined by the N centers $(c_i)_{1 \leqslant i \leqslant N}$ of the squares. These centers being given, it is therefore possible to determine, by taking the middles of two successive centers, the coordinates of the points A_i and B_i for each square, which correspond to the start and end of the curve Γ_i in square $\mathcal{C}_i$. Consideration of the mortises and tenons orients the circuit, and it is necessary to consider this orientation for pieces 5 and 6. Let us choose an orientation of the mortises and tenons (which amounts to orienting the circuit) in the following way: if the circuit is traversed in increasing order of the square indices, $1, 2, \ldots,$ then in each square, the first extremity of the piece corresponds to the female connector and the second corresponds to the male one.

We denote by $p_i \in \{1, ..., 6\}$, the type of piece concerned in square $\mathcal{C}_i$. We will write respectively A_i and B_i (elements of $\mathcal{H}_i$), for the start of the curve Γ_i, corresponding to the female connector, and the end of the curve Γ_i, corresponding to the male connector. The number p_i therefore depends only on the points A_i and B_i. For example, if these two points are two successive vertices, the piece is of type 4.

Another way to see this is to notice that each piece is totally determined by the relative position of the square containing the previous piece and the one containing the following piece, as well as the nature of the points A_i and B_i (that is, being a vertex or middle). To this end, for $i \in \{2, ..., N-1\}$, we consider the angle

$$\alpha_i = (\widehat{\overrightarrow{c_{i-1}c_i}, \overrightarrow{c_i c_{i+1}}}) \in [0, 2\pi[. \tag{7}$$

See Figure 9. If we let

$$\forall i \in \{2, ..., N-1\}, \quad \alpha_i = \frac{k_i \pi}{4}, \tag{8}$$

then the sole possible values of k_i describe the set $\{0, 1, 2, 6, 7\}$.

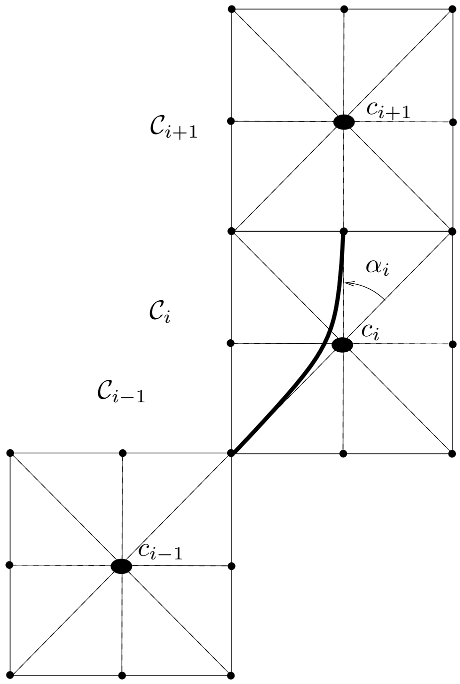


Figure 9: Each piece is defined by the two squares delimiting it and the square to which it belongs.

The relations between these elements are given in Table 1. In Figure 9, the example shows the case of a piece corresponding to $k_i = 1$, that is, of type 6.

A_i	B_i	k_i	type of piece
middle	middle	0	1
middle	middle	2 or 6	2
vertex	vertex	0	3
vertex	vertex	2 or 6	4
middle	vertex	1 or 7	5
vertex	middle	1 or 7	6

Table 1: Piece numbers p_i as a function of A_i, B_i and k_i

Remark 1. *In the case where $k_i \neq 0$, the value of k_i allows the determination of the turning direction of the piece in question (right or left). For $k_i = 1$ or $k_i = 2$, the piece turns towards the left, and for $k_i = 6$ or $k_i = 7$, it turns towards the right. One may therefore also associate a sign to the numbers of the curved pieces. Only the absolute value is important for counting of the different types of piece, but we will see later that it may be necessary to keep this sign in order to orient the direction in which this piece turns.*

Description of all circuits

Firstly, we describe the search for all circuits with N pieces for which the center of the first piece is arbitrarily equal to the origin, and center of the last is given by $(x, y) \in \mathbb{Z}^2$.

The second square, which is necessarily neighboring the origin, may be therefore chosen among 8 possible squares. For each of these choices, one may choose freely the values of k_i, $2 \leqslant i \leqslant N-1$ from $\{0, 1, 2, 6, 7\}$, which fixes the values of c_i, $1 \leqslant i \leqslant N$, as well as the values of the angles α_i, $2 \leqslant i \leqslant N-1$. We are given, moreover, the first point A_1 of the first curve (in $\mathcal{H}_1$) and the last point B_N of the last curve (in $\mathcal{H}_N$). The point B_2 is known; from this we deduce the number of piece p_1. Likewise, p_N is known. For all i, $2 \leqslant i \leqslant N-1$, the natures of all of the points A_i and B_i and the value of k_i are known, from which we deduce the value of p_i using Table 1. Of all of the circuits thus defined, we will keep only those corresponding to $c_N = (x, y)$.

Thus, by varying a certain number of independent parameters, we are capable of enumerating all of the circuits, in a geometric (the determination of the c_i) and constitutive (determination of the p_i) way, going from the origin to a given point.

If one now seeks all of the circuits which form a loop, one will similarly consider the center of the first square to be arbitrarily equal to the origin. The last square c_N can only be one of the 8 neighbors of the first one. By symmetry and rotation, one may simply choose $c_N \in \{(1, 0), (1, 1)\}$. For each choice of c_N, we apply that which we have seen above to determine all of the circuits going from the origin to c_N. In this case, the vertices A_1 and B_N are necessarily known and equal, since they will necessarily be the vertex or the common middle of the first and last square. Note that one could also set, in a similar manner to (7) and (8),

$$\alpha_1 = (\widehat{\overrightarrow{c_N c_1}, \overrightarrow{c_1 c_2}}) \in [0, 2\pi[, \tag{9a}$$

$$\alpha_N = (\widehat{\overrightarrow{c_{N-1} c_N}, \overrightarrow{c_N c_1}}) \in [0, 2\pi[, \tag{9b}$$

and

$$\forall i \in \{1, N\}, \quad \alpha_i = \frac{k_i \pi}{4}. \tag{10}$$

We will keep only the circuits such that k_1 and k_N belong to the set $\{0, 1, 2, 6, 7\}$. We hence deduce the values of the N integers $(p_i)_{1 \leqslant i \leqslant N}$. Finally, of all of these circuits, we will keep only those for which the total number of each piece of each type is less than N_j.

This method, based on a parameter sweep, obtained as the Cartesian product of finite sets, is very costly in time, and is quickly limited for values of N which are too large (we will return to this point in Section 3.6). In [5, 7, 8, 10, 11, 13, 17], there exist some much more subtle and parallelizable techniques for the enumeration of all self-avoiding or polygons walks. However, we have already pointed out in the introduction the essential differences between our circuits and self-avoiding walks or polygons, which may render the use of the methods from [11] ineffective here. Furthermore, the geometric trajectories of the circuits need to be determined in order to satisfy the constraint concerning the N_j. These circuits will also need to undergo eliminations to take into account the repetition of isometric circuits (see Section 3.3), as well as the satisfaction of a local constraint, which is not that of self-avoiding walks or polygons (see Section 3.4). Finally, we consider it important, in addition to counting all the circuits, to present them all, for small values of N at least.

Consideration of the isometries

Let us begin by studying a simple example. We will choose a small value of N without regard for the constraints imposed by N_j.

Example 2. *If we plot all the feasible circuits with $N = 5$ pieces and $N_j = +\infty$, we obtain the 10 circuits in Figure 10. In this figure, one can in fact see two different circuits repeating several times. In each of the two sets of circuits, one finds the same circuit up to a direct isometry. Two circuits are isometric (this isometry being direct) if and only if they both possess the same signed number of pieces (see Remark 1), up to cyclic permutations and up to direction of travel. We note that the total number of circuits examined equals 2000.*

If we keep only those which are different, up to a direct isometry, we obtain the 2 circuits in Figure 11. In this figure, one notices that the first circuit is the image of the second under an indirect isometry. The numbers of the pieces are identical, up to cyclic permutations and up to direction of travel. Moreover, to take this indirect isometry into account, it is necessary to replace the signed numbers of curved pieces with their opposites.

If we keep only those which are different up to an isometry, we obtain the unique circuit in Figure 12.

More generally, we draw all the circuits obtained for N and N_j given .

A consideration of the direct isometries will result in the comparison of all of the obtained circuits. If two among them possess the same signed numbers of pieces, up to cyclic permutations, then one of the two will be eliminated.

Secondly, a consideration of the indirect isometries will be performed. Similarly, if two circuits possess the same signed numbers of pieces, but which are opposite (for the curved pieces), up to cyclic permutations, then one of the two will be eliminated. To take all of the indirect isometries into account, it will also be necessary to eliminate the circuits by also comparing the indices with permutations of the type N, $N-1$, ..., 2, 1, which amounts to considering the traversal of the circuit in the opposite direction. In this case, one will replace the signed number of pieces ± 5 by ∓ 6 and vice-versa. The two pieces 5 and 6 are in effect identical; only the orientation changes. This elimination will be legitimate if the number of available pieces of types 5 and 6 are identical, , which will always be true in the following (see Remark 3).

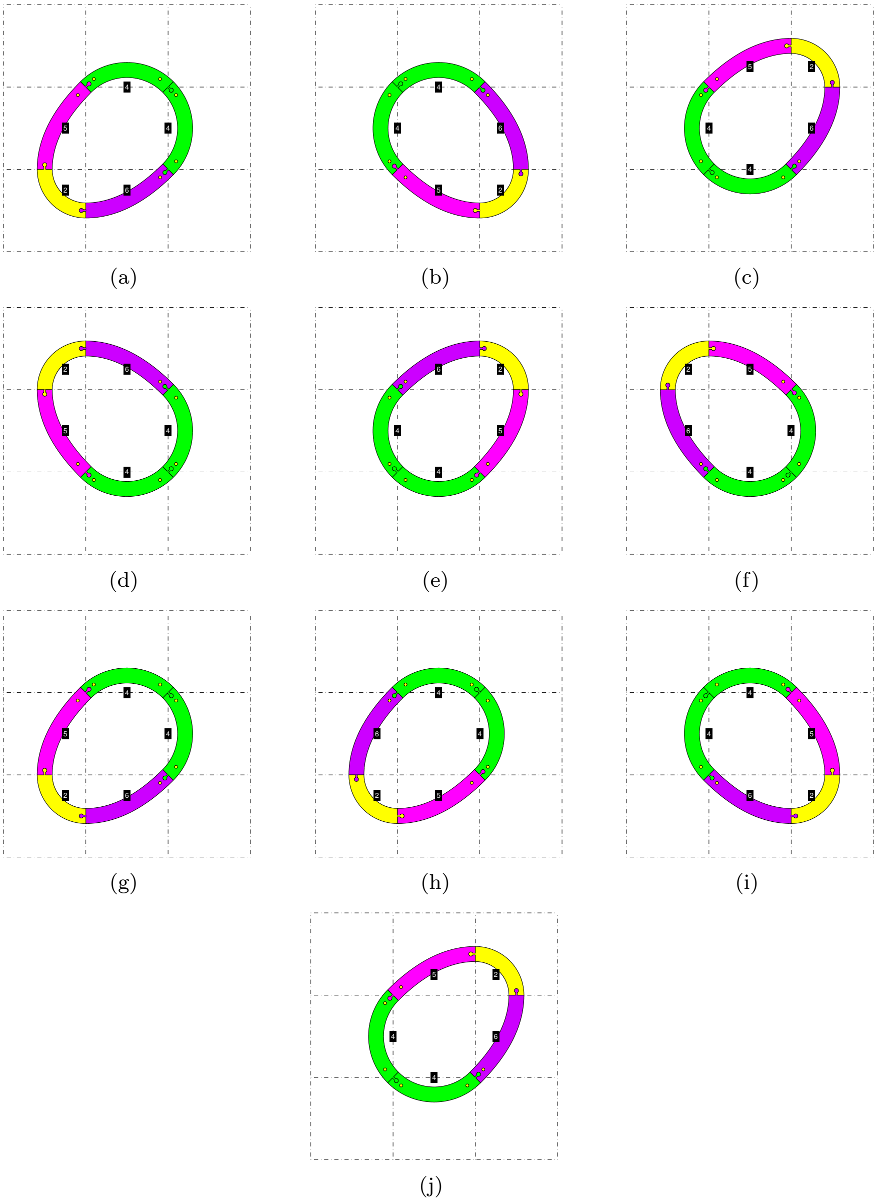
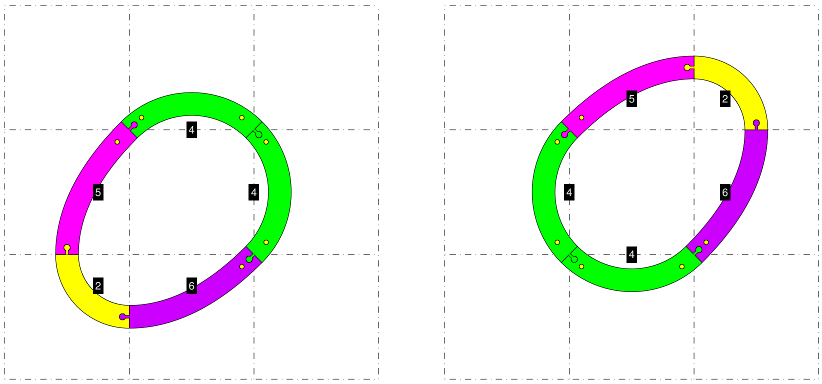


Figure 10: All of the 10 circuits kept from the set of 2000 possible circuits.



(a) (b)

Figure 11: All of the 2 circuits kept from the set of 2000 possible circuits.

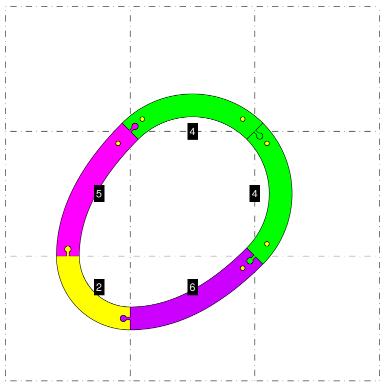
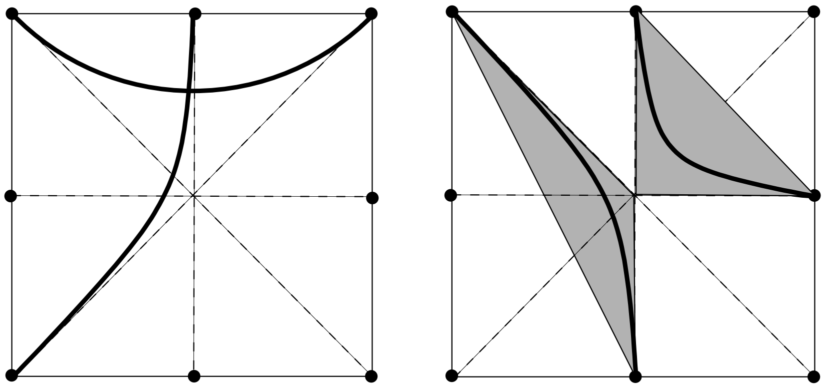


Figure 12: The sole circuit kept from the set of 2000 possible circuits.

Remark 3. *In the case of traditional self-avoiding polygons, the number of squares is necessarily even, which is no longer true here. Nonetheless, we can say that, in every circuit, the numbers of pieces of types 5 and 6 are equal. Indeed, note that, from Table 1,*

- *the pieces of type 1, 2, 3 and 4 connect together two points of the same nature (middles of sides or vertices of squares);*
- *the pieces of type 5 connect a middle to a vertex (in this order);*
- *the pieces of type 6 connect a vertex to a middle (in this order);*

A circuit departs from a point and returns to the same point. All of the points corresponding to the extremities of the pieces used in the circuit are either vertices or middles. It follows that that there are as many pieces connecting a middle to a vertex (in this order) as pieces connecting a vertex to a middle (in this order). Otherwise, the point of departure would not be of the same nature as the point of arrival.



(a) Case where the two midlines are necessarily secant: $K_a = \{2\}$, $K_b = \{0, 4, 5, 6, 7\}$.

(b) Case where the two midlines are necessarily disjoint.

Figure 13: Two pieces within a single square.

Remark 4. *Note that, in the enumeration of traditional self-avoiding polygons, only translations are taken into account in the isometries. Our enumeration problem is therefore quite different to the one in the literature. In the case where both notions coincide, this implies that the configurations that one will obtain will be a priori less numerous than those in the literature.*

Consideration of local constructibility constraints

In the research into self-avoiding walks and polygons, a very important additional local constraint is considered: the squares must be pairwise distinct. Here, this constraint isn't imposed, since only the fact of being able to produce circuits which are constructible with the real pieces counts. These constraints are not exhibited by Example 2, since the small number of pieces considered does not provide unconstructible circuits, but these constraints come up in larger circuits, exemplified below.

First of all, it is necessary that, aside from each of the pairs of contiguous pieces, which therefore have a unique extremity in common, none of the pieces has an extremity in common with other pieces which aren't contiguous. If this extremity is a vertex, this criterion is easy to write, and is not detailed here. If this extremity is a middle, and if two non-contiguous pieces have this extremity in common, they necessarily belong to the same square, which is the case that we now study.

Two pieces may belong to the same square on the condition that they are disjoint (or have tangent edges).

We therefore take two pieces in the same square $\mathcal{C}_i$, and we wish to verify that they are disjoint. Several cases present themselves:

1. They have at least one extremity in common. In this case, they are not disjoint.

2. Their extremities are pairwise distinct.

   (a) We denote by $P_1 = A_i$ and $P_2 = B_i$ (respectively P_1' and P_2') the extremities of the first (respectively second) piece. The two points P_1 and P_2 belong to the border $\partial\mathcal{C}_i$ of the square. They therefore define two connected components (for the induced topology) denoted $\mathcal{P}_a$ and $\mathcal{P}_b$. If

   $$(P_1' \in \mathcal{P}_a \text{ and } P_2' \in \mathcal{P}_b) \text{ or } (P_2' \in \mathcal{P}_a \text{ and } P_1' \in \mathcal{P}_b), \tag{11}$$

   then the two vertices of the second piece are on both sides of the midline of the first piece, and by continuity of the midlines, they have a point in common and, necessarily, in this case, the two pieces are not disjoint. Conversely, if (11) does not hold, one can show that the midlines are necessarily disjoint (see case 2b).

   Let us clarify this. We describe the location of the extremities P_1 and P_2 of the first piece by two integers, κ_1 and κ_2, in $\{0, ..., 7\}$ in the following way: if $\vec{I}$ designates the first vector of the chosen orthonormal basis, and c_i is the center of the square, then

   $$\left(\widehat{\vec{I}, \overrightarrow{c_i P_1}}\right) = \frac{\kappa_1 \pi}{4}, \quad \left(\widehat{\vec{I}, \overrightarrow{c_i P_2}}\right) = \frac{\kappa_2 \pi}{4}.$$

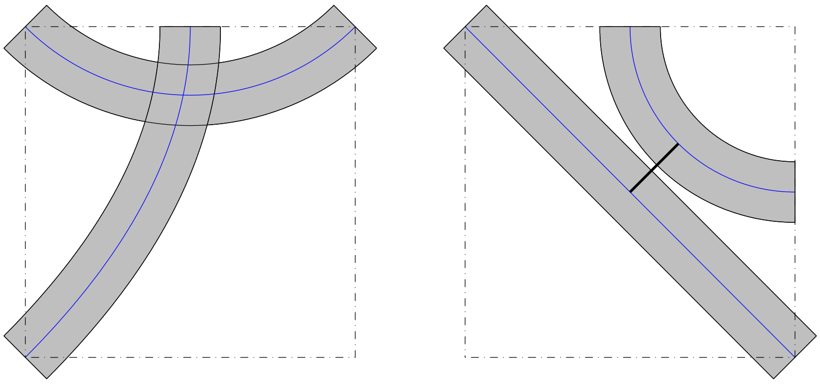
   Define κ_1' and κ_2' for the second piece. The set $\{0, ..., 7\}$ can be partitioned as follows: $\{0, ..., 7\} = K_a \cup K_b \cup \{\kappa_1\} \cup \{\kappa_2\}$, such that all of the vertices corresponding to the indices of K_a (respectively K_b) are consecutive in the square. Property (11) is equivalent to

   $$(\kappa_1' \in K_a \text{ and } \kappa_2' \in K_b) \text{ or } (\kappa_2' \in K_a \text{ and } \kappa_1' \in K_b). \tag{12}$$

   If this holds, the two pieces are not disjoint. See, for example, Figures 13(a) and 14(a), which illustrate this case.

   (b) Let us now assume that (12) does not hold. In this case, κ_1' and κ_2' both belong to either K_a or to K_b. If none of the pieces is straight, the cardinality of K_a and K_b is necessarily in $\{1, 2\}$ or in $\{4, 5\}$. Thus, κ_1' and κ_2' cannot belong to the smaller set, and we are necessarily in the case where the set of vertices contained between the two extremities of the first piece and the set of vertices contained between the two extremities of the second piece are disjoint. See for example Figure 13(b), which illustrates this case. A Bézier curve is necessarily included in the control polygon $A_i c_i B_i$. It is the same for the midlines of the circular pieces. Thus, in this case, each of the middle curves is included in two triangles which only have the center c_i of the square in common, through which none of the midlines pass (since the straight midlines are not considered). This reasoning is still valid if one of the midlines is straight. In short, in this case, the midlines of the two pieces are disjoint.

It thus follows that, if the two pieces under consideration are coincident, we are in case 1. Otherwise we are either in case 1 or 2a, in which the pieces are not



(a) Case where the two midlines are necessarily secant.

(b) Configuration where the distance between the two midlines is the smallest.

Figure 14: Two pieces within the same square.

disjoint. Finally, if we are in the last case, 2b, the midlines of the two pieces are disjoint. In this case, if we write Γ_i and Γ_i' for the two curves (which are in the same square), one may consider

$$\delta = \sqrt{\inf_{(M,M')\in\Gamma_i\times\in\Gamma_i'} d^2(M,M')}, \tag{13}$$

where $d(M,M')$ is the Euclidean distance between the points M and M'. The pair (M,M') describes a compact subset of $\mathbb{R}^2\times\mathbb{R}^2$ and d is continuous. This lower bound exists and is necessarily attained at a pair of points (M_0, M_0') of $\Gamma_i\times\Gamma_i'$. Since the two curves Γ_i and Γ_i' are disjoint, the number δ is necessarily strictly positive. In addition, one can show that for every pair of curves $\Gamma_i\times\Gamma_i'$ which fall into in this case, the real number δ is necessarily attained at a pair of points which cannot be at the edge of the square. In this case, since $(M,M')\mapsto d^2(M,M')$ is a differentiable function, its differential is zero there, which results in the perpendicularity of the straight line (M_0M_0') to the tangent to the curve Γ_i (respectively Γ_i') at the point M_0 (respectively M_0').

Computationally, all of these properties have been verified by conducting a sweep of all the possible pairs of curves $\Gamma_i\times\Gamma_i'$, which represents 1600 cases to study. We have determined the pair of curves which correspond to the smallest possible distance δ, given by (in the case of a square of unit length):

$$\delta_{\text{min}} = 0.20711, \tag{14}$$

which corresponds to the configuration in Figure 14(b). This expression equals

$$\delta_{\text{min}} = \frac{1}{2}\left(\sqrt{2}-1\right). \tag{15}$$

By construction of the edges, and by the perpendicularity properties seen above, at a constant distance equal to the half-width $e/2$ of the piece, two pieces will be disjoint if and only if

$$\delta_{\min} \geqslant e. \tag{16}$$

The case $\delta_{\min} = e$ corresponds to the case where the edges of both rails are tangent, and is still acceptable[1]. The choice of a standard cross-section, compatible with the Brio®-type miniature vehicles, corresponds to e given by (4). We therefore verify that (16) holds, which means that with the chosen cross-section, every pair of curves which are not in the case where the midlines necessarily cut across each other (case 1 or 2a), gives rise to a situation where the two pieces are disjoint, as in the case indicated by Figure 14(b)). We note that in the case where the smallest distance is attained, the smallest distance between the two edges is given by $\delta_{\min} - e \geqslant 0$, which (multiplying by the reference length given by (5)) numerically gives:

$$\xi = (\delta_{\min} - e)L = 0.51493 \text{ cm},$$

which is very small in the end, with respect to the value given by (5)!

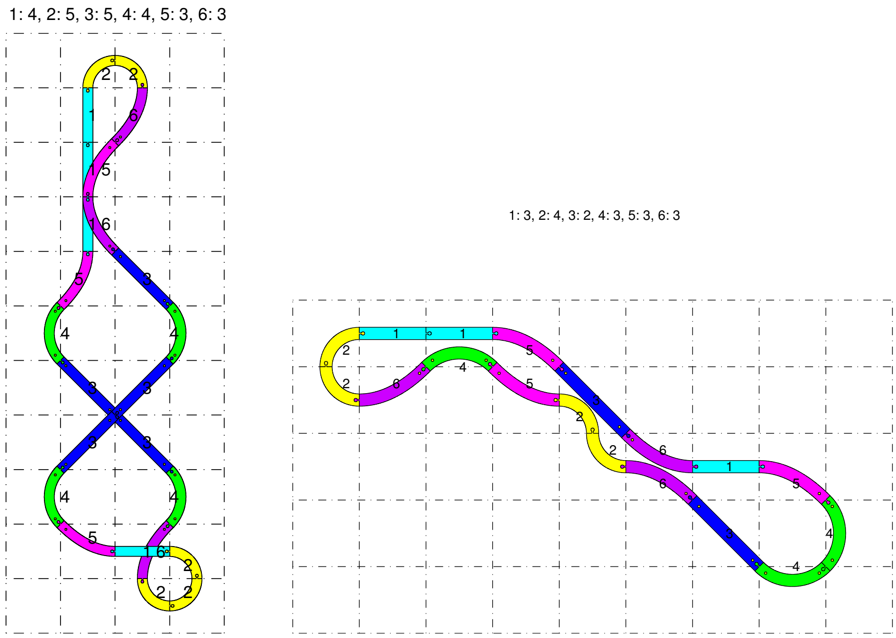
By induction, the complete study where several pieces belong to the same square has been implemented. Note that, for the values of N used in this article, there not exist square containing more of three squares.

In the patent [1, 2], the inclusion of switches, bridges, and crossings was considered; it suffices that these elements are also included in squares and satisfy the principles of construction. In the first instance, only planar and simple (without switches or crossings) pieces have been realised. For circuits including other element than these, this study of local constraints will therefore need to be reconsidered. See Section 6.2.

We conclude by two examples showing unconstructable and constructible circuits, demonstrated computationally.

Example 5. *In Figure 15(a), we chose three examples of non-disjoint pieces, covering the three cases seen above. In contrast, in Figure 15(b), the circuit is constructible with three squares in which two disjoint pieces appear each time.*

[1]If the design of the pieces is perfect, as no imperfection is permitted. In practice therefore, to be safe, one will prefer to choose $\delta_{\min} > e$.



(a) A non-constructible circuit.

(b) A constructible circuit.

Figure 15: Two circuits.

Adopted definitions

Definition 6 (perfectly looping walk)**.** *We will call a perfectly looping walk, a path* Γ *of class* $\mathcal{C}^1$ *in* $\mathbb{R}^2$*, defined in Section 2.1. This path is defined up to isometries, up to a cyclic permutations and up to direction of travel. In an abuse of terminology, we will also call a perfectly looping walk that which allows us to define the path* Γ*, i.e, any of:*

- *the sequence of the centres of the squares* $(c_i)_{1\leqslant i\leqslant N}$ *occupied by the path* Γ*, satisfying all of the constraints given in Sections 3.2, 3.3 and 3.4, this sequence being defined up to isometries, up to cyclic permutations and up to direction of travel;*
- *the sequence of angles* $(\alpha_i)_{1\leqslant i\leqslant N}$ *defined in Sections 3.1 and 3.2, this sequence being defined up to multiplication by* -1*, up to cyclic permutations and up to direction of travel;*
- *The sequence of signed piece numbers* $(p_i)_{1\leqslant i\leqslant N}$ *defined in Section 3.1, this sequence being defined up to exchange of* ± 5 *with* ∓ 6*, up to cyclic permutations and up to direction of travel.*

Note as well that a perfectly looping walk depends on N_j and on the width e which, in this article, is chosen to be less than the critical width equal to the maximum of e_0 defined by (2) and $\delta_{\min}$ defined by (14) and (15). In the slightly different case where $e > e_0$, e must remain less than $1/2$; in the latter case, the connection rules, given in Section 3.4, are to be modified, which is also taken into account in the algorithms used. If we decide to include the pieces of types 7 and 8, e_0 is then given by (3).

Definition 7 (circuit)**.** *In the remainder of this article, a circuit is therefore the set of geometric pieces which is based on the geometric construction of a perfectly looping walk.*

Computational limitations and algorithmic complexity

In [11], the number of self-avoiding walks was able to be exactly determined up to $N = 71$; the author obtained 4 190 893 020 903 935 054 619 120 005 916 paths! In [5], the number of self-avoiding polygons could be exactly determined up to $N = 130$; the author obtains 17 076 613 429 289 025 223 970 687 974 244 417 384 681 143 572 320 polygons! Unfortunately, as noted above, these parallelizable methods could not be implemented here. All of the algorithms presented have been programmed in Matlab ®. Two versions were planned: the first is vectorial (and therefore parallelizable), and avoids the use of loops, which is relatively fast. However, the tables used are quickly of a significant size, as well as the total number of circuits to be studied. Up to $N = 9$, these calculations are possible. Beyond that, the memory size is too great. It is necessary to move on to calculations with loops, which are much longer, but which avoid the storage of large tables corresponding to possible circuits. Up to $N = 11$, the calculations are reasonable. Beyond $N = 11$, the calculations were not carried out.

The exhaustive enumeration of possible circuits makes use of Cartesian products of finite sets, and is therefore of complexity $\mathcal{O}(A^N)$, which, in any case, limits computer calculations in theory.

In [5, 7, 8, 10, 13, 17], an estimation of the number of self-avoiding polygons is given for $N \to +\infty$:

$$q(N) \sim A\mu^N N^{\gamma-1}, \tag{17}$$

where μ is called the connective constant, γ a critical exponent and A a critical amplitude. The proposed value of μ is the same as the one corresponding to self-avoiding walks (see [10]):

$$\mu \approx 2.638. \tag{18a}$$

The value of γ corresponding to square lattices is given by

$$\gamma - 1 \approx -\frac{5}{2}, \tag{18b}$$

and finally, we have (see [10])

$$A \approx 0.0795774715. \tag{19}$$

The estimation (17) cannot be used here as is, since we have seen that the search for self-avoiding walks and polygons is not identical. Nevertheless, we will improperly use this approximation to evaluate the number of circuits for larger values of N in Section 4.

Some examples of the enumeration of circuits

We reconsider some examples similar to Example 2, by giving the circuits in the sences of definitions 7 and 6.

Example 8. *If we draw some of the feasible circuits with* $N = 8$ *pieces and* $N_j = +\infty$, *we obtain the* 10 *circuits in Figure 16. Note that Figures 16(a), 16(b) and 16(c) correspond to circuits which are used only the parts 1 et 2. We shall return to these particular circuits.*

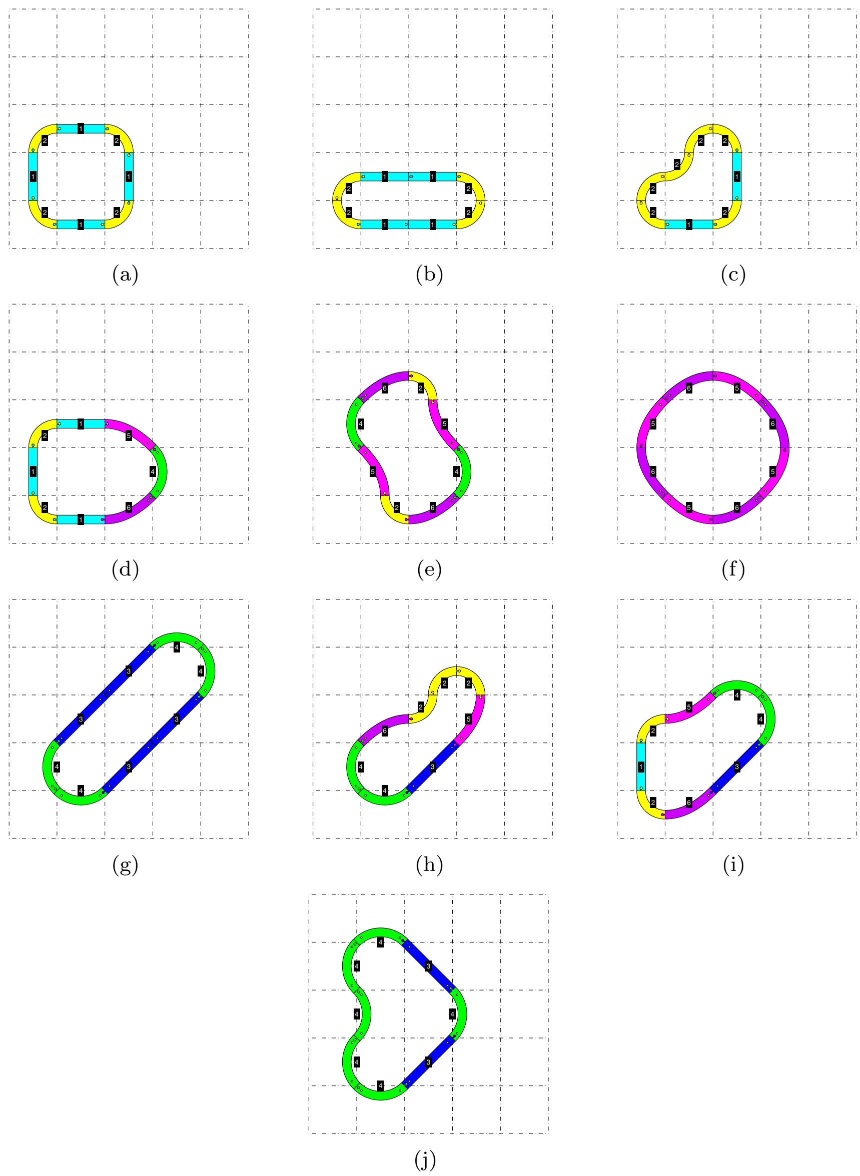


Figure 16: 10 of the 33 circuits retained from the set of 250000 possible circuits.

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Example 9. *If we draw some of the feasible circuits with* $N = 11$ *pieces and* $N_j = 4$*, we obtain the* 10 *circuits in Figure 17.*

Comparison with traditional systems and the classical theory of self-avoiding polygons (square tiling)

Traditional systems, such as *Brio* ®, offer a multitude of shapes of track pieces, which are all circular or straight, but never parabolic. Naturally, one cannot compare the studied circuits with these types of system, which do not offer designs which can be varied, scaled and modulated at will. However, in some such systems there exist in particular eighths of a circle, which assembled in pairs gives a quarter-circle, whose radius is equal to one of the lengths of the straight track pieces. In other words, the pieces of the studied systems numbered 1 and 2, used alone[2], allow the creation of simple circuits which could be created with the traditional systems. These circuits have straight pieces which can only be perpendicular to each other. The forms are less varied and above all, the number of possible circuits offered is much smaller than those of our system (see Sections 3.9 and 4).

We now choose to show a circuit formed solely from pieces 1 and 2. We note that, in this case, adjacent squares may only have one common side, and that this is very close to the case of self-avoiding polygons, but pieces in the same square may also coexist. We also note that the number of pieces used is necessarily even, exactly in the case of self-avoiding polygons.

Example 10. *We choose* $N = 8$ *pieces,* $N_j = +\infty$ *if* $j \in \{1, 2\}$*, and zero otherwise, and we draw all the feasible circuits with* $N = 8$ *pieces. We obtain the* 4 *circuits in Figure 18. Note that we find naturally some of circuits with* 8 *pieces of Example 8 (see Figures 16(a), 16(b) and 16(c)).*

Example 11. *As in the example 10, we draw all the feasible circuits with* $N = 10$ *pieces. We obtain the* 7 *circuits in Figure 19.*

[2]One may also consider pieces 3 and 4 which are homothetic to pieces 1 and 2 with the ratio $\sqrt{2}$.

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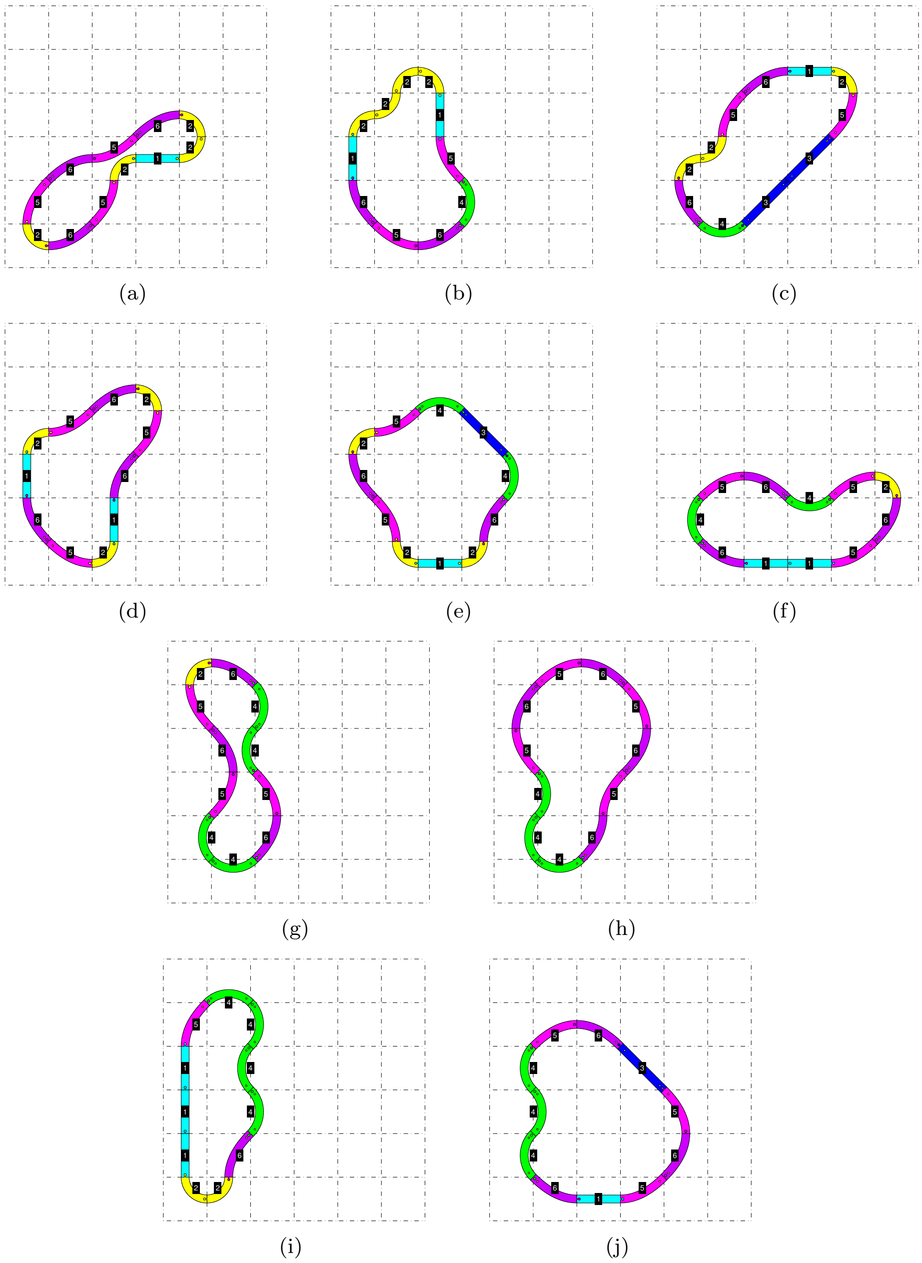
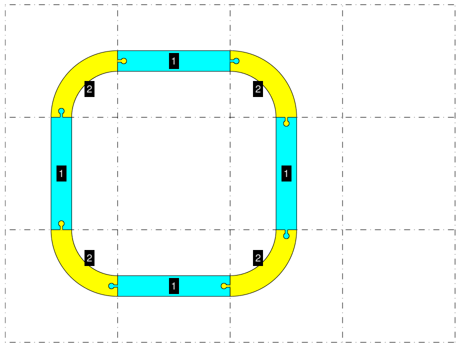
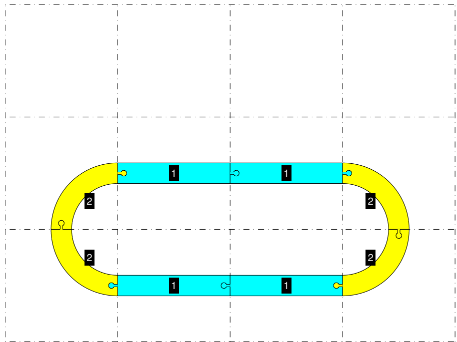


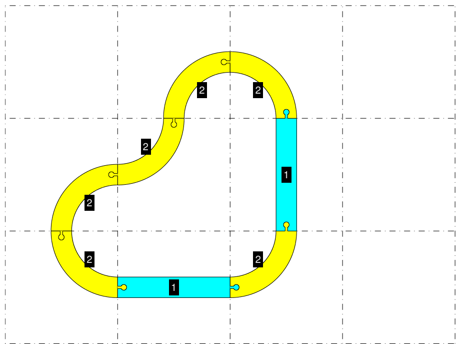
Figure 17: 10 of the 753 circuits retained from the set of 31250000 possible circuits.



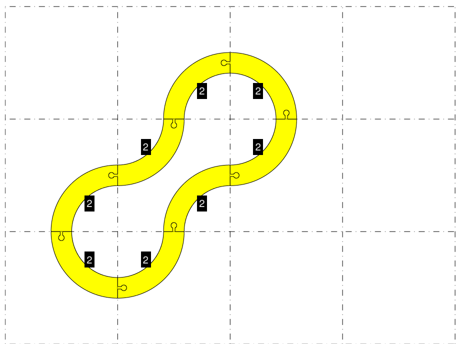
(a)



(b)



(c)



(d)

Figure 18: All of the 4 circuits kept from the set of 250000 possible circuits.

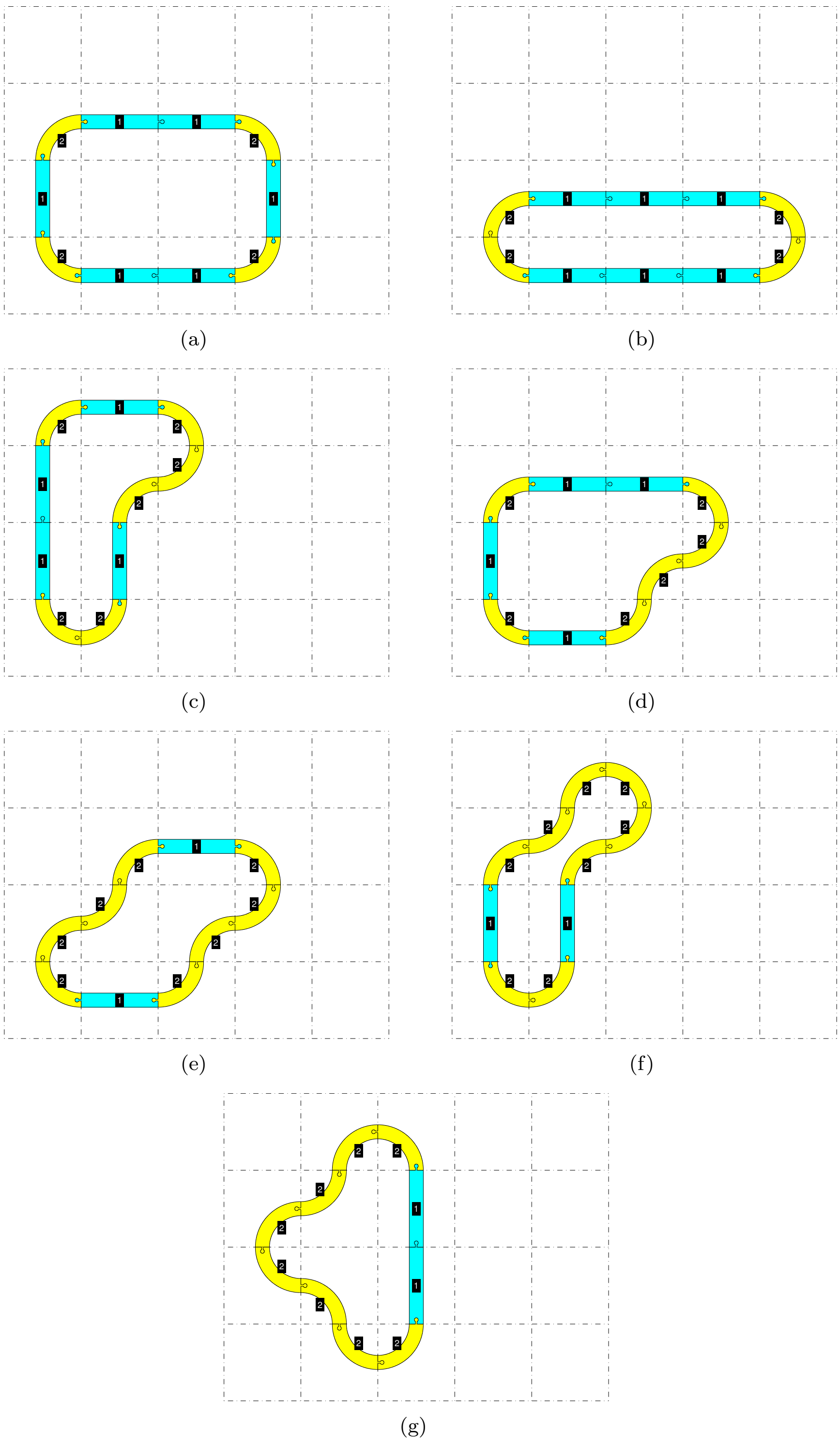


Figure 19: All of the 7 circuits kept from the set of 6250000 possible circuits.

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Determination of the number of circuits

N	$N_j = +\infty$	$N_j = 4$
1	0	0
2	0	0
3	0	0
4	2	2
5	1	1
6	5	5
7	7	6
8	33	28
9	74	63
10	304	244
11	986	753

Table 2: Numbers of circuits corresponding to $N_j = +\infty$ and $N_j = 4$

In Table 2 we give the numbers of circuits corresponding to $N_j = +\infty$ et $N_j = 4$. This last case corresponds to the distributed *Easyloop* boxes.

N	Self-avoiding polygons	traditional *Brio* system	*Easyloop* system
4	1	1	2
5	0	0	1
6	2	1	5
7	0	0	6
8	7	4	28
9	0	0	63
10	28	7	244
11	0	0	753

Table 3: The numbers of self-avoiding polygons, the (non-zero) numbers of circuits for traditional *Brio* system and the *Easyloop* system

Finally, in Table 3, the numbers of self-avoiding polygons, corresponding to a square lattice (see [15, table p. 396]) or `http://oeis.org/A002931/b002931.txt` and a comparison between traditional systems (see Section 3.8) and the *Easyloop* system are proposed. For the *Easyloop* system, only the number of constructible circuits up to an isometry is displayed. The number N_j equals 12 if $j = 1, 2$, and zero otherwise.

Traditional circuits are very close to self-avoiding polygons, except for the following two differences already mentioned above: the permitted isometries are all the isometries of a square, and a square may be used multiple times by the circuit. The common point is that for N odd, the number obtained is zero.

We draw the circuits obtained in examples 10 and 11 in the form of closed polygons. See Figures 20 and 21 respectively.

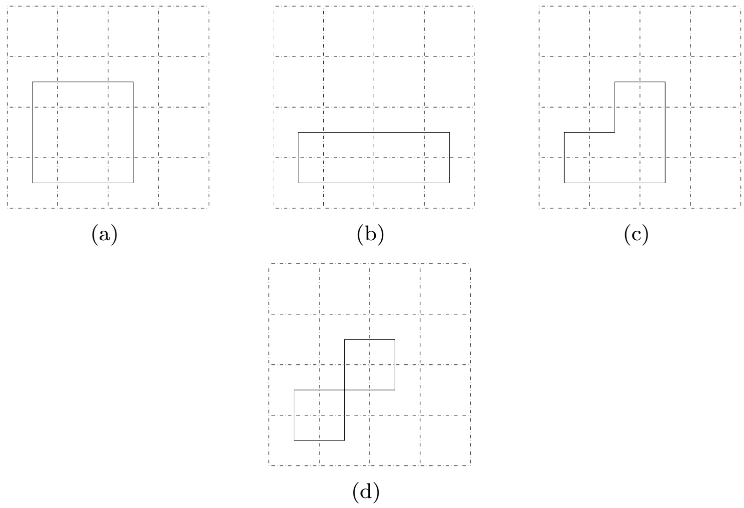

Figure 20: The 4 circuits with 8 pieces, drawn under polygons form.

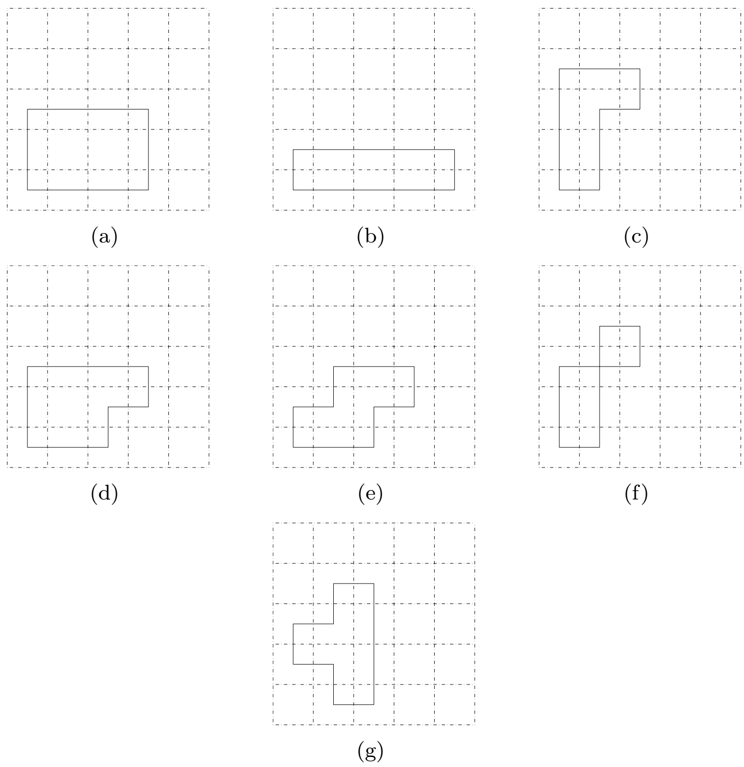

Figure 21: The 7 circuits with 10 pieces, drawn under polygons form.

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- For $N = 8$ (see Figure 20), we obtain 4 traditional circuits and 7 self-avoiding polygons. As noted in `http://oeis.org/A002931`, the 7 self-avoiding polygons correspond to the 1, 2 and 4 rotations (by angle $\pi/2$) of the circuits in Figures 20(a) 20(b) and 20(c) respectively. The circuit in Figure 20(d) does not correspond to any self-avoiding polygon, since one of the squares is occupied by two pieces. We have thus indeed found $7 = 1 + 2 + 4$.
- For $N = 10$ (see Figure 21), we obtain 7 traditional circuits and 28 self-avoiding polygons. In fact, the two circuits in Figures 21(a) and 21(b) each provide, by 2 rotations (by angle $\pi/2$), 2 self-avoiding polygons. The 2 circuits in Figures 21(c) and 21(d) each provide, by 4 rotations (by angle $\pi/2$) and one reflection, 8 self-avoiding polygons. The circuit in Figure 21(e) provides, by one rotation (by angle $\pi/2$) and one reflection, 4 self-avoiding polygons. The circuit in Figure 21(g), provides, by 4 rotations (by angle $\pi/2$), 4 self-avoiding polygons. The circuit in Figure 21(f) does not correspond to any self-avoiding polygon, since one of the squares is occupied by two pieces. We have thus indeed found $28 = 2 \times 2 + 2 \times 2 \times 4 + 4 + 4$ self-avoiding polygons.

Estimation of the number of circuits with a significant number of pieces

In the case where N is greater than 11, the calculations take too long, and it is not possible to use the enumeration of the circuits. We use, with much impropriety, the estimation given by (17), which comes from [17, 10, 13, 7, 8, 5]. We will take the number of circuits given in Section 3.9, and we will make use of it to evaluate the constants A, μ and γ in formula (17). This evaluation is replaced by an equality and the coefficients A, μ and γ are determined by solving a least squares system, which becomes linear when we take the logarithm.

	$N_j = +\infty$	$N_j = 4$
$\gamma - 1$	-8.75998	-8.69817
μ	9.13739	8.69023

Table 4: The values of $\gamma - 1$ and μ obtained using Table 2.

Let us now estimate $\gamma - 1$ and μ using the different results from Table 2. See Table 4. The obtained values are naturally different from the values given in (18). This is normal since the estimation (17) has been replaced by an equality, and naturally, nothing *a priori* validates this equality. We note that the signs of the coefficients are consistent with those in the literature.

The retained values of the coefficients A, μ and $\gamma - 1$ in formula (17) in the case of the *Easyloop* boxes, corresponding to $N_j = 4$, therefore correspond to the last column in Table 4 and are given by

$$A = 4.5900\,10^1, \quad \gamma - 1 = -8.69817, \quad \mu = 8.69023. \tag{20}$$

In case (20), the estimated numbers of circuits are then $\{0, 0, 0, 2, 2, 3, 8, 21, 65, 226, 857\}$, which is close to the exact numbers of circuits ($\{0, 0, 0, 2, 1, 5, 6, 28, 63, 244, 753\}$). For $N = 24$, we obtain

$$q(24) \approx 1560511691458. \tag{21}$$

We then obtain the curve shown in Figure 22.

To compare the *Easyloop* circuits with traditional systems, one obtain with the same way

$$q(24) \approx 130229, \tag{22}$$

which is still much smaller than (21).

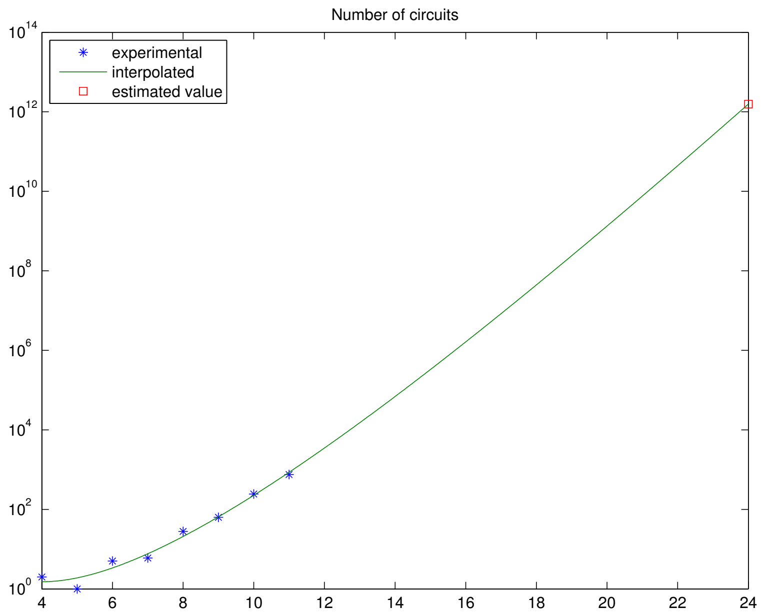


Figure 22: Estimation of the number of circuits in the case $N_j = 4$.

Random construction of circuits with a large number of pieces

For values of N smaller than 11, we are capable of obtaining all of the constructible circuits, and in particular to show them. Another objective of a manufacturer would be to offer a catalogue of train-track designs which may contain circuits with any N. Unfortunately, beyond $N = 11$, this is no longer conceivable. Manual designs are possible, but tiresome, and non-programmable. We thus propose in this section a way to automatically generate circuits for given N and N_j for values larger than $N = 11$, without having to create all of the possible circuits as proposed in Section 3.2, by relying on a random method.

For a given $N \geqslant 1$, we consider $r, s \in \mathbb{N}^*$ such that $N = r + s$. We are capable of determining all of the circuits with r pieces starting from the origin by describing a Cartesian product of finite sets. To avoid this long stage, we will simply randomly choose q circuits by choosing the parameters in this Cartesian product. For each of these circuits, the last square occupied by the last piece, is not necessarily equal to the origin. We take $R \in \mathbb{N}^*$ and we keep from these circuits only those for which the absolute value of the abscissa and the ordinate is less than R. For each of these retained circuits, we are capable of determining all of the circuits with s pieces starting from the last square and returning to the origin. We therefore consider all of the circuits obtained by the concatenation of the circuits with r pieces going from the origin to any square with the circuits with s pieces returning to the origin. Finally, of these circuits, we keep only those for which they types of pieces are less than N_j. We also apply the selection of the isometries and the constructibility constraints. One has hence obtained a certain number of circuits constructible with N pieces, without having had to to construct the Cartesian product of the circuit parameters determining all of the possible circuits, whose cardinality is too great. Naturally, to increase the chances of success, one must choose r, s, q and R as large as possible. Computationally, it is necessary that these numbers not be too great. The random determination will therefore consist of choosing these parameters appropriately. One may create such circuits oneself using the executables distributed for Windows, quoted on page 41.

Example 12. *We choose $N_j = 4$ and the following parameters*

$$r = 12, \quad s = 5, \quad q = 18, \quad R = 8.$$

We obtain the random circuit with 17 pieces given in Figure 23.

We have obtained some circuits in a random way, being able to take values of N strictly larger than 11, and, finally, in a much shorter time.

Generalizations

Shape of the tiling

We have seen that for a square tiling, the number of curves necessary to connect each point of $\mathcal{H}_i$ to every other distinct point in $\mathcal{H}_i$ was equal to 5 or 6, according to whether or not one takes the pieces numbered 7 and 8. This number depends intrinsically on the number of points in $\mathcal{H}_i$ and on the cardinality of the group of the isometries leaving the square invariant.

The question arises whether or not the method of constructing the rails of the studied circuits can be applied to types of tiling other than the square, and if one is capable of determining the number of basic curves, here equal to 5 or 6, uniquely from the tiling and the points $\mathcal{H}_i$ considered. This generalization is also mentioned for self-avoiding walks in [12], which is a simpler case since the circuits may only follow the edges of the tiles constituting the tiling of the plane.

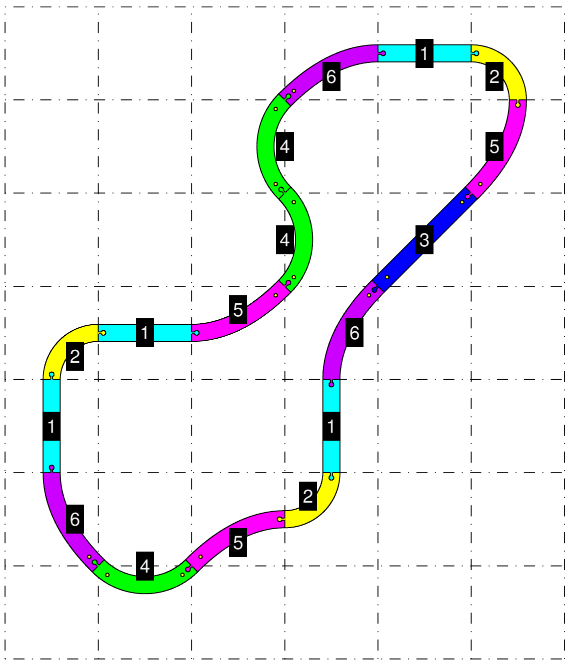
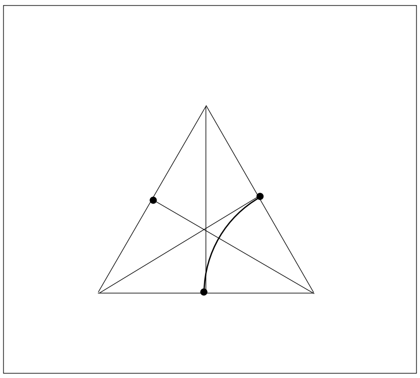
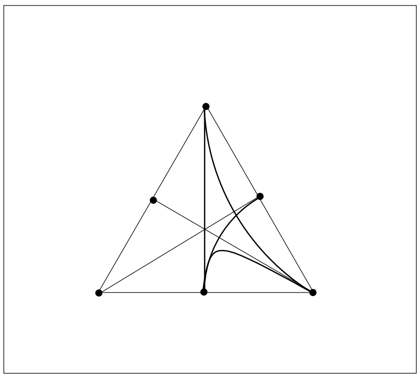


Figure 23: A random circuit with 17 pieces.



(a) : with the 3 middles



(b) : with the 3 middles and the 3 vertices

Figure 24: Other possible tilings: the equilateral triangle

For example, one may consider tiling the plane with equilateral triangles, taking only the 3 middles of the 3 sides (see Figure 24(a)) or the 3 middles and the 3 vertices of the triangle (see Figure 24(b)). We impose that the curve passes through two distinct points of this set $\mathcal{H}_i$, while being tangent to the straight line connecting this point with the center of the triangle. In the first case, a single curve is necessary, while in the second, 3 are. Other solutions can be envisaged, with other types of possible tilings.

Can the number of necessary curves be expressed as a function of the tiling polygon and the nature of the points $\mathcal{H}_i$? The enumeration of circuits, constructed using these methods, seems once again to be an open problem.

Presence of switches

We keep to the case of square tiling. As described earlier, if the presence of switches is not anticipated, the pieces of circuits must not have extremities in common. One can, on the contrary, allow certain extremities be common to several pieces, which simply corresponds to allowing switches.

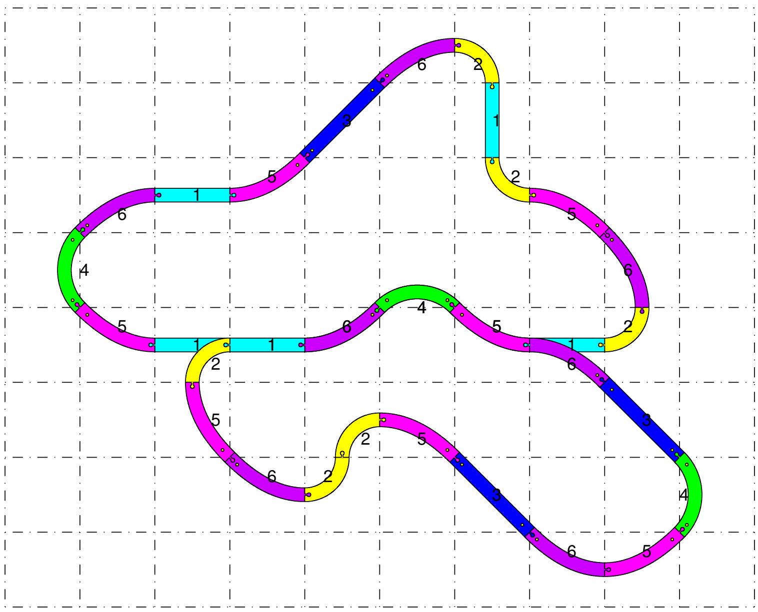


Figure 25: An example track design with 2 switches

In this case, the oriented circuits may contain several loops, which makes them directed graphs, whose enumeration is a much more arduous problem due to the multiplicity of the types of possible switches and therefore the types of possible graphs. See Figure 25, which shows an example of a circuit with switches.

Conclusion

The question "Is it possible to tally all of the circuits which can be realized from a given number of pieces?" is simple to express, and more difficult to resolve. We have succeeded in this article: the enumeration and the construction of such circuits is possible and have been implemented computationally up to $N = 11$. Beyond that, an estimation of the number of possible circuits has been provided (see (21), corresponding to the case $N_j = 4$, and (22)). We note that these numbers correspond to the number of circuits which contain exactly 24 pieces.

If we want to tally all of the circuits which contain at most 24 pieces, it is sufficient to sum the last column in Table 2, which gives 1102 circuits, then to apply the estimation for N varying from 12 to 24, with the estimated parameters given by (20), which gives in total

$$1873804310490, \tag{23}$$

that is, a total of more than

$$\text{one trillon feasible circuits with 24 pieces.} \tag{24}$$

In addition, a random circuit construction has been offered, allowing values of N strictly larger than $N = 11$ to be obtained. Some executables and a catalogue of circuits are available online.

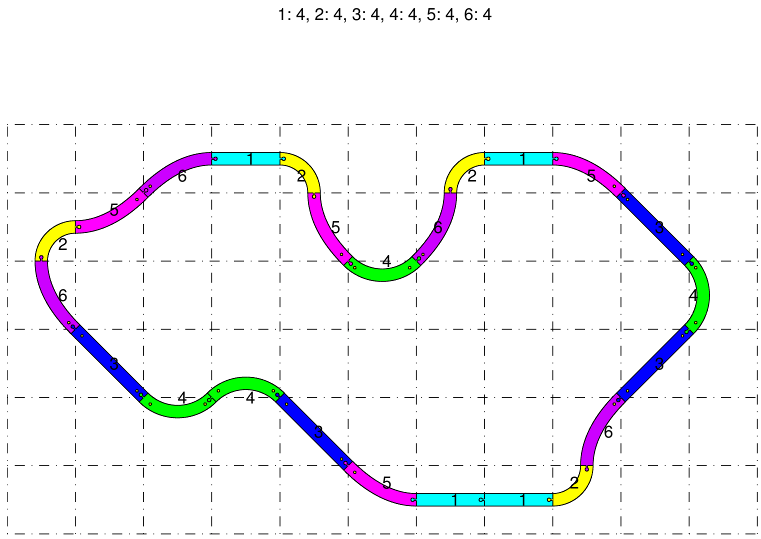


Figure 26: An example track design with 24 pieces.

Finally, we note that one circuit corresponding to $N_j = 4$, and containing exactly the maximum number of pieces (24) has been created by hand. See Figures 26.

It is interesting that the traditional theory of self-avoiding paths corresponds almost to existing trains circuits (see Section 3.8) while the patend studied system corresponds to the notion of perfectly looping walk.

It remains to improve the circuit enumeration algorithms to obtain higher values of N, in the deterministic case, by trying, for example, to avoid the very long enumeration of possible circuits; is a direct construction of constructible circuits possible, without going through this enumeration? An application of the parallelizable techniques proposed by G. Slade, I. Jensen, or A. J. Guttmann might be tried on the circuits in order to increase the values of N for which the circuit enumerations are exact.

It would be interesting to prove if estimation (17) is valid, with an eventual calculation of the constants A, μ and γ. The generalization raised in Section 6 allows the creation of other types of circuits, but also an attempt to understand the algebraic nature of the system proposed with squares.

URLs of softwares and catalogues available on Internet

http://utbmjb.chez-alice.fr/recherche/brevet_rail/MCRInstaller.exe
http://utbmjb.chez-alice.fr/recherche/brevet_rail/creecircuit.exe
http://utbmjb.chez-alice.fr/recherche/brevet_rail/creecircuitaleat.exe
http://utbmjb.chez-alice.fr/recherche/brevet_rail/dessinecircuit.exe
http://utbmjb.chez-alice.fr/recherche/brevet_rail/mode_emploi_rail_demo.pdf

http://utbmjb.chez-alice.fr/recherche/brevet_rail/catalogue.pdf
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Articles

where are (pseudo)science fool's hoax articles in april from?

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Abstract: *In this paper, we discuss in detail what is behind April Fool's Day.*

Keywords: April fool's day.

Introduction

April Fool's Day, sometimes called All Foll's Day, is one of the most light-hearted days of the year. April 1 has long been celebrated as a day to celebrate, well, foolishness to be exact. More specifically, April Fools' Day is about making other people look stupid with practical jokes.

The origin of April Fools' Day is uncertain. Somebody considers it as a celebration related to the turn of the seasons. Ancient cultures, including those of the Romans and Hindus, celebrated New Year's Day on or around April 1. It closely follows the vernal equinox (March 20 or March 21). In medieval times, much of Europe celebrated March 25, the Feast of Annunciation, as the beginning of the new year.

Others believe it stems from the adoption of a new calendar. In 1582, Pope Gregory XIII ordered a new calendar (the Gregorian Calendar) to replace the old Julian Calendar. The new calendar called for New Year's Day to be celebrated January 1. That year, France adopted the reformed calendar and shifted New Year's day to January 1. According to a popular explanation, many people either refused to accept the new date, or did not learn about it, and continued to celebrate New Year's Day on April 1. Other people began to make fun of these traditionalists, sending them on "fool's errands" or trying to trick them into believing something false. Eventually, the practice spread throughout Europe. But we have no direct historical evidence for this explanation, only conjecture.

It is worth noting that many different cultures have had days of foolishness around the start of April, give or take a couple of weeks. The Romans had a festival named Hilaria on March 25, rejoicing in the resurrection of Attis. The Hindu calendar has Holi, and the Jewish calendar has Purim. Perhaps there's something about the time of year, with its turn from winter to spring, that lends itself to lighthearted celebrations. To these days, April Fools' Day is observed throughout the Western world. Practices include sending someone on a "fool's errand", looking for things that don't exist; playing pranks; and trying to get people to believe ridiculous things.

Although it might seem that April has experienced a slight decline of Fame in recent years, April 1 is still a welcome opportunity for usually serious media to banter with impunity. With the April tradition media can go beyond the seriousness and let some sensation to the world. There is nothing more exciting than catching a million people out at a time. If we investigated the influence of the media, we could not forget the date of April 1. In this day, the number of TV and radio stations and newspapers are testing their readers, listeners and viewers when catching their attention on the grotesque reports. The newspaper sensations in the form of serious-looking reports are on a daily basis.

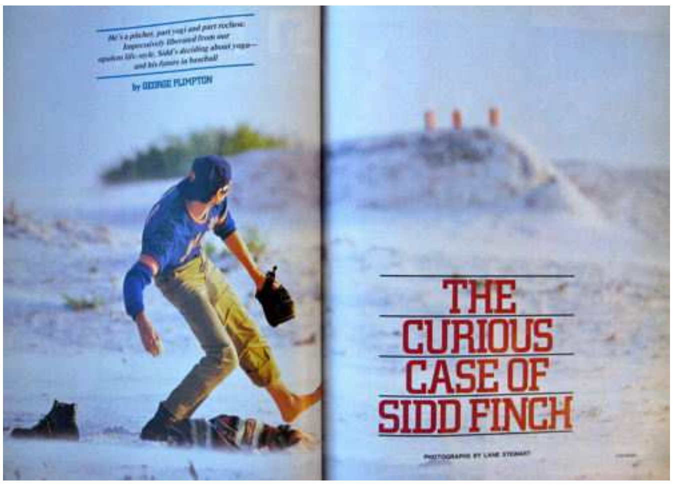
Even the smallest local newspaper tries to cheat their readers by informing them about the blast of the chimney in the local factory, saving the collapsing bridge or the unexpected arrival of a celebrity in the local discos. People attracted by these sensational titbits are gathering in front of the screens to find out soon that they were just taken in by the bunch of journalists who are more than happy to report how many people fell for their bait in the following issue.

We must also be vigilant following days. Although the first in a joking occupy mainly daily periodicals, including Internet, some journals do not want to be left behind and publish the April issue. The readers have to be alert only April 1 but basically the throughout month.

And they grip on not only but also in the choice of hoaxes. The most success jokes will go down the history of April hoaxes. Their international rankings can be found on the website of The Museum of Hoaxes. In the first place there is a report of the British BBC television from 1975. Contribution in the transmission informed viewers that Swiss farmers grew spaghetti on trees due to warm weather and had a record harvest this year. Although it may seem highly improbable, many viewers believed it.

Figure 1: The Swiss Spaghetti Harvest.

For the best-rated newspaper sensation is considered an article about incredible rookie baseball Mets's player which was published in Sports Illustrated Magazine, in 1985. The boy named Sidd Finch could reportedly pitch a baseball about 65 mph faster than previously recorded speed for a pitch. But this was not all, Finch had "learned the art of the pitch" in Tibet where he learned the teachings of the "great poet-sain Lama Milaraspa". Mets's fans were amazed by this rumor and clamored for more information. However, to their great disappointment, they read in the next issue of the magazine, they were victims of April fools hoax.



New Mets's player.

Basically, we can divide the journalistic April fool's hoax into several categories: seriously conceived message, which you can hardly see through; report with a clear overstatement, in which you can just having fun with the adaptation or absolutely humorous messages that can be true, just they do not have any place in the current edition. There are no reliable "rules of April" how we can recognize a hoax report. Everything depends on the sophistication of journalists and presence of mind of the reader. Sometimes we can use some tools such as print text upside down or using apparently fiction names and titles, but the author of the hoax article throw such a lifebuoy exceptionally. Every day, we are witness of events that we do not dare to believe for many reasons, even if they are sometimes true. So how we can know what is true and what is not? The chapter itself is, when scientific journals make fun of us; for example, reports of breakthroughs and inventions.

In a time, in which we live, the science become more progressive every day. Media constantly spewing on us messages about new scientific results. But how we can distinguish that discovery is crucial or extremely stupid? And do we have the ability to do it?

There are plentiful of pranks or hoaxes. In the following chapters, we will gradually trade the April fool's hoaxes from mathematicians of physicists. We will show some examples of such scientific April hoax articles. You can also read where the authors draw inspiration and what happens when readers take such a discovery seriously.

The great Moon hoax

After reading the introductory chapter, we have idea how an April fool's hoax article might look like. In this chapter, we introduce the first ever article which passed into history and triggered an avalanche of newspaper sensation.

Historically, the first false news sensation, which went down in history, is associated with the New York daily press *The Sun*[1]. In August 1835 *The Sun* published a series of briefings on astronomical observation of Sir John Frederick William Herschel that took the newspaper and scientific worlds by the ears.

The first angle of the plot appeared on *The Sun* on August 21, 1835. The report was allegedly taken from the English magazine *Edinburgh Courant* [27]: "*We have just learnt from an eminent publisher in this city that Sir John Herschel, at the Cape of Good Hope, has made some astronomical discoveries of the most wonderful description, by means of an immense telescope of an entirely new principle.*"

This announcement only serve as an initial report to the much more complicated series of articles that began appearing in the newspaper four days later. Everyday news brought readers news of astronomical observations. All

[1] *The Sun* was a New York newspaper that was published from 1833 until 1950. The founder and first editor of the newspaper was Benjamin Day. He came up with the idea of daily newspapers with an extremely low price, so he decided to create and fund his press based on the results from the sale of individual issues. Profit from each issue had to cover production costs. To newspapers sold well, spreadsheet content to offer simple and fun to read. Day emphasis on sensational news reports about crimes, sex and so on [21].

articles cited the results published in a supplement to the English scientific magazine.


THE SUN.

Number 615.] NEW YORK, TUESDAY MORNING, AUGUST 25, 1835. [Price One Cent.

GREAT ASTRONOMICAL DISCOVERIES

At the Cape of Good Hope.

Front page of daily New York *The Sun*: August 25, 1835.

Articles describing the landscape of the Moon, which seemed to telescope observers in the same similarities, as a viewer on Earth offering distance less than a hundred meters. The continuation of the story focused not only on the lunar landscape, but also for animals and creatures that lived on the Moon. It talked about different life forms on the moon, including such fantastic animals as unicorns, two-legged beavers and furry, winged humanoids resembling bats. The articles also offered vivid description of the moon's geography, complete with massive craters, enormous amethyst crystals, rushing rivers and lush vegetation. Then, suddenly, it followed a short statement that further discoveries failure prevented the telescope. This story ended.

Figure showing beings from the Moon.

Exotic landscapes, flora and fauna of the Moon has issued the first article became a sensation overshadowing reports in other newspapers. People all over New York to discuss whether the story is true. Readers were completely taken in by the story, however, and failed to recognize it as satire. A number of New York newspapers reprinted article, or at least its excerpts in order to they did not lose their readers (see [27]):

The Daily Advertiser wrote that: "No article has appeared for years that will command so general a perusal and publication. Sir John has added a stock of knowledge to the present age that will immortalize his name and place it high on the page of science."

The Times said that everything in the *The Sun* story was probable and plausible, and had an "air of intense verisimilitude".

The New York Sunday New advised the incredulous to be patient: "Our doubts and incredulity may be a wrong to the learned astronomer, and the circumstances of this wonderful discovery may be correct".

The craze over Herschel's supposed discoveries even fooled a committee of Yale University scientists, who traveled to New York in search of the *Edinburgh Journal* articles.

The articles were an elaborate hoax. It was a perfect mystification, which succumbed not only readers, but newspapers and some scientists. The only thing on the whole stunt was real, was the figure of John Herschel. Sir John Frederick William Herschel was the greatest astronomer of his time, moreover he was the son of the celebrated astronomer Sir William Herschel. In truth, in January 1834, he went to South Africa and established an observatory near Cape Town, with the intention of completing his survey of the sidereal heavens by examining the southern skies as he had swept the northern, thus to make the first telescopic survey of the whole surface of the visible heavens. Nevertheless Herschel had not really observed life on the moon, nor and he accomplished any of the other astronomical breakthroughs credited to him in the article. In fact, Herschel was not even aware what it was happening in New York. That the such discoveries had been attributed to him, he found out until much later. At first he was rather amused, and only complained that his own observations, sadly, never will nor half as entertaining as they are described in the paper. Later, however, he was much annoyed because he had to constantly face questions from people who thought that the articles are true [35]: "*I have been pestered from all quarters with that ridiculous hoax about the Moon – In English, French, Italian and German!!*"

The author of this newspaper stunt was reportedly reporter Richard A. Locke[2], who in August of that year began working for the newspaper *The Sun.* There are persistent rumors that Locke confessed to his friend Finn authorship of articles in a weak moment. Reveal the secrets of a friend who works in a

[2] Richard Adam Locke (1800–1871) was an English journalist, writer and later editor of *Somerset paper.* In 1835, he first met with the editor Benjamin Day and began working for *The Sun* [35].

competing newspaper, was proved a poor choice. The next day, the newspaper *Journal of Commerce* published the news that a series of monthly articles about discoveries is a hoax, and Locke identified the author as a fiddle. There are also speculations that by writing articles involving more people. In connection with the mentioned articles is most often named yet one man: Jean-Nicolas Nicollet[3], French astronomer passengers at the time in America. However, there is no specific evidence about who was the real author of those sensational articles, and even Locke never publicly admit authorship.

Regarding the intention of the entire newspaper fiddle, assumptions are somewhat nebulous. The first option is entirely pragmatic: Locke's aim could be to create a sensational story that would increase the sales of *The Sun*. Another reason could be targeted ridicule from some rather extravagant astronomical theories, which were published at that time. It comes into consideration also the option that Locke was inspired by a story by Edgar Allan Poe[4] about similar inhabitants of the moon called *The Unparalleled Adventure of One Hans Pfaall*.

Whatever the intentions of any of Locke, his hoax sparked sharp criticism from the other New York newspapers. For the "robust" newspapers of that time, which came from the Enlightenment concept of the press as a medium of education for the "common people", it was similar to the stigma content "relegation" task newspaper. Despite the fact that the New York newspaper condemned all competition from fraud, *The Sun* for a long time he was not ready to disappoint their readers and confess. Until September 16, 1835, more than two weeks after the conclusion of the story, was imprinted long article on the topic of authenticity discoveries. At the end of this article *The Sun* the whole affair has concluded that although the articles about discoveries on the moon initially written as a satire to entertain readers, later unexpectedly encountered new circumstances, which may confirm the authenticity of some breakthroughs, and therefore it is necessary explore everything properly again. This statement apparently after previous experience nobody believed, but because readers and most newspapers fell for a ruse, it is better not to pursue the case further too and take the whole affair rather humorously. "That the public were misled, even fop an instant" Poe declared in his critical essay on Locke's writings [27], "merely proves the gross ignorance which, ten or twelve years ago, was so prevalent on astronomical topics."

However, it is clear that gambling confidently readers did not erode sales of the newspaper *The Sun*, quite the contrary. Thanks to a sensational articles, the number of his prints has increased dramatically and even after the discovery of fraud had not fallen. *The Sun* every day began to publish more or less fanciful reports about various attractions, stories and tidbits from the world of crime, sex and so on. Over the next year cost of *The Sun* several times higher than

[3]Jean-Nicolas Nicollet (1786–1843), also known as Joseph Nicolas Nicollet, the French geographer, astronomer and mathematician who is most notable thanks to the mapping of the upper stream Mississippi in 1830 [18].

[4]Edgar Allan Poe (1809–1849) was an American Romantic poet, novelist, literary theorist and essayist. He was the author usually fantastic and mystical stories and founder of the detective genre [4].

the average cost of a conventional printed in the USA. No wonder then that this convincing success prompted several attempts to infiltrate into the same area newspaper market.

The Indian rope trick

Another in a series of successful newspaper sensation came from *Chicago Daily Tribune*[5]. On August 9, 1890 the newspaper published a report describing the breathtaking illusionist performances of the Indian fakir [34].

The story is narrated by a young amateur photographer Frederick S. Ellmore, who traveled to India with his friend George Lessing. During his stay in Gaya he attended a performance of a local snake charmer. The performance was stunning. The young travelers were most attracted by a rope trick. The fakir took a ball of gray twine in a hand. Taking the loose and between his teeth he with a quick upward motion, tossed the ball into the air. Instead of coming back to him it kept on going up and up until out of sight and there remained only the long swaying end. At the same moment a young boy appeared beside the fakir. He began climbing it and vanished in clouds. A moment later the twine disappeared.

The story was also added to the pictures. While the fakir was going through his performances Lessing was to make a rapid pencil sketch of what he saw while Ellmore at the same moment would take the photographs with kodak. Strangeness was that the scene outlined in Lessing's sketches did not agree with the Ellmore's photographs. Despite the fact that the sketch showed the boy climbing the twine, the camera said there was no boy and no twine. Therefore Ellmore concluded that the fakir had hypnotized the crowd into believing the trick had been performed(see [5]): "...his eyes were remarkable both for their brilliancy and their intense depth, if I may so term it. They seemed to be almost jet black and were set unusually deep in his head. When we stepped into the little circle about him those eyes took us in from sole to crown...I'm compelled believed that my theory is absolutely correct - that Mr. Fakir had simply hypnotized the entire crowd, but couldn't hypnotize the camera." In the conclusion of the article he promised to sent the copies of the pictures to the London Society for Psychical Research to closer investigation.

[5] *Chicago Daily Tribune* is a major daily newspaper based in Chicago, Illinois, United States. It was founded in 1847 and formerly self-styled as the "World's Greatest Newspaper". In the year 1854–1901 Joseph Medill was a managing editor of the newspaper.

Picture from *Chicago Daily Tribune* with Lessing's sketch and Ellmore's photograph.

Both readers and other newspapers believed the article. The story quickly spread and gained credibility, first in the United States, then in Great Britain and very soon it was translated into many European languages. Over the next four months this article provoked so much debate that the editor *Daily Tribune* decided to clarify the whole affair and admit that the story was completely fabricated. On December 6, 1890 the short notice was published at the bottom of a *Queries and Answers* column (see [37]): "The article on hypnotism referred to in the above query was written for the purpose of presenting a theory in an entertaining form... The principal character was Mr. F. S. Ellmore (sell-more), and the writer considered that the name would suggest to a careful reader that ir was a 'sell'." How many reader devoted attention to the confession, it is hard to estimate. Although press transfers the messages better than other media, simultaneously it allows people the disclaimer to just skip. Whatever the readers may read the disclaimer or not, the story about the Indian rope trick definitely did not fit.

Over the ensuing decades, this trick has inspired much controversy in magic and psychical research circles. Other eyewitness accounts to the trick were presented, but finally, they collapsed under investigation. Despite all the statements of the witnesses were false, with every repetition the story became more real and realistic not only rumor[6]. Magicians and illusionists bet among themselves who will perform the air-open show as the first. In 1930's, Lt Col Elliot of the London Magic Circle, when offering a substantial reward for an outdoor performance, found it necessary to define the trick. He demanded that (see wiki): "the rope must be thrown into the air and defy the force of gravity, while someone climbs it and disappears." Challenges to perform the trick in the open air, not on a stage, were taken up but never met by magicians.

We must admit that this newspaper sensation was really successful. Many people (including scientists) did not want to believed that the story was not

[6]Rumor is a tall tale of explanations of events circulating from person to person and pertaining to an object, event, or issue in public concern. It contains nothing from which we could assess the veracity (see [20]).

inspired by a reality and so they traveling to India inquired about the trick and helped to spread the legend there; soon Indians were defending their trick against claims it had originated centuries prior in China. In 1996 Peter Lamont[7] a historian and performer of magic dealt with the origin of the trick and concluded that there were no known references to the trick predating 1890, and later stage magic performances of the trick were inspired by article. By now the rumor about the Indian rope trick lived its own life and with time diverse accounts of the trick begun to appear in print which differ in the degree of theatricality displayed by the magician and his helper[8]. According to the version of the story, the tricks they used to be attributed to various Indian or Moroccan fakirs.

The question still is, how did we discover the identity of the story's anonymous author? In 1891 Andrew Stewart, the editor of a popular British weekly *People's Friend*, had read various copies of the original story and sent a letter to the *Daily Tribune* seeking more information of the rope trick. One John E. Wilkie[9] responded to his letter (see [24]): "I am led to believe... that the little story attracted more attention than I dreamed it could, and that many accepted it as perfectly true. I am sorry that any one should have been deluded." And the letter was signed, with no obvious sense of irony, "sincerely yours, John E. Wilkie". Richard Hodgson of the American Society for Psychical Research has on the question of authorship a different opinion. R. Hodgson did not credit Wilkie either, but instead and noted that (see [24]): "[t]he story of the boy climbing the rope and disappearing is, in one form or another, very old". This would be another reason why Wilkie would not be remembered as the man who launched the legend. For while he was responsible for making the story famous, the story itself was not entirely original. There had been, for centuries, many stories of ropes, cords, chains and the like rising into the air, of humans and animals climbing to the top and disappearing. Such stories, in fact, can be found in the mythologies and folklore of several cultures, not only in India but in Europe and China, in North America and Australia.

And even if we knew the real author of this hoax, today we can only guess, what was the main impetus for the writing this hoax. For example, Lamont believes Wilkie's ruse was inspired be debates at the time about conjurors and psychic phenomena.

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[7] Peter Karl Lamont is a research fellow at Edinburgh University. He is also historian and performer of magic (see [29]) In 1996, he pulls of a neat trick himself in making 264 pages appear on so slim a topic. Moreover, his book contains the names the alleged hoaxer and follows the tricky caroms the legend took over the years [24].

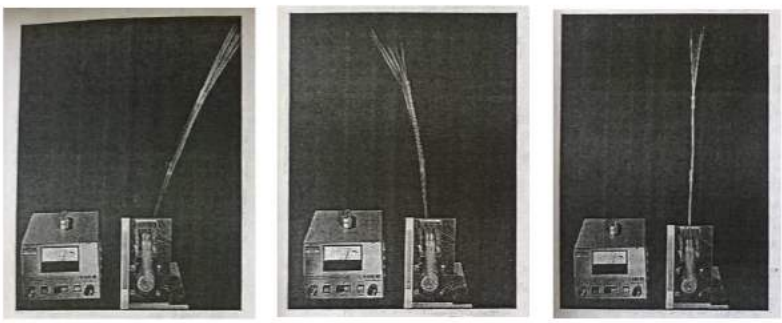
[8] One of them threw a rope into the air which hitched itself up to apparently nothing in the sky above; one could see the rope going straight up as far as one could see anything... A small boy then swarmed up this rope, becoming smaller and smaller, till he likewise vanished from sight, and a few minutes later bits of his (apparently mangled) remains fell from the sky, first an arm, then a leg, and so on till all his component parts had descended; these the juggler covered with a cloth, mumbled something or other, made a pass or two and behold! there was the boy smiling and whole before us (see [24]).

[9] John Elbert Wilkie (1860–1934) was an American journalist and Director of the United States Secret Service from 1898 to 1911. His father, Franc B. Wilkie,was an editorial writer at the newspaper; the two traveled to Europe and served as the *Chicago Times*' European correspondents. In 1911 John E. Wilkie joined the staff of the *Chicago Tribune*, where he initially served as financial editor and later city editor (viz [17]).

Very curiously is that although Wilkie have dreamed trick which the magician have not succeeded in doing it till now, but it does not mean that it is totally unrealistic. When Wilkie unleash his imagination, surely he did not know that one hundred years later, something similar will realize through scientific and technological methods.

Very similar analogy is described by the Swiss mathematician Daniel Bernoulli. In 1738 he published the article on pendulum motion. Bernoulli was interested in multiple pendulums that means N pendulums suspended from one another. He discovered that this system can oscillate at any one of N different natural frequencies $f_1, \ldots, f_N$, where f_1 denotes the smallest and f_N the largest. In the lowest-frequency mode the pendulums swing to and more of less together, much as if they formed one long, single pendulum. The other way around, in the highest-frequency mode, adjacent pendulums swing in opposite directions at any given moment.

In 1992 a mathematician David Acheson and a physics Tom Mullin began to study more closely the Indian rope trick. They turned everything upside-down and discovered that it is possible to take N linked pendulums, turn then upside-down, so that they are all precariously balanced on top of one another, and then stabilize them in this position by vibrating the pivot up and down. They conclusion said that (viz [2]): "the trick can always be done, so long as the pivot is vibrated up and down by a small enough amount and at a high enough frequency." By the experiment they even discovered, that with a 50 cm inverted triple pendulum, for example, the pivot was vibrating up and down through 2 cm or so at a frequency of about cycles per second. Moreover the inverted state is quite stable because when they push them over by as much as 40 degrees or so and they would still gradually wobble back to the upward vertical.



The upside-down pendulum

In 1993 they published their results in *Nature* magazine and two years later they made a brief appearance, with the experiment, on the British BBC television on the TV programme *Tomorrow's World*. Sometimes, however, fictions and fantasies of journalists seem closer to reality than advances of science. While the *Daily Tribune* was enjoyed greater sales after publishing hoax article, Acheson and Mullin did not get popularity. After the TV program, the BBC received a telephone call of from outraged viewers who claimed that (viz [2]): "[the] balancing act was obviously impossible, and contrary to the laws of physics" and he seemed "genuinely upset with *Tomorrow's World* for 'lowering their usually high standards' and 'falling prey to two tricksters from Oxford'.

Gravity nullified

Let's return back to the sensational newspaper articles. Fascination with the power of newspapers keep also subsequent periods. Another newspapers the idea of news sensationalism further developed and promoted it with a far more aggressive. Over time heals completely withdrawing from publishing completely fictional reports, as undermining the credibility of the sheet. Newspaper stunts were replaced by sensationalist headlines and graphic illustrations that complement the only real event.

All this has led to the fact that at the beginning of the 20th century, people began to look at the newspaper with a certain disdain due to the lack of objective information and pandering to readers. Efforts to regulate the work of journalists and provide relevant information to readers culminated in the release of the ethical code of journalists[10]. The ethical code serves primarily as ensuring moral support for journalists and their readers and sets a limit on what is good, moral and what is not. Although none of the ethical code has legal force, but its compliance is mandatory and violations can be punished. Penalties for violations are different in every country. Generally, a person in violation of the Code may be sued, and the result is a matter of court proceedings - from moral sanctions such as a reprimand, despite the temporary suspension, to the exclusion of professional associations.

This regulation of the press is the reason why today's press, especially if we focus on professional scientific journals to his readers a little more forgiving and canards deleted only once a year, on April. However, in this day readers really need to be alert. A chance to catch their readers from time to time will not miss even reputable scientific journals.

Moreover it is becoming harder and harder to recognize when it is a hoax, and when it is a real discovery. Additionally, if we are not experts in the field and do not understand the issues, we have little chance to detect fraud and not be fooled. On April, Other times, serious scientific journals, often use (or abuse) the reader's ignorance about technical topics.

On April 1927 a German journal *Radio Umschau*[11] publish a report called *Uberwindung der Schwerkraft? Ein neuer Erfolg der Quarzkristallforschung* about discovery of antigravity device.

[10]The first ethical code of journalists was named the *French Charter of the duties of journalists* and it was publish in France in 1918. In 1926 followed the Ethics Code in USA. Furthermore, in 1947, again in USA, it was publish the Hutchins Commission on press freedom and in 1950's *International Federation of Journalists* was founded by journalistic organizations of the USA and Western Europe.

[11]*Radio Umschau*.

The discovery was made in a newly established central laboratory of the Neuuartadline-Werke in Darreskein, Poland, by Krowsky and Frost. While experimenting with piezoelectric properties of quartz crystals they discovered that the constants of very short waves, carried on by means of quartz resonators, a piece of quartz which was used, showed a clearly altered appearance. It means quartz crystal changed the entire structure. Moreover it lost its weight, had become practically negative, and its was able to levitate.

Fig. 1. Im Quarzkristall-Laboratorium.

Ueberwindung der Schwerkraft?

Ein neuer Erfolg der Quarzkristallforschung.

the article from German journal *Radio-Umschau*

It is possible that the Polish scientists were actually at the birth of methods to overcome gravity? It is true that science goes forward every day and we must admit that theoretically everything is possible. With the development of technology we do not know the day or the hour, when it will be found for the way gravity to break and perhaps a similar manner to that described in the article. However, it did not happen, and although the Frost's and Krowsky's experiment would surely bring a revolutionary discovery for science and humanity, the facts in the article are not true.

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One of the readers who was very amused interested in the article, was Hugo Gernsback[12]. Gernsback is still highly praised for his contribution to the science fiction genre. Due his enthusiasm for science fiction, or maybe just because, he was a great opponent of pseudoscience such as astrology spiritualism and especially the attack on alternative medicine. In 1913 he founded the magazine *Science and Invention* which was given up until 1929. The magazine was focused on science and technology, especially for amateur science experiments, construction of radio and notable inventions. There were often published speculative articles about upcoming technologies and even science fiction stories (see [26]). So when April fools article about the discovery of antigravity device appeared in the German magazine, Gernsback was so impressed article, he could no resist the joke and did not share this with your readers as well.

And because history repeat itself Gernsback used the same schema such as the *New York Sun* – he published the report called Gravity Nullified (Quartz Crystal Charged by High Frequency Currents Lose Their Weight) with the remark that (see [12]): "this report appears in a reliable German journal, Radio Umschau."

ÃlÃ¡nek Gravity Nullified z Ã¨asopisu Science and Invention

In this case, the editor propably aware that many readers take the article seriously, and so he tried to throw a lifeline in the form of final notes (see [12]): "Don't Fail to See Our Next Issue Regarding This Marvelous Invention." Above

[12]Hugo Gernsback (1884–1967), born Hugo Gernsbacher, was a Luxembourgian American inventor, writer, editor, and magazine publisher, best known for publications including the first science fiction magazine. His contributions to the genre as publisher were so significant that, along with the novelists H. G. Wells and Jules Verne, he is one person sometimes called "The Father of Science Fiction". In his honor, annual awards presented at the World Science Fiction Convention are named the "Hugos" (see [19]).

sentence should tell readers that this is only joking article. Also, if the readers better view individual images, they found that their labels do not completely correspond to what is shown on each image. Unfortunately, not all readers were so receptive and farsighted, and so a large proportion of them would fall.

In the following October issue of tfhe magazine *Science and Invention* the article appeared which was called Gravity Nullified - A Hoax, in which the editors tried to put everything into the correct perspective and explain to readers that the article was only hoax and they cannot take this seriously (see [12]): "As a matter of fact, most of the statements are true, with the exception, of course, of those statements referring to the expanded crystal and to the loss of weight caused by the supposed high frequency currents." One of the disadvantages of periodicals is that are not daily published and disclaimer of the article may come up next month, so readers lived a lie for quite a long time. Moreover the Achilles' heel of this disclaimer certainly is that it is disprovable. It means that the original report claim that anything goes and disclaimer suddenly claims that it is not. Generally, we can say that every statement about something that does not exist, has in terms of verifiability terrible handicap. In addition, for the article, where we are not able to self-assess his truthfulness on the based on our knowledge and skills, we come to the knowledge of the basic paradox: the belief in denial follows the same logic as the belief in the original statement. In both cases, the point is to believe one's word. Questions of the readers: "What to believe?", depends largely on what "Where it says?" And if readers evaluated as a credible source of information, and the authors themselves acknowledge the fact that in the future may be all different, we can not be surprised that some readers were so excited about an experiment that did not want to believe that it is a hoax.

And therefore the deception did not left entirely without consequences. In 1981, in the February issue of the journal *Planetary Association for Clean Energy* John G. Gallimore published an article entitled Anti-Gravity Properties of Crystalline Lattices. Gallimore informed the readers that in the summer of 1927, two Polish scientists Kowsky and Frost described the specific anti-gravitational properties of crystals. The report about their discover appears in magazines *Science and Invention* and *Radio Umschau* shortly after their experiment, some photographs of the tests was published as well.

Also David Hatcher Childress and W. P. Donavan are convinced of the truth of this discovery. In the book *The Anti-Gravity Handbook*, they polemise whether the magazine misunderstood when the article declared false.

Perfection, however, that the joke was delivered by authors who publish on the Internet. For example, the above mentioned W. P. Donavan (also acting under the name Bill Donavan) wrote in his work *Glimpses of Epiphany* that he is not only deep convinced that the results of Frost's and Krowsky's experiment are true, but also that "it seems that something got out, that wasn't supposed to get out" and now it is an effort to conceal everything.

Of course, we can speculate what the editor followed through publishing this article. Someone accuses the magazine that the article was published with the intention to profit from higher sales.

The article eventually became one of the more popular jokes publications and occupies pride of place on the website of The Museum of Hoaxes. Needless to say, the editor's note in the explanatory article Gravity Nullified - A Hoax did not speak clearly to the contrary (see [12]): "Scientific hoaxes are no novelty. One of the most famous, which was no exposed as quickly as this one, appeared in no less than the *New York Sun*. At that time, in August, 1835, a certain professor was supposed to have submitted his report on a fantastic moon people to the *Edinburgh Journal of Science*, to which manuscript the *New York Sun* obtained the first rights, and the article ran consecutively over a period of the time." On the other hand, he also notes (see [12]): "The moral is that we should not believe everything that we see, but do a little original thinking ourselves, because we may never know, otherwise, what are facts and what are not." Therefore, it is also possible that the article should have more educational aspect.

Six sensational discoveries that somehow have escaped public attention

In 1975, in the April issue of the Scientific American magazine was published the article about six sensational discoveries of recent years [8]. The readers were informed about a huge discovery in the number theory, about the finding of the counterexample four-colour-map theorem, about local flaw in the special theory of relativity, about revolutionary chess-playing programme, about discovery of page from Leonardo da Vinci 's notebook, and about the invention of psychic-energy-working motor.

In fact any of these revolutionary discoveries were not of great importance to the world. Although they were based on true facts, lots of information has been thought through. And in many cases the names of scientists, who have studied the given problem, were altered. All conclusions in the article were completely fabricated.

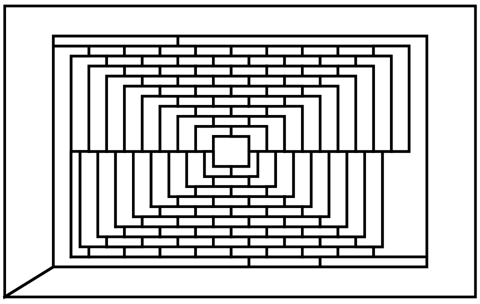
The author of this article was Martin Gardner[13] who published column called Mathematical Games for a long time. Mathematics became his lifelong passion, but he was also an expert magician, a well-known sceptic or a leading figure in refusing pseudoscientific theories ranging from modern diets to flying saucers. All of these hobbies are reflected in his April hoax article.

Four-color-map theorem

Discovery of the counterexample of the four-colour-map theorem created a considerable stir among the readers. In autumn 1974, the American

[13] Martin Gardner (1914â2010) wrote columns in *Scientific American* magazine for long twenty five years and published more than 70 books. In spite of, or perhaps because of, lacking proper mathematical education, Gardner's articles and books influenced generation of people. Thanks to his boundless enthusiasm and careful choice of topics, his articles got the general public interested in math. He also had other hobbies apart from mathematics. He wrote a lot of books concerning magic, philosophy, or commented on other authors' books.

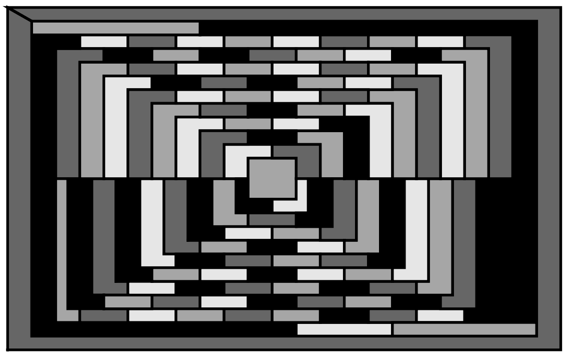
mathematician William McGreor managed to construct an example of a planar map with 110 countries (picture below), where minimum of five colours is needed for colouration.



The planar map constructed by William McGregor

Since 1740s all mathematicians tried to prove the fact that four colours are enough to colour any political map in the way that any of two neighbouring states would have the same colour. For decades they were unable to come up with the longed-for evidence of the theorem, so they tried to construct at least a counterexample. When examining small maps it was shown that four colours were enough for colouration. It seemed, however, that it would be about much more complicated maps. In early 1974, the mathematician Jean Mayer managed to prove that theorem is true for a planar map that contains a maximum of 95 states. Everybody expected that the problem would be soon resolved. Various symposiums on graph theories were held and the problem of the four colours was discussed everywhere. Gardner took the advantage of the atmosphere which longed for resolving the problem, and offered readers a simple solution.

The map, which is showed in the picture, was designed by a correspondent William McGregor (his real name). If you try to colour the map, you will discover soon, that four colours are enough.



William McGregor's map coloured by foour colours.

In fact, the four colour theorem was proved in 1976 by Kenneth Appel and Wolfgang Haken. It was the first major theorem to be proved by the use of a computer (see [1]).

Gardner received thousands of letters from his readers who sent him copies of the coloured maps where only four colours were used. Some of them claimed they spent days before working it out. A large number of readers, including mathematicians, fell for the Gardner's joke as many articles at that time mentioned.

When Norman Kent Roth published the article called *Map colouring* in December 1975, he was snowed under with letters from readers who kept informing him that *Scientific American* had already published a map disproving the four colour theorem whose author was Martin Gardner.

In December, 1976, a British mathematician George Spencer-Brown announced that he had found the proof of four colour theorem without using a computer. Although the experts finally agreed that the evidence contained errors, his announcement did not pass unnoticed. In January 17th, 1977, Canadian journal *Vancouver Sun* published a letter from a woman in British Columbia, in which she protested against Brown's proof because of Gardner's article.

In 1978 *Artificial Intelligence* magazine published the article whose author stated that he managed to colour McGregor's map using a computer programme. The author obviously did not realize it was a hoax.

Not only Gardner, but also the staff of *Scientific American* magazine, was little bit taken aback by some reactions of the readers. The following letter, signed by a mathematician Ivan Guffvanoff III. at the University of Wisconsin, was a bit frightening for the staff of Scientific American magazine, unless they realized that it, too, was a joke (see [9]): "This is to inform you that my lawyer will soon be contacting you for a damage case of $25 million. In the mathematics section of your April 1975 issue, Martin Gardner wrote that the four-colour problem had been solved. I have been working on this problem for 25 years. I had prepared a paper to be submitted to American Mathematical Monthly. The paper was over 300 pages in length. In it I had proved that the answer to the four-colour problem was no and that it would take five colours instead of four. Upon reading Gardner's article, that someone else would publish the solution before I could, I destroyed my paper. Last week I read in Time magazine that Gardner's article was a farce. I did not read Gardner's entire article, only the part on the four-colour problem, so I was not aware of the farce. Now that I have destroyed my article, it will not be possible to reproduce all 300 pages, since the work has extended over such a long time. I therefore believe that damages are due me. I believe that Gardner's article was the most unprofessional article I have ever seen in yours or any other journal. This kind of activity is below the dignity of what I thought your magazine stood for. I am not only suing you but I am cancelling my membership, and I will ask all my friends to cancel theirs".

A reference of Gardner's article can also be found in the Italian magazine *Rendiconti*. In 1975, mathematician Serge Benjamino published series of articles in which he showed that McGregor's map can be coloured using four colours.

Ramanujan's constant

Another piece of news Martin Gardner came up with was a surprising discovery in number theory claiming that the number $e^{\pi\sqrt{163}}$ is integer.
This exciting result was discovered thanks to American mathematician John Brillo in 1974. He was supposed to find an ingenious way of applying Euler's constant to prove that

$$e^{\pi\sqrt{163}} = 262\,537\,412\,640\,768\,744\,.$$

Mathematicians of the 18th century were already interested in number $e^{\pi\sqrt{163}}$. This number was discovered by Indian mathematician Srinivasa Ramanujan[14]. In any case, it is not an integer. S. Ramanujan occupied himself with several similar powers of Euler's number in the article *Modular equations and approximations to* π. But it was clear to him that all numbers were transcendental numbers (viz [30]): "[from equations] we can find whether $e^{\pi\sqrt{n}}$ is very nearly an integer for given values of n, and ascertain also the number of 9's or 0s in the decimal par…"
John Brillo, to whom this hoax is attributed, is a play on the name of the distinguished number theorist John Brillhart.

Gardner's idea, that the prime number 163 manages to convert the expression to an integer, implies from the fact that the number 163 is, in many respects, interesting. For example the number 163 is one of the Heegner numbers. To understand better the importance of Heegner numbers we recall some qualities of complex numbers.

There are many possibilities how to express complex numbers. The most famous method is by using the Gaussian integers. By *Gaussian integer* is meant a complex number $z = a + bi$ an integer when a, b are integers:

$$\mathbb{Z}[\sqrt{-1}] = \mathbb{Z}[\mathrm{i}] = \{\mathrm{z} = \mathrm{a} + \mathrm{bi} : \ \mathrm{a}, \mathrm{b} \in \mathbb{Z}\}\,.$$

Gaussian numbers form a square lattice in the complex plane. The mappings between complex numbers and Gaussian numbers are of one-to-one correspondence. It means that every complex number is paired with just one Gaussian number and the other way round. Gauss also discovered that every Gaussian number can be uniquely factored into Gaussian primes[15]. Gaussian primes are of shape

$$\begin{cases} a + b\mathrm{i}, & \text{when } a^2 + b^2 = p \text{ is prime, or} \\ up, & \text{when } u = \{\pm 1, \pm\mathrm{i}\} \text{ a } p \text{ is prime of shape } 4k+3. \end{cases}$$

[14]Srinivasa Ramanujan (1887–1920) was an Indian mathematician with a wide range of interests such as heuristic aspects in number theory, mathematical analysis, infinite series (see [32]).
[15]Gaussian number z.

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For example::

$$\begin{aligned}
&2 = (1+i)(1-i) = 1^2 + 1^2,\\
&3 \text{ is prime},\\
&5 = (2+i)(2-i) = 2^2 + 1^2,\\
&7 \text{ and } 11 \text{ is prime},\\
&13 = (3+2i)(3-2i) = 3^2 + 2^2 \text{and so on.}
\end{aligned}$$

An alternative system how to define the complex âwhole numbersâ is through Eisenstein integers. Like Gaussian numbers form a square lattice in complex plane, Eisenstein integers form a triangular lattice. Every number is of shape $z = a + \omega b$ when $a, b \in \mathbb{Z}$ and $\omega = \mathrm{e}^{2\pi\mathrm{i}/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$ we call *Eisenstein integer*:

$$\mathbb{Z}[\sqrt{-3}] = \mathbb{Z}[\omega] = \{z = a + \omega b : \ a, b \in \mathbb{Z}, \ \omega = \mathrm{e}^{2\pi\mathrm{i}/3}\}\,.$$

So the Eisenstein integers also have unique factorization and every nonzero Eisenstein integer is uniquely the product of Eisenstein primes[16]. Eisenstein primes are of the shape

$$\begin{cases} a + \omega b, & \text{when } a^2 - ab + b^2 = p \text{ is 3 or prime of shape } 3k+1, \text{ or} \\ up, & \text{when } u = \{\pm 1, \pm\omega, \pm\omega^2\} \; and \; p \text{ is prime of shape } 3k+2. \end{cases}$$

It was not entirely clear whether complex integers can be always uniquely factored into prime numbers. We know that this is true for numbers containing $\sqrt{-1}$ nebo $\sqrt{-3}$. How we can factorize the numbers of shape $a + b\sqrt{-5}$? This is not unique factorization in this system of numbers. For example 6 factorizes in two different ways:

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})\,.$$

Neither of the numbers $2, 3, 1 + \sqrt{-5}$ or $1 - \sqrt{-5}$ cannot be further factorized. We have two different ways of factorization. We can ask the question: Which negative numbers in a number system $\mathbb{Z}[\sqrt{-d}]$ can be uniquely factorized? The answer is *Heegner numbers*:

$$-1, -2, -3, -7, -11, -19, -43, -67, -163.$$

For a long time mathematicians were aware of these nine numbers but the question was whether there were more numbers meeting the requirements. At the beginning of the 20th century they came to the conclusion that if other number did exist, it would be only one. It gave rise to "tenth discriminant problem". In 1936 Hans Arnold Heilbronn and Edward Linfoot showed that if other discriminant existed, it would be bigger than 10^9. In 1952 mathematician Kurt Heegner[17] produced evidence that such a tenth discriminant didn't exist and the list of nine was complete. Unfortunately, the experts didn't accept his

[16] Eisenstein number z.

[17] Kurt Heegner (1893–1965) was a German mathematician who was famous for his discoveries in number theory (see [23]).

proof because they expressed doubts about its validity. In 1966–67, two young mathematicians, Harold Stark and Alan Baker, both gave independent proofs. H. Stark also focused on Heeger's proof and two years later he confirmed its validity.

Heegner numbers have a lot of interesting qualities. It is given the formula

$$n^2 - n + k,$$

when $k > 1$. This formula represents primes for the consecutive numbers $n = 1, 2, \ldots, k-1$ as long as $1 - 4k$ is one of the Heegner numbers. Heegner number we have in case $k = 2, 3, 5, 11, 17$ a 41.

$n^2 - n + 2$	$n = 1$	2
$n^2 - n + 3$	$n = 1, 2$	$3, 5$
$n^2 - n + 5$	$n = 1, 2, 3, 4$	$5, 7, 11, 17$
$n^2 - n + 11$	$n = 1, 2, \ldots, 10$	$11, 13, 17, 23, 31, 41, 53, 67, 81, 101$
$n^2 - n + 17$	$n = 1, 2, \ldots, 16$	$17, 19, 23, 29, 37, 47, 59, 73, 89, 107, 127, 149, 173, 199, 227, 257$
$n^2 - n + 41$	$n = 1, 2, \ldots, 40$	$41, 43, 47, 53, 61, 71, 83, 97, 113, 131, 151, 173, 197, 223, 251, 281, 313, 347, 383, 421, 461, 503, 547, 593, 641, 691, 743, 797, 853, 911, 971, 1033, 1097, 1163, 1231, 1301, 1373, 1447, 1523, 1601$

Another remarkable fact of Heegner numbers is that the numbers $\mathrm{e}^{\pi\sqrt{\mathrm{d}}}$ are getting closer to integers, the bigger Heegner number d is:

$$\begin{aligned}
\mathrm{e}^{\pi\sqrt{43}} &= 884\,736\,743,999\,777\ldots \\
\mathrm{e}^{\pi\sqrt{67}} &= 147\,197\,952\,743,999\,998\,66\ldots \\
\mathrm{e}^{\pi\sqrt{163}} &= 262\,537\,412\,640\,768\,743,999\,999\,999\,999\,250\,07
\end{aligned}$$

Chess-playing programme

Readers who like playing chess must have found a big chess discovery interesting. In 1973 the Artificial Intelligence Laboratory in the Massachusetts Institute of Technology designed a special-purpose chess-playing computer.

The programme known as MacHic, was made by Richard Pinkleaf with the help of ex-world-chess-champion Mikhail Botvinnik of the U.S.S.R. Unlike most chess-playing programmes, MacHic used methods of artificial intelligence â a special learning machine that profited from mistakes by keeping records of all games in its memory and thus was steadily improving. In 1974, after many games of chess, the programme arrived at a surprising result: "It had established, with a high degree of probability, that pawn to king's rook 4 is a win for White." This was quite unexpected because such an opening move used to be regarded as poor. The machine MacHic constructed a "game tree" and analysed which position were about to win.

The chess-playing programme, described by Martin Gardner, was built by the Artificial Intelligence Laboratory in the Massachusetts Institute of Technology.

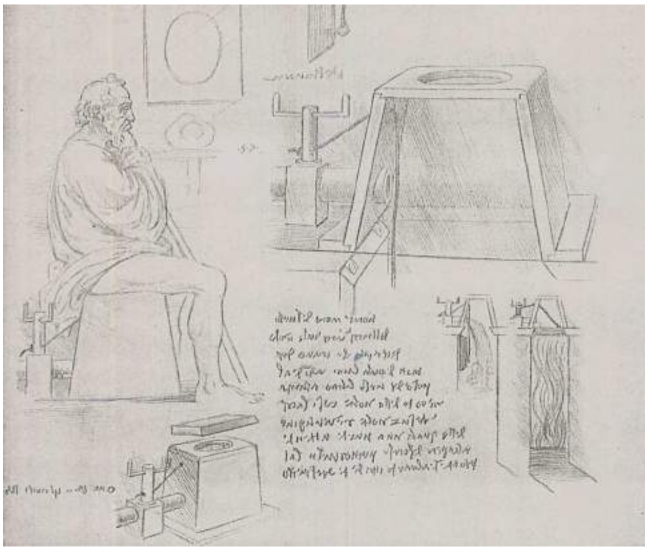
In fact, it was designed between 1966-67; and its creators were Richard D. Greenlatt and Donald E. Eastlake III. and it was known as Mac Hack or also the Greenblat chess programme. The truth is that it was a revolutionary chess programme at the time. This programme was the first one, which could simulate human conditions while playing chess. It was also the first programme, which was able to compile and analyze the game and thanks to these qualities it was able to win the game against a man. However, it never counted the probability of winning in different positions.

Unfortunately, people did not know much about computers at the time Gardner's article came out. And that's why high number of readers took this article seriously.

Discovery of the missing page from Leonardo da Vinci's notebook

For lovers of art and various inventions Gardner published a groundbreaking discovery of Leonardo da Vinci.

In the 1960s the famous manuscript of Leonardo da Vinci, known as *Codex Madrid I*, was found in the National Library in Madrid. In the manuscript there were some sketches and treatises on theoretical and applied mechanics. Later, it was discovered that one page was missing. For years, the nature of the missing page was speculated about. Augusto Macaroni of the Catholic University of Milan thought that the page might have dealt with some type of flushing mechanism, because the sketch was in a section on hydraulic devices. In December 1974 the missing page was finally found (see figure below). It turned out that A. Macaroni was right. Ramón Paz y Bicuspid came across the missing page when he browsed the 15th-century treatise on the Renaissance art of perfume making. The sketch became a great discovery, because the drawing established Leonardo as the first inventor of the valve flush toilet.



The missing page from Leonardo da Vinci's notebook.

The drawing, which is shown in the illustration, inspired Gardner to write his "discovery". In fact, Leonardo da Vinci's sketch was drawn by Anthony Ravielli, a graphic artist well known for his superb illustrations in books on sports, science, and mathematics. Gardner claims that: "Many years ago a friend of Ravielli's had jokingly made a bet with a writer that Leonardo had invented the first valve flush toilet. The friend persuaded Ravielli to do a Leonardo drawing in brown ink on faded paper. It was smuggled into the New York Public Library, stamped with a catalogue file number, and placed in an official library envelope. Confronted with this evidence, the writer paid off the bet".

The writer obviously was not the last one who was taken in by this joke. Gardner's hoax also gained entrance to Wikipedia. We can find this information under the reference to the European toilet paper holders (see [38]): "An important gap in the history of toilet paper receptacles was filled in 1974, with the discovery of a missing page from the Codex Madrid I, a notebook of Leonardo da Vinci found in Madrid's National Library in the 1960s. Ramon Paz y Bicuspid found the missing page, which verified a long-held belief that Leonardo had invented the first valve flush toilet. As you can see from the sketch (at right), the valves involved clearly double as toilet paper holders, and one of them is conveniently within an arm's-length of the seat."

The fact is that Leonardo da Vinci was one of the broad-based inventors and dealt with sewer system. One of the major advances in urban hygiene, which Leonardo wanted to implement, was an underground collection and disposal of household and street waste. This was the main cause of serious health hazards all over Europe and of several pestilence epidemics. When in 1484–86 pestilence epidemics hit Milan, Leonardo tried to create a plan for the ideal city for the French King Francouis I., where people would live better and healthier than in existing cities. The project, created by Leonardo da Vinci in 1516, was called Romorantin. A crucial role in this project played an underground collection and disposal of household and street waste. Romorantin, on the bank of the river Sauldre, is the capital of the Sologne region in France. The project also included a palace for the king with the series of flush toilets, including run-off channels in the walls inside and a ventilation system going through the roof. Unfortunately, as well as his plans for flying machines and military tanks, this project was destroyed and declared nonsense.

The name Augusto Macaroni is a wordplay on Augusto Marinoni, a da Vinci specialist at the Catholic University of Milan. Ramón Paz y Bicuspid is a play with words on Ramón Paz y Remolar, the man who actually found the two missing da Vinci notebooks.

Logical flaw in the special theory of relativity

Stunning is also the discovery of the mistake in the theory of relativity.

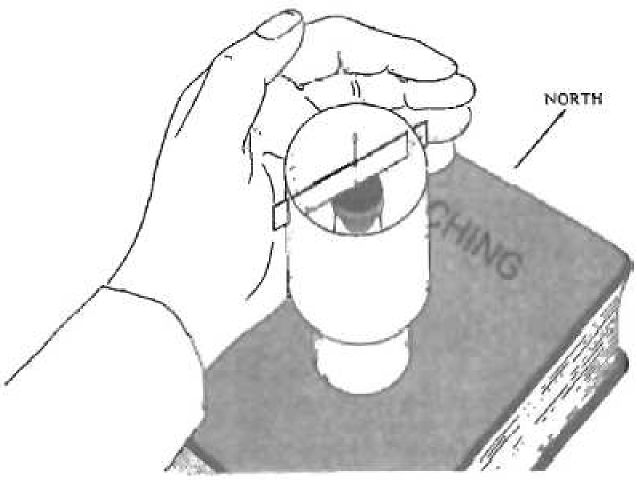
The crucial "thought experiment" is described in the paragraph about the mistake in the theory of relativity. It proves that a meter stick travels at a high speed along horizontal plate with a circular hole with diameter 1m centered. The plate is parallel with the stick's path and moves perpendicularly to it. In terms of system associated with a moving stick the situation appears that the front end of the stick exceeds a hole long before the rear section of the stick enters the hole, so the stick would not fall into the hole. These two situations are equivalent, however, and therefore the basic assumption of the theory of relativity is broken.

This relativistic paradox is well known. It is often referred to as the Meterstick and the hole paradox or Fast walking man paradox. The solution of this paradox lies in admitting the fact that rigid stick is relativistically unacceptable. The rigid sticks do not exist in relativity. The front part of the stick slightly bents at the entrance of the hole and bent stick can pass through the opening.

The psychic motor

At the end of the article there was another blockbuster waiting for the hungry readers and it was a great discovery in parapsychology in the form of a simple motor that runs on psychic energy.

The motor was constructed by Robert Ripoff, the noted Prague parapsychologist and founder of the *International Institute for the Investigation of Mammalian Auras*, in 1973. Construction of the motor is not difficult â we need paper, needle, glass bottle, and the *Bible* or the *I Ching.* And then we have no choice but to lend a hand on it, make our mind blank and focus our mental energy on the motor until the psychic energy coming from our aura takes effect and the motor starts to rotate slowly.



The psychic motor from *Scientific American* magazine.

"Psychic motor" that is shown in the picture above, is a modification of Ripoff rotor, which was described by Hugo Gernsback in the magazine *Science and Invention* (see [11]). Readers of the magazine were asked to send the explanation what makes the cylinder turn. The best response should be rewarded with $20.

$50.00 Prize Contest - Psychic Motor.

AT the right we have what is called a "psychic motor," or a motor to demonstrate what has been termed "animal magnetism." Of course there is no magnetism to it. A magnet does not effect the instrument at all.

The device shown in the illustration is easily constructed. A piece of writing paper two and one-half inches wide is glued so as to form a cylinder approximately two inches in diameter. Two holes are made diagonally opposite each other and a piece of straw pushed through them. This piece of straw or tooth-pick should extend on either side approximately a quarter of an inch. A minute drop of glue secures it to the cylinder. A needle is passed down through the center of the straw. The entire cylinder is pivoted on the needle point on top of a glass stoppered bottle. When the right hand approaches the cylinder it will be found to rotate in one direction, and when the left hand is held near the cylinder it will rotate in the opposite direction.

For the best letter explaining why the cylinder rotates, which explanation should be made in pictorial form, as nearly as possible, a first prize of $25.00 will be paid. For the second best an award of $15.00 will be made. For the third, a prize of $10.00 will be given, and for the letter ranking fourth there will be a check for $5.00. Contestants are not limited to the number of answers they may send.

In event of a tie, an identical prize will be given each. This contest closes in New York on January 10th, and all material must be in our hands by that time. Address answers to "Psychic Motor" Editor, care of this magazine.

The article about the psychic motor from *Science and Invention* magazine.

In fact, the motion can be caused by any of three forces: slight air currents in the room, convection currents produced by heat of the hand, and currents cause by breathing. The three forces could be combined in many unpredictable ways, so we cannot say nor influence, in which side the cylinder would rotate.

Martin Gardner was considered to be one of the leading polemics against pseudoscientific and fringescientific theories, astounding discoveries, the paranormal and everything what became later known as pseudoscience. He was a founding member of the *Committee for the Scientific Investigation of Claims of the Paranormal*, in short CSICOP in 1976. The mission of the committee was to promote scientific inquiry, critical investigation, and the use of reason in examining controversial and extraordinary claims. From 1983 until 2002 Gardner wrote a column in magazine *Skeptical Inquirer*. Gardner was able to influence people's opinions and also mitigate the damage caused by pseudoscientists. "Bad science contributes to the steady dumbing down of our nation", declared Gardner (see [7]). In his articles, he always tried to put all these misleading and confusing information appearing in the media straight. Although his critics considered him as a very serious man, Gardner had a playful mind. He was often rather amused than outraged of many "amazing discoveries". He and H. L. Mencken said that (see [7]): "one horselaugh is worth ten thousand syllogisms." And he hoped that his readers would understand his article.

Several psychic motor appeared on sale after Gardner's hoax. Most readers, however, did not take this discovery seriously.

When Gardner published his article, he did not have the faintest ideas of the great acclaim he was going to get. Gardner claimed that the main purpose of the article was to entertain his readers. To his surprise, both the public and also professionals did not understand his joke and took the article seriously. He got thousands of letters from mathematicians and physicists. Many readers pointed out that he made a mistake in his article — but only in the field they specialized in, but the rest they considered unquestionable. Speaking up for the cheated readers, Gardner's article was a truly brilliant hoax in full details. When readers read only one paragraph, about the topic they were interested in, it was not clear at first sight that it was a hoax.

Gardner was one of the few authors who signed the article and proudly claimed responsibility for it. Benevolent readers forgave him soon. Everyone loved Gardner's columns for their originality, playfulness and witty spirit.

Legislating the value of π

Human imagination, however, will never be wild enough for some politicians' decisions. In 1998 *New Mexicans for Science and Reason*[18] magazine published the report that the Alabama state legislature passed a law redefining a mathematical constant π.

The law was passed on March 30, 1998, redefining the value of π to exactly integer three. The law took the state's scientific community by surprise ([25]): "It would have been nice if they had consulted with someone who actually uses π." Mathematicians form University of Alabama tried to explain to state legislature that π is a universal constant, and cannot be arbitrarily changed by lawmakers. In addition, we can never express it exactly because π is an irrational number, which means that it has an infinite number of digits after the decimal point. However, the mover did not listen to any arguments (see [25]): "I think that it is the mathematicians that are being irrational, and it is time for them to admit it. The Bible very clearly says in I Kings 7:23 that the altar font of Solomon's Temple was ten cubits across and thirty cubits in diameter, and that it was round in compass." On the contrary, he called into question the usefulness of any number that cannot be calculated exactly, and suggested that not knowing the exact answer could harm students' self-esteem (see [25]): "We need to return to some absolutes in our society, the Bible does not say that the font was thirty-something cubits. Plain reading says thirty cubits. Period." The members of the state school board supported the change in value of π, but they believed that the old value should be retained as an alternative (see [25]): "...the value of π is only a theory, and we should be open to all interpretations."Their idea was that students would be given the freedom to decide for themselves what value π should have.

Somehow bewildering is the fact that this joke was probably inspired by true events. In 1897, the legislature of Indiana attempted to give π an exact value by law. The Indiana Pi Bill was fabrication of Edwin J. Goodwin. This man was convinced he had found the solution of "squaring the circle" - an ancient problem whose task is to construct a square with a pair of compasses and a pair of scissors. This problem had been infatuated many thinkers, but it had already been proven impossible in 1882. Among other, Goodwin's contradictory explanation contained argument regarding relating to the diameter of a circle (see [31]): "...the fourth important fact, that the ratio of the diameter and circumference is as five-fourths to four."

We know that the ratio of the diameter to the circumference is equal to π, so

[18] *New Mexicans for Science and Reason* is a non-profit group with the goals of promoting science, the scientific method, rational thinking, and critical examination of dubious or extraordinary claims (see [25]).

Goodwin was effectively dictating a value for π according to the following recipe:

$$\pi = \frac{\text{circumference}}{\text{diameter}} = \frac{4}{\frac{5}{4}} = 3,2$$

Goodwin reportedly said (see [31]): "Indiana schools could use his discovery without charge, but that the state and he would share the profits from royalties charged to other schools who wished to adopt a value of 3,2 for π." The technical nature of the bill so baffled the politicians that it being passed without any objection.

However, considerations of whether the number π can be expressed as an integer constant, are completely misleading. The number π, which is approximately $3,141\,592\,6$, is the length of the circumference of a circle whose diameter is exactly 1. Generally speaking, a circle of diameter d has a circumference of $\pi\, d$. The number π cannot be expressed as a fraction and it really is an irrational number. The simplest proof that p is irrational uses calculus, and it was founded by Johann Lambert [19] in 1770. More strongly, π is transcendental. It means that it does not satisfy any algebraic equation that relates it to rational numbers. In 1882, this property was proved by Ferdinand Lindemann, [20] also using calculus. The fact that π is transcendental implies that the classical geometric problem of 'squaring the circle' is impossible.

Although there were lots of evidences suggesting deception in the article, for example, the fictitious names of people who wanted to enforce the law, thanks to the Internet the article was passed on to many different countries. In a short time the article spreaded in the world. But, as the story was forwarded from person to person, all of the deliberate hints disappeared. The names, which were originally fictitious, began to be gradually replaced by the names of particular living people working at the university or in the Alabama state legislature. Gradually, the newer and better versions of the original article began to appear, cutting their way through the internet. All funny hints from the article disappeared and it became more and more credible. As the general public was more and more acquainted with the article, the Alabama state legislature began to receive hundreds of letters and phone calls from people who protested against this legislation.

Conclusion

The previous chapters contain cross-section of fool hoax articles from the past to the present. The fact is that both the daily press or in the serious magazine, the authors of fool hoax articles follow a similar goal - to entertain its readers

[19]Johann Heinrich Lambert (1728â1777) was a Swiss mathematician, physicist, philosopher and astronomer. He is best known for proving the Irrationality of π. He was the first to introduce hyperbolic function into trigonometry. Also, he made conjecture regarding non-Euclidean space. In Photometria Lambert also formulated the law of light absorption â the Beer-Lambert law (see [15]).

[20]Carl Louis Ferdinand von Lindemann (1852â1939) was a German mathematician, noted for his proof, published in 1882, that π is a transcendental number. His methods were similar to those used nine years earlier by Charles Hermite to show that , the base of natural logarithms, is transcendental (see [6]).

and force them to critically assess the content of the report and while watching TV or reading newspapers to ask themselves whether the given report may be true or not. But the question is, if this is not a realistic goal.

Not all information can be verified and not all are connected to the lack of knowledge among the recipients. And thus created simple or complex judgments about the facts. Every person, even the great professionals, has many ideas about different things, people, institutions, countries, etc. Modern man can not cover all the knowledge and information that surround it. Expertise based on knowledge in a particular field then exposes humans to receive and formed the only opinion in other fields. These opinions - unlike knowledge - have a strong irrational character and are not only view but also a conviction.

In everyday life we rarely verify the information that we hear from the media. Social life is based on trust that the task of verifying the report is commissioned by someone. If we read the report in the newspaper, we assume that it is proven, though for we have no proof. We rely on responsibility and professional duties of journalists.

The authors of the hoax articles are always surprised and shocked by the apparent lack of will verify the facts, which are given in the article (see [20]): "The journalistic profession the verifying, what is then further distributed to thousands of people, is a fundamental requirement. Leading French journalist Jean Lacouture reminds aptly that the role of the journalist is not so much to spread the message about the birth or death of the king, but rather to refute or confirm rumors that one or the other would accelerate, accompanied or distorted." The fact that it is necessary to inculcate future journalists verification reflex, mean that this kind of reaction is not spontaneous. Why do we take for normal that nobody â or almost nobody â not verifies the information in the newspaper? There are some situations when we try to verify information from a newspaper or other media, especially if the report is born on us, for example stock ticker, military or other major decisions. On the other hands, the cases if we act without apparent risk eliminate the need for verification. If we are not forced to make any decisions, the motivation for authentication is missing. Only the professional skeptics (such as journalists) or those with their own personal interests can make an effort to explore more information.

Moreover, we cannot argue that the people believe everything what the media give them. If we read the report in the newspaper or magazine, tacitly assume that the news passed the filter in some groups, such as the editorial board, before the news came to us. If it was a hoax, it would have not believed so many people would not get to us.

For the media, which created a ethical code and behave according to him, this code can be considered a guarantee objectivity and proper etiquette. Still, however, the rules of behavior by the ethical code are not in principle for the public enforceable. It depends only on the media; whether they will act as powerful without liability or whether they will aspire higher and look for what is one of the deepest level of humanity â the good and the associated truth. The problem of the April articles is that although the media behave unethically in that moment, it still remains within the law. Then the public can do nothing else but be a little indulgent and taken the April cells with reserve.

The question is whether, in the upcoming years, the meaning of April articles will tend to decrease, as even completely serious news often seem unbelievable and people are confused whether the April Fool's Day is not in December. But now, we can do nothing but to look forward to April and new kinds of joke the professionals, for sure, are coming up with.

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Math and Fun with Algorithms

FINDING THE YEAR'S SHARE IN DAY-OF-WEEK CALCULATIONS

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Abstract: *The dominant part in the mental calculation of the day of the week for any given date is to determine the* year share*, that is, the contribution of the two-digit year part of the date. This paper describes a number of year share computation methods, some well-known and some new. The "Parity Minus 3" method, in particular, is a new alternative to the popular "Odd+11" method. The paper categorizes the methods of year share computation, and presents simpler proofs of their correctness than usually provided.*

Keywords: day of the week, calendar algorithms, doomsday method, first Sunday algorithm, mental arithmetic, year share.

Introduction

Finding the day of the week (DOW) for any given date is by now a trivial computational problem. While writing a program from scratch for computing the DOW is not difficult, most programming languages include libraries that provide routines for this purpose. Moreover, all existing computer operating systems and "office" applications have built-in facilities for this computation. For a discussion of several DOW algorithms best suited for programming, see the Wikipedia article "Determination of the day of the week" [1]. A very comprehensive reference on DOW calculations in general is the German book *Enzyklopädie der Wochentagsberechnung* by Hans-Christian Solka[2].

While there is scarcely any need anymore for a new DOW algorithm for computers, the interest in *mental calculation* algorithms continues unabatedly. The best known, and undoubtedly one of the best, of such methods is the "Doomsday Rule" [3] invented in 1973 by John Conway. Another method with much merit, and for most people the easiest to use, is the "First Sunday

Algorithm"[4] introduced by Robert Goddard in 2009[1]. Such methods proceed by isolating and finding the contributions of the century, two-digit year, month, and day parts of the date, and then adding or subtracting these contributions to form a total. The total is an integer which represents the serial of the DOW being sought in some ordering of the days. (A common convention is to assign the numbers 0, 1, 2, ..., 6 to Sunday, Monday, Tuesday, ..., Saturday, respectively.) If the total happens to not be in the range 0, 1, 2, ..., 6, it is simply reduced modulo 7 to yield a value in that desired range. Note that since the total is formed by adding or subtracting various terms, each of those terms can also be individually reduced modulo 7 as soon as it is computed or at any stage during summation.

The century and month part contributions typically amount to constants that one memorizes, and the contribution of the day part typically consists of counting weekdays forward or backward from a date determined by previous steps. It is the contribution of the two-digit year part that takes most time (the largest number of seconds!) in mental calculation. This is the case in both the Doomsday Rule and the First Sunday Algorithm.

The DOW of any fixed date within a year (i.e., a month-day combination) is advanced one day by every common year and two days by every leap year. For a (two-digit) year y within a century, the DOW advances y days because of common years and $\lfloor \frac{y}{4} \rfloor$ days because of leap years, making the total DOW advance from the century year to the year y equal to $y + \lfloor \frac{y}{4} \rfloor$, that is,

$$\left\lfloor \frac{5y}{4} \right\rfloor \tag{1}$$

We use the term *year share* to refer to the expression (1) that represents the contribution of the two-digit year part of a date to its DOW. Of course, any expression congruent modulo 7 to (1) serves our purpose equally well. In some methods the year share is subtracted while forming the total sum, so they compute a value which equals (or is congruent modulo 7 to) the negative of the expression (1).

When a new method of computing the year share is proposed, it has to be proven correct by showing that its result is congruent modulo 7 to the expression (1), or to minus this expression when that's what the method claims to compute. While the correctness of many methods is quite obvious and hardly requires a proof, for some methods it is not immediate. But in the published account of these methods, the proofs are often missing or are unnecessarily complicated. We have tried to provide correctness proofs by very similar, simple arguments. It is hoped that by treating the methods in a uniform way the present approach will provide a better understanding of the methods and will be helpful in devising new methods and better variants of old methods.

[1] Hans-Christian Solka has kindly informed me that methods equivalent to First Sunday were published by the German magician C. Willmann in 1896, by E. Rogent and W.W. Durbin in 1927, and by some other authors later as part of *Dominical letter* research. The references, to which I don't have access, are in [2].

Methods of Year Share Computation

A number of alternative methods of year share computations have appeared in the literature. These methods work essentially by substituting some expression for the variable y in (1) and transforming the resulting expression into one that is easy to evaluate by mental arithmetic. The transformations try to minimize the steps in the integer divisions by 4 and reduction modulo 7, and, of course, to maximize the work with small integers.

We describe a selection of known methods here and also suggest a few new ones (Methods 2, 5c, 5f, 7, and 10). We have placed the methods into three categories of (1) "special" methods that avoid division by a divisor other than 2, (2) division by integers larger than 2, and (3) operations on the individual digits of the two-digit year. The category (3) is really a subcategory of (2) because its methods involve division by 10. However, the methods in (3) require somewhat different mental arithmetic.

Special methods

The overriding advantage of these methods is that you need to keep manipulating only a single variable (testing it, increasing or decreasing it, halving it). By contrast, the methods of the next sections require you to remember and work with several numbers.

1. **Odd+11 Method**

   This method, evolved from an idea proposed by Michael Walters in 2008, is described more formally in a 2011 paper by Chamberlain Fong and Michael Walters[8]. Of all the present alternatives for year share computation, this method seems to be the quickest.

   The "Odd+11" operation takes a given year y (between 0 and 99, inclusive) as input and produces the negative of the year share as output, by proceeding as follows:

   i. Set the value of YS to that of y. In symbols, $YS \leftarrow y$.
   ii. If YS is odd, then increase it by 11, i.e., $YS \leftarrow YS + 11$, else leave it unchanged.
   iii. Halve YS, i.e., $YS \leftarrow YS/2$.
   iv. If YS is odd, increase it by 11, i.e., $YS \leftarrow YS + 11$, else leave it unchanged.

   Fong and Walter[8] add two more steps of doing $YS \leftarrow YS \bmod 7$ and then $YS \leftarrow 7 - YS$ to turn the result into a positive year share. We omit these steps since the negative year share is what the "First Sunday Algorithm" needs anyway, and, moreover, any result in the mod 7 congruence class is acceptable.

   To show that the result of applying "Odd+11" to the input y is congruent modulo 7 to the negative of (1), we first express y as a polynomial as follows: Divide y by 4; call the quotient a; divide the remainder (whose value is between 0 and 3, inclusive) by 2; call the new quotient b and the new remainder c. We can now write

$$y = 4a + 2b + c, \tag{2}$$

where $0 \leqslant b \leqslant 1$ and $0 \leqslant c \leqslant 1$.

Let us first apply the steps of "Odd+11" to this value of y.

i. $$YS = 4a + 2b + c.$$

ii. If YS is odd, i.e., if $c = 1$, then increase YS by 11, else leave YS unchanged. This is the same as writing

$$YS = 4a + 2b + 12c.$$

iii. Halve YS, i.e.,
$$YS = 2a + b + 6c.$$

iv. If YS is odd, i.e., if $b = 1$, then increase YS by 11, else leave YS unchanged. This is the same as writing

$$YS = 2a + 12b + 6c. \tag{3}$$

Next, let us evaluate the negative of the expression (1) for the value of y given by (2).

$$-\left\lfloor \frac{5y}{4} \right\rfloor = -\left\lfloor \frac{20a + 10b + 5c}{4} \right\rfloor = -\left(5a + 2b + c + \left\lfloor \frac{2b + c}{4} \right\rfloor\right) = -5a - 2b - c \tag{4}$$

As (3) and (4) differ by a multiple of 7 (viz. $7a+14b+7c$), they are congruent modulo 7.

Fong and Walters's own proof in [8] is longer and more complicated.

2. **Parity Minus 3 Method**

This new method is inspired by and very similar to "Odd+11", but involves smaller integers. The "Parity Minus 3" operation takes a given year y (between 0 and 99, inclusive) as input and produces the negative of the year share as output, by proceeding as follows:

i. Set the value of YS to that of y. In symbols, $YS \leftarrow y$.

ii. Check and remember YS's parity (odd or even). If YS is odd, then decrease it by 3, i.e., $YS \leftarrow YS - 3$, else leave it unchanged.

iii. Halve YS, i.e., $YS \leftarrow YS/2$.

iv. If YS's parity (odd or even) has changed, decrease YS by 3, i.e., $YS \leftarrow YS - 3$, else leave it unchanged.

Examples:

(a) $y = 24$
Step 1: $YS = 24$. Step 2: Even, hence $YS = 24$. Step 3: Halve, so $YS = 12$. Step 4: Even, so parity unchanged, hence answer is $YS = 12$.

(b) $y = 37$
Step 1: $YS = 37$. Step 2: Odd, hence $YS = 37 - 3 = 34$. Step 3: Halve, so $YS = 17$. Step 4: Odd, so parity unchanged, hence answer is $YS = 17$.

(c) $y = 58$
Step 1: $YS = 58$. Step 2: Even, hence $YS = 58$. Step 3: Halve, so $YS = 29$. Step 4: Odd, so parity changed, hence answer is $YS = 29 - 3 = 26$.

(d) $y = 79$
Step 1: $YS = 79$. Step 2: Odd, hence $YS = 79 - 3 = 76$. Step 3: Halve, so $YS = 38$. Step 4: Even, so parity changed, hence answer is $YS = 38 - 3 = 35$.

To show that the result of applying "Parity Minus 3" to the input y is congruent modulo 7 to the negative of (1), we first express y as a polynomial as follows: Divide y by 4; call the quotient a; divide the remainder (whose value is between 0 and 3, inclusive) by 2; call the new quotient b and the new remainder c. We can now write

$$y = 4a + 2b + c, \tag{5}$$

where $0 \leqslant b \leqslant 1$ and $0 \leqslant c \leqslant 1$.

Let us first apply the steps of "Parity Minus 3" to this value of y.

i.
$$YS = 4a + 2b + c.$$

ii. If YS is odd, i.e., if $c = 1$, then decrease YS by 3, else leave YS unchanged. This is the same as writing

$$YS = 4a + 2b - 2c.$$

iii. Halve YS, i.e.,
$$YS = 2a + b - c.$$

iv. Since each of b and c is either 0 or 1, the new parity of $YS = 2a + b - c$ is odd or even depending, respectively, on whether $b \neq c$ or $b = c$. The old parity of YS determined in Step 2 of the method was odd or even depending on whether $c = 1$ or $c = 0$. Thus the new parity is different from or same as the old one depending on whether $b = 1$ or $b = 0$. YS is to be decreased by 3 if the parity has changed, i.e., $b = 1$, and is to be left unchanged if $b = 0$. This is the same as writing

$$YS = 2a - 2b - c. \tag{6}$$

Next, let us evaluate the negative of the expression (1) for the value of y given by (5).

$$-\left\lfloor \frac{5y}{4} \right\rfloor = -\left\lfloor \frac{20a+10b+5c}{4} \right\rfloor = -\left(5a+2b+c+\left\lfloor \frac{2b+c}{4} \right\rfloor\right) = -5a-2b-c \tag{7}$$

As (6) and (7) differ by a multiple of 7, they are congruent modulo 7.

Division by various integers

3. **Division by 12**

In the original version of the Doomsday Rule[3], Conway states the method (originally due to Lewis Carroll, see Gardner[5]) to compute the year share as follows:

"add the number of *dozens* [... in y], the *remainder* after [... the dozens are taken out], and the number of *fours* in the *remainder*".

That is, the year share is

$$\left\lfloor \frac{y}{12} \right\rfloor + y \bmod 12 + \left\lfloor \frac{y \bmod 12}{4} \right\rfloor. \tag{8}$$

To prove that the method computes the year share correctly, let us write

$$y = 12q + r, \text{ where } 0 \leqslant r < 12.$$

Now (8) can be rewritten as

$$q + r + \left\lfloor \frac{r}{4} \right\rfloor \tag{9}$$

With $12q + r$ substituted for y in (1), year share equals

$$\left\lfloor \frac{5y}{4} \right\rfloor = \left\lfloor \frac{60q+5r}{4} \right\rfloor = 15q + r + \left\lfloor \frac{r}{4} \right\rfloor. \tag{10}$$

Since (9) and (10) differ by a multiple of 7, they are congruent modulo 7.

4. **Division by 4**

This method called *Highest Multiple of Four* by YingKing Yu[6] uses a very simple, easy to remember calculation to produce the negative of the year share. Solka[2] credits Carl Willmann with a much earlier equivalent method. One thinks of the two-digit year as the sum of a multiple of 4 and a remainder which is 0, 1, 2, or 3. Then the method computes "half of that multiple of 4, minus the remainder". Incidentally, for some people "the closet multiple of 4 not larger than the given year" is easier to remember as the current or previous "leap year", "Olympics year", or the "US Presidential Election year".

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To prove that the method works correctly, let us write $y = 4q + r$, where $0 \leqslant r < 4$. Then the output of the method is $2q - r$. That the value of the negative of the expression (1) when $4q + r$ is substituted for y is congruent modulo 7 to the expression $2q - r$ is shown as follows:

$$-\left\lfloor \frac{5y}{4} \right\rfloor = -\left\lfloor \frac{20q + 5r}{4} \right\rfloor = -(5q + r + \left\lfloor \frac{r}{4} \right\rfloor) = -5q - r \equiv 2q - r \pmod 7$$

5. **Division by other integers**

Divisions by 4 and 12 furnish nice methods, as we have seen above, because these divisions result in formulas that require simple arithmetic. Division by 10 also has nice properties and will be covered in the next section. The divisors 5, 11, 16, and 17 also lead to simple formulas for the year share. Solka[2] gives formulas for several divisors, including 8, 12, 16, 20, and 24, grouped together into a "universal approach" section.

In general, for a divisor d, we write $y = dq + r$, where $0 \leqslant r < d$, and evaluate the year share expression (1), i.e., $\left\lfloor \frac{5dq+5r}{4} \right\rfloor$ (or its negative). After reducing $5d$ modulo 28, we can expand the expression into the form $aq + r + \left\lfloor \frac{bq+r}{4} \right\rfloor$ for some integers a and b. We can further play with this expression in various ways, e.g., increase a by any integer k and compensate for that change by decreasing b by $4k$. Below we show only those divisors d between 5 and 20 for which a and b have values $0, +1$, or -1, because any other value would require extra multiplications that would complicate mental arithmetic. For the sake of comparison, we include the formulas for divisors 4 and 12 given in the previous sections.

(a) $d = 4$:

$$\text{Negative year share} = 2q - r.$$

(b) $d = 5$:

$$\text{Negative year share} = q - r - \left\lfloor \frac{q + r}{4} \right\rfloor.$$

(c) $d = 11$:

$$\text{Positive year share} = r + \left\lfloor \frac{r - q}{4} \right\rfloor.$$

Note that the dividend $r - q$ in the expression $\left\lfloor \frac{r-q}{4} \right\rfloor$ can be negative. The evaluation of integer quotients in such cases requires some extra care. This is discussed at the beginning of Section .

(d) $d = 12$:

$$\text{Negative year share} = q + r + \left\lfloor \frac{r}{4} \right\rfloor.$$

(e) $d = 16$:

$$\text{Positive year share} = -q + r + \left\lfloor \frac{r}{4} \right\rfloor.$$

(f) $d = 17$:

$$\text{Positive year share} = r + \left\lfloor \frac{q + r}{4} \right\rfloor.$$

Methods operating on the year's individual digits

The advantage of these methods is that they involve arithmetic with smaller numbers than those arising in the methods of the previous category, and dividing these numbers by 4 or reducing them modulo 7 is quite easy to do mentally.

The earliest such method is credited by Solka[2] to L.T. Sakharovski. This method, published in 1957, has assigned codes to the tens and units digits of the year, and these codes are added together to get the year share. As our interest is mainly in year share *computation*, we will not reproduce Sakharovski's table (given in [2]), and will describe five methods that do this computation in various ways.

Method 6 is different from the other methods in this section because it operates on the tens and units digits not of the two-digit year but of the highest multiple of four not exceeding the year. Method 7 seems to be the simplest, but in Method 10 we work with only one number at a time (while keeping one sign in memory), similarly to "Odd+11" or "Parity Minus 3".

Some of the methods in the present category involve integer divisions with the dividends allowed to be negative. (However, the divisor is required to be positive.) Our needed congruences will hold only if we define the integer quotient in such a division as follows:

$$\left\lfloor \frac{p}{q} \right\rfloor = \begin{cases} \left\lfloor \frac{p}{q} \right\rfloor, & \text{if } p \geqslant 0 \text{ (and } q > 0) \\[2ex] -\left\lfloor \frac{|p|}{q} \right\rfloor, & \text{if } p < 0 \text{ and } |p| \bmod q = 0 \text{ (and } q > 0) \\[2ex] -\left(\left\lfloor \frac{|p|}{q} \right\rfloor + 1 \right), & \text{if } p < 0 \text{ and } |p| \bmod q > 0 \text{ (and } q > 0) \end{cases}$$

Methods 7 through 10 have much in common and are essentially variations on the same theme. Suppose the tens and units digits of the two digit-year part are, respectively, t and u. That is $y = 10t + u$. Let's evaluate the negative of the year share expression (1) in terms of t and u:

$$-\left\lfloor \frac{5y}{4} \right\rfloor = -\left(\left\lfloor \frac{50t + 5u}{4} \right\rfloor \right) \tag{11}$$

Any expression derived by adding a multiple of 28 to the numerator of the fraction in (11) is obviously congruent modulo 7 to (11), and is hence just another expression for the negative year share. Examples are:

$$-\left(\left\lfloor \frac{22t + 5u}{4} \right\rfloor \right) \tag{12}$$

$$-\left(\left\lfloor \frac{-6t + 5u}{4} \right\rfloor \right) \tag{13}$$

Method 7, 8, and 9 are derived from (12) with or without the negative sign, and Method 10 in essence evaluates (13) directly.

6. **Computing with Year's Individual Digits (Eisele)**

The following method by Martin Eisele (citation in [2]) is unique as it operates on the tens and units digits not of the given year but of the largest multiple of four not exceeding that year. Let y be the two-digit year part of the date. Let q and r be the integer quotient and remainder when y is divided by 4, that is,

$$y = 4q + r, \text{ where } 0 \leqslant r < 4. \tag{14}$$

Let t and u be the tens and units digits of $4q$, that is,

$$4q = 10t + u, \text{ where } 0 \leqslant u < 10. \tag{15}$$

Then, according to Eisele, the year share is

$$2t - \frac{u}{2} + r. \tag{16}$$

Note that u must be even since by (15) $10t + u$ is a multiple of 4. Thus we have

$$2q = 5t + \frac{u}{2}, \text{ where } u = 0, 2, 4, 6, \text{or } 8. \tag{17}$$

To show the correctness of Eisele's method, we verify that the year share expression (1) with $4q + r$ substituted for y is congruent modulo 7 to (16):

$$\begin{aligned}
&\left\lfloor \frac{5y}{4} \right\rfloor = \left\lfloor \frac{20q + 5r}{4} \right\rfloor, \text{ by (14)} \\
&= 5q + r + \left\lfloor \frac{r}{4} \right\rfloor = 5q + r \equiv -2q + r \pmod 7 = -5t - \frac{u}{2} + r, \text{ by (17)} \\
&\equiv 2t - \frac{u}{2} + r \pmod 7
\end{aligned}$$

A method by Alexander Harringer (see [2]) turns out to be a variation of the above method. For computing the year share, Harringer proposes the formula

$$2t + 3u + r. \tag{18}$$

instead of Eisele's (16). Notice that (18) and (16) differ by the quantity $\frac{7u}{2}$. As u is an even number, this quantity is a multiple of 7, and hence the formulas by Harringer and Eisele are congruent modulo 7.

7. **Computing with Year's Individual Digits (Aa)**

Let the tens digit and units digit in the year part be t and u, respectively. That is, $y = 10t + u$. Then the negative of the year share can be found in this way:

i. Compute $\left\lfloor \frac{2t+u}{4} \right\rfloor$.
ii. Add u to above.
iii. Subtract the above sum from $2t$.
This is the year share.

Example. Suppose *year* is 59. Then:

i. From $t = 5$ and $u = 9$, we compute $2t + u = 19$ and then its quarter which is 4.
ii. Adding $u = 9$ to it, we get 13.
iii. Subtracting this from $2t = 10$, we get $10 - 13 = -3$. This is the negative year share. We can reduce it modulo 7 immediately to 4 or just leave it as -3 to be reduced modulo 7 in a later step of DOW calculation.

The result of performing the steps of the method with input digits t and u is

$$2t - \left(\left\lfloor \frac{2t + u}{4} \right\rfloor + u \right) \tag{19}$$

To show that this is the negative year share, we evaluate the negative of expression (1) with $10t + u$ substituted for y.

$$\begin{aligned} -\left\lfloor \frac{5y}{4} \right\rfloor &= -\left\lfloor \frac{50t + 5u}{4} \right\rfloor = -\left(12t + u + 2t + \left\lfloor \frac{2t + u}{4} \right\rfloor\right) \\ &= -12t - \left(\left\lfloor \frac{2t + u}{4} \right\rfloor + u \right) \end{aligned} \tag{20}$$

Since (19) and (20) differ by a multiple of 7, they are congruent modulo 7.

8. Computing with Year's Individual Digits (Fong)

The following method by Chamberlain Fong[7] operates on the year's individual digits and computes the positive year share. (Fong also credits YingKing Yu with this method, and cites Yu's work in [7].) Let t and u be, respectively, the tens and units digits of the year. Then the method computes the year share as

$$2t + 10(t \bmod 2) + u + \left\lfloor \frac{2(t \bmod 2) + u}{4} \right\rfloor . \tag{21}$$

To prove that the year share so computed is correct, we proceed as follows. Substituting $t = 2t_1 + t_2$, where $0 \leqslant t_2 \leqslant 1$, we write (21) as

$$2(2t_1 + t_2) + 10t_2 + u + \left\lfloor \frac{2t_2 + u}{4} \right\rfloor = 4t_1 + 12t_2 + u + \left\lfloor \frac{2t_2 + u}{4} \right\rfloor . \tag{22}$$

Substituting $t = 2t_1 + t_2$, hence $y = 10t + u = 20t_1 + 10t_2 + u$, we evaluate the expression (1) for year share as follows:

$$\left\lfloor \frac{5y}{4} \right\rfloor = \left\lfloor \frac{100t_1 + 50t_2 + 5u}{4} \right\rfloor = 25t_1 + 12t_2 + u + \left\lfloor \frac{2t_2 + u}{4} \right\rfloor \tag{23}$$

Since (22) and (23) differ by a multiple of 7, they are congruent modulo 7.

9. Computing with Year's Individual Digits (Wang)

This method by Xiang-Sheng Wang[9] computes the positive year share. Let t and u be, respectively, the tens and units digits of the year. Then the method computes the year share as

$$u - t + \left\lfloor \frac{u}{4} - \frac{t}{2} \right\rfloor . \tag{24}$$

To prove that the year share so computed is correct, we evaluate the negative of the year share expression (1) with $10t + u$ substituted for y.

$$\left\lfloor \frac{5y}{4} \right\rfloor = \left\lfloor \frac{50t + 5u}{4} \right\rfloor = \left\lfloor \frac{4u + 52t + u - 2t}{4} \right\rfloor = u + 13t + \left\lfloor \frac{u - 2t}{4} \right\rfloor$$
$$= u + 13t + \left\lfloor \frac{u}{4} - \frac{t}{2} \right\rfloor. \tag{25}$$

Since (24) and (25) differ by a multiple of 7, they are congruent modulo 7.

Note that the fraction in (24) can be negative, so its floor has to be evaluated carefully following the procedure stated at the beginning of Section .

10. **Computing with Year's Individual Digits (Ab)**

Here is another method that operates on the individual digits of a two-digit year. Let the tens digit and units digit in the year part be t and u, respectively; that is, $y = 10t + u$. Then the negative year share is found in this way:

i. Compute $5u - 6t$, and let its absolute (positive) value be a. Also remember its sign ('plus' or 'minus').
ii. Compute $b = \left\lfloor \frac{a}{4} \right\rfloor$. If there was a non-zero remainder, and the sign in the previous step was 'minus', increase b by 1.
iii. Affix the opposite sign to b. (That is, make it $-b$ if the sign was 'plus', and $+b$ i.e., just b if the sign was 'minus'.) This result is the negative year share.
   If this value turns out to be negative or larger than 6, we can reduce it modulo 7 immediately or just leave it as is to be reduced modulo 7 in a later step of DOW calculation.

Example. Suppose *year* is 87. Then:

i. From $t = 8$ and $u = 7$, we compute $5u - 6t = 5 \times 7 - 6 \times 8 = 35 - 48 = -13$. So we have $a = 13$ and sign = 'minus'.
ii. By dividing a by 4, we get $b = 3$. Since there was a nonzero remainder in the division, and the remembered sign is 'minus', we add 1 to b, making $b = 4$.
iii. Since the remembered sign is 'minus', we attach the opposite sign to b, making it +4, i.e., 4. So the negative year share is 4.

The result of performing the steps of the method with input digits t and u is

$$-\left\lfloor \frac{5u - 6t}{4} \right\rfloor \tag{26}$$

To show that this is the negative year share, we evaluate the negative of expression (1) with $10t + u$ substituted for y.

$$-\left\lfloor \frac{5y}{4} \right\rfloor = -\left\lfloor \frac{50t + 5u}{4} \right\rfloor = -\left\lfloor \frac{5u - 6t + 56t}{4} \right\rfloor \equiv -\left(\left\lfloor \frac{5u - 6t}{4} \right\rfloor + 14t\right) \tag{27}$$

Since (26) and (27) differ by a multiple of 7, they are congruent modulo 7.

Concluding remarks

This article is concerned only with the year share part of DOW calculation, not with any other details of the DOW computation methods. The year share part is where most of the calculation time is spent. We have tried to describe the methods with a uniform, systematic approach, and have provided simple proofs of their correctness.

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Games and Puzzles

QUANTUM DISTRIBUTION OF A SUDOKU KEY

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Abstract: *Sudoku grids are often cited as being useful in cryptography as a key for some encryption process. Historically transporting keys over an alternate channel has been very difficult. This article describes how a Sudoku grid key can be secretly transported using quantum key distribution methods whereby partial grid (or puzzle) can be received and the full key can be recreated by solving the puzzle.*

Keywords: sudoku, quantum key distribution methods.

Sudoku grids

A Sudoku grid is a 9×9 array which is further subdivided into "mini-grids" of size 3×3, with each of the 81 cells of the grid containing the values 1 to 9 such that each appears exactly once in each row, column and mini-grid. A Sudoku puzzle contains an incomplete assignment of values to a grid, with the goal of the puzzle being to complete the assignment. The values shown in the puzzle are referred to as givens or predefined cells and are unmovable. Published puzzles should contain a selection of givens chosen carefully to ensure that a solution is unique and a range of types of reasoning may be required to find the solution of a puzzle. A Sudoku puzzle, and its solution is presented in Figure 1.

						2		
	8				7		9	
6		2				5		
	7			6				
			9		1			
				2			4	
		5				6		3
	9		4				7	
		6						

(a) Sudoku Puzzle

9	5	7	6	1	3	2	8	4
4	8	3	2	5	7	1	9	6
6	1	2	8	4	9	5	3	7
1	7	8	3	6	4	9	5	2
5	2	4	9	7	1	3	6	8
3	6	9	5	2	8	7	4	1
8	4	5	7	9	2	6	1	3
2	9	1	4	3	6	8	7	5
7	3	6	1	8	5	4	2	9

(b) Sudoku Grid

Figure 1: An example sudoku puzzle and its solution [6].

Although published Sudoku puzzles are generally 9×9 in size, other dimensions can be used, and for every non-prime dimension n there is an $n \times n$ Sudoku grid [4]. However not every size of Sudoku has unique size mini-grids. As examples, a 6×6 Sudoku (known as Rudoku) can have mini-grids of size 3×2 or 2×3 (although these are essentially simply a rotation from one to the other), and a 12×12 Sudoku can have mini-grids of size either 3×4 or 6×2 (leading to very different puzzles).

Recently Sudoku grids have been popular for use in cryptographic systems [3, 5, 7, 10, 11] using a variety of methods. In this article we are not going to go into any detail regarding encryption methods using Sudoku grids, instead this article will detail how a key formed using a Sudoku grid can be shared using quantum key distribution using a method similar to the BB84 protocol described in [2].

Throughout this article we will refer to a sender of the message (Alice) and a receiver (Bob). Alice and Bob are two commonly used placeholder names. The names are used for convenience; for example, "Alice sends a message to Bob" is easier to follow than "Party A sends a message to Party B".

Quantum key distribution

Quantum key distribution relates to a branch of cryptography based on physics and quantum mechanics first published in [1]. Quantum cryptographic methods no longer considered to be a secure as once thought, however there are modifications to be able to cope with some attacks, for example [9]. In this section we will describe the BB84 protocol proposed by Charles H. Bennett and Gilles Brassard [2], in which a key is sent securely over a quantum channel.

For a long time physicists and scientists tried to determine if light was a wave or a particle, since it seemed to have properties relating to both. It was eventually determined that photons are the fundamental particle of light and they have the unique property in that they are both a particle and a wave. In order to simplify the explanation of quantum key distribution here we will assume that there are only four polarisation (vibrations) of photons, up-down (denoted $|$), left-right ($-$), top left-bottom right ($\backslash$) and bottom left-top right ($/$). The idea is to use photons to transmit data rather than computers. Hence, to transmit a key, a message, or anything you wish to send, you could send it via a fiber optic cable as a sequence of photons.

By placing a polaroid in the path of light it is possible to ensure the emerging beam of light consists only of photons having the same polarisation (direction of vibration) by stopping those photons which are vibrating in the wrong direction (or in some cases altering the direction so that the photon leaving the polaroid is in the right polarisation even if it wasn't when it was approaching the polaroid).

The receiver of the beam of light is able to detect the polarisation of the photons only if they use the correct detector. So, for this simple example, Bob will be able to detect $|$ and $-$ photons if he uses a rectilinear detector (shaped like $+$) and able to correctly detect $\backslash$ and $/$ if he uses a diagonal filter (shaped like $\times$).

However, in some instances a photon vibrating in $\backslash$ or $/$ may travel through a rectilinear detector, randomly changing into either $|$ or $-$, and similarly in some instances a photon vibrating in $|$ or $-$ may travel through a diagonal detector, randomly changing into either $\backslash$ or $/$. A random half don't travel through but a random half do (and then these are then reoriented).

These photons can be used in quantum key distribution to send a binary key over an alternate channel:

1. Alice begins by transmitted a bit-stream of 1s and 0s represented by a polarised photon, according to either the rectilinear (horizontal/vertical) or diagonal polarisation scheme. In the rectilinear scheme 1 is represented by $|$ and 0 by $-$; in the diagonal scheme 1 is represented by $\backslash$ and 0 by $/$.
2. Bob has to measure the polarization of these photons, but since he has no idea of the scheme that Alice used he randomly swaps between his $+$ and $\times$ detectors, sometimes he gets it right sometimes wrong. If Bob uses the wrong detector he may well misinterpret Alice's photon.

   At this point Alice sent 1s and 0s and Bob has detected some correctly and some incorrectly.
3. Alice communicates with Bob on a traditional alternate channel (e.g. a telephone) and tells him the scheme for the photons, but not how she polarized them. So she might say the first was $+$ but she will not say whether it was $|$ or $-$. Bob then tells Alice of which he guessed correct.
4. Finally Alice and Bob ignore all photons for which Bob used the wrong scheme and use only those for which he used the right scheme. This generates a new short sequence of bits for which only Alice and Bob know the values.

In Figure 2 an example is given which includes Alice's original bit sequence and her polarisation schemes, Bob's detections schemes, the measurements he has made and the final retained bit sequence corresponding to only those bits where Alice and Bob used the same scheme. These three stages have allowed Alice and Bob to share 11001001.

Alice's Bit Sequence	1	0	1	1	0	0	1	1	0	0	1	1
Alice's Polarisation Scheme	$+$	$+$	$\times$	$+$	$\times$	$\times$	$\times$	$+$	$\times$	$+$	$+$	$\times$
BOB's Detection Scheme	$+$	$\times$	$+$	$+$	$\times$	$\times$	$+$	$+$	$\times$	$+$	$\times$	$\times$
Bob's Measurement	1	0	0	1	0	0	1	1	0	0	0	1
Retained Bit Sequence	1	-	-	1	0	0	-	1	0	0	-	1

Figure 2: An example of BB84 quantum key distribution scheme [8].

The really big advantage of doing this comes via quantum mechanical properties. Heisenberg's Uncertainty Principle states that even the act of observing such photons affects them and their polarity. So, if an eavesdropper (Eve) even tries to look at (never mind even try to change!) the communication, she will destroy its configuration, and hence the fact that someone has tried to intercept or eavesdrop will be immediately obvious to the receiver.

Quantum key distribution of sudoku grids

The BB84 protocol detailed in [2] is adapted here (in a simplified version) in order to transmit a Sudoku grid over a quantum channel. In order to coincide with the method described in Section we will consider transmitting a 4×4 Sudoku grid (for example Figure 3(a)). Using two polaroid filters $+$, and $\times$ and four vibrations of photon. This time the direction of vibration will be used to represent a value between 0 and 4 where 0 is represented by $|$, 1 by $-$, 2 by $\backslash$ and 3 is represented by $/$. In order to transmit the grid over the quantum channel the following actions are performed:

1. Alice transmits the 4×4 Sudoku grid as a string of 16 photons, first representing the top left value as a photon, then working through each value of the grid in turn until she reaches the bottom right.
2. Bob has to measure the polarization of these photons, but since he has no idea of the scheme he randomly swaps between his $+$ and $\times$ detectors, sometimes he gets it right sometimes wrong. If Bob uses the wrong detector he may well misinterpret Alice's photon.

   At this point Alice sent her Sudoku grid and Bob has detected some correctly and some incorrectly.
3. Alice communicates with Bob on a traditional alternate channel (e.g. a telephone) and tells him the scheme for the photons, but not how she polarized them. So she might say the first was $+$ but she will not say whether it was $|$ or $-$. Bob then tells Alice of which he guessed correct.
4. Bob recreates the Sudoku grid with as much information as he has managed to detect using the correct filters. Since Bob will not have detected all the values then what remains is a Sudoku puzzle.
5. Bob solves the Sudoku puzzle in order to determine the Sudoku grid key which was sent by Alice.

In the event that Sudoku puzzle does not have a unique solution Bob can contact Alice on a traditional channel to request more information to help him to determine which solution Alice has. The information divulged to Bob at this point, if overhead by an eavesdropper, is not sufficient that the eavesdropper would be able to construct the Sudoku grid without the additional information that Bob has determined over the quantum channel.

Since this process is based on the BB84 protocol then this method will allow Alice and Bob to know if their communication has been eavesdropped.

Example of quantum distribution of a sudoku grid

Alice and Bob are sharing a message but first Alice needs to send her key to Bob on an alternate channel. Alice has only a small message and so has chosen to use a 4×4 Sudoku grid (Figure 3(a)). She sends this grid to Bob in the form of photons where 0 is represented by $|$, 1 by $-$, 2 by $\backslash$ and 3 is represented by $/$, this string is given in Figure 3(b) (the scheme Alice will be using - rectilinear ($+$) or diagonal ($\times$) is also given in Figure 3(b)).

0	1	2	3
3	2	1	0
2	0	3	1
1	3	0	2

(a) Sudoku Key

Value	0	1	2	3	3	2	1	0	2	0	3	1	1	3	0	2
Alice's Scheme	+	+	×	×	×	×	+	+	×	+	×	+	+	×	+	×
Photon Stream	|	−	\	/	/	\	−	|	\	|	/	−	−	/	|	\

(b) Photon Stream

Figure 3: Alice's sudoku key and photon stream representation.

Bob sets up his photon detector to receive the stream of photons that Alice has sent, he chooses between diagonal and rectilinear randomly (his choice is given in Figure 4(a)). He then calls Alice on an alternate channel to ask her what schemes she used to generate the photons. In those instances where Bob and Alice are using the same scheme Bob will correctly interpret the value sent by Alice. Bob can use these values to construct a Sudoku puzzle grid (Figure 4(b)), and ignore those values for which they used different schemes. He can solve the Sudoku puzzle in order to generate the Sudoku key (Figure 4(c)).

Photon Stream	|	−	\	/	/	\	−	|	\	|	/	−	−	/	|	\
Bob's Scheme	+	×	+	+	×	×	+	×	+	+	×	+	×	×	+	+
Alice's Scheme	+	+	×	×	×	×	+	+	×	+	×	+	+	×	+	×
Value	0				3	2	1				3	1		3		2

(a) Photon Stream

0			
3	2	1	
		3	1
	3		2

(b) Sudoku Key

0	1	2	3
3	2	1	0
2	0	3	1
1	3	0	2

(c) Sudoku Key

Figure 4: Bob's interpretation of the photon stream and sudoku puzzle/grid.

In this instance Bob does not need any extra information in order to solve the Sudoku puzzle and so Alice and Bob have shared a key over an alternate channel. Alice can now use this key to encrypt a test message and send it to Bob. If Bob can accurately decrypt the test message using the Sudoku grid then they know that Eve has not been eavesdropped in their communication. Had Eve been eavesdropping in their communication it would have altered the photon stream received by Bob and therefore his Sudoku grid would not match Alice's making accurate decryption impossible for Bob.

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Games and Puzzles

THE MAXIMUM QUEENS PROBLEM WITH PAWNS

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Abstract: *The classic n-queens problem asks for placements of just n mutually non-attacking queens on an $n \times n$ board. By adding enough pawns, we can arrange to fill roughly one-quarter of the board with mutually non-attacking queens. How many pawns do we need? We discuss that question for square boards as well as rectangular $m \times n$ boards.*

Keywords: chess, n-queens problem, combinatorics.

The n-queens problem

Take a standard chessboard and as many queens as possible. How many queens can you put on the chessboard, with at most one queen in each square, so that no queen "attacks" (i.e. is a vertical, horizontal, or diagonal move away from) any other queen? Figure 1 shows a placement of eight mutually non-attacking queens. Finding such arrangements is the classic "8-queens problem", first proposed by M. Bezzel in 1848. In 1869 this problem was generalized to the "n-queens problem" of placing n mutually non-attacking queens on an $n \times n$ board. The n-queens problem has solutions for $N = 1$ and $N \geqslant 4$. Hundreds of papers have been written on this problem and its variations, including versions on other board shapes (including cylinders, toruses, and three-dimensional boards) and other piece types (including bishops, rooks, and fairy pieces) . We refer interested readers to the book *Across the Board: The Mathematics of Chessboard Problems* by J. J. Watkins [5], the n-queens survey article by J. Bell and B. Stevens [1], and the online n-queens bibliography maintained by W. Kosters [4].

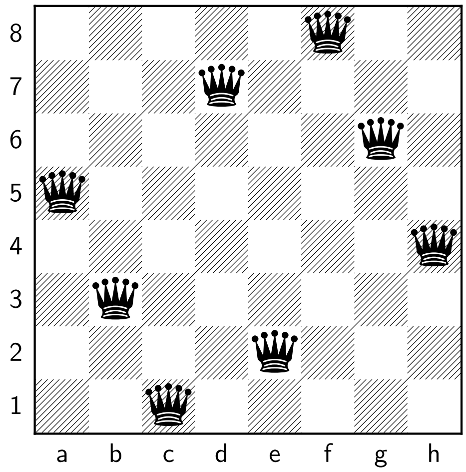

Figure 1: Eight mutually non-attacking queens on a standard chessboard.

Peppering the problem with pawns

Can we do better? Can we put more than eight queens on a standard chessboard or more than n queens on an $n \times n$ chessboard? If we follow the rules of the classic problem, the answer is clearly "no": each queen attacks every other square on its row (or column), so there is at most one queen per row (or column) and thus at most n queens on an $n \times n$ board.

We now change the rules in order to allow more queens on the board. Recall that in chess, queens do not move through other pieces. If we put a pawn in a square between two queens that are in the same row, column, or diagonal, those queens no longer attack each other. If we allow the placement of some pawns, how many mutually non-attacking queens can we place on the board? This is the "maximum queens problem", posed in 1998 by K. Zhao in her disseration [6].

For $n \geqslant 6$, we can put $n + 1$ queens on the board with 1 pawn [3, Theorem 1]. Figure 2 shows an example arrangement. Zhao proved that we need 3 pawns to put 6 queens on a 5×5 board [6]. For each $k > 0$, we can place $n + k$ mutually non-attacking queens on an $n \times n$ board with k pawns, if n is sufficiently large [2, Theorem 11].

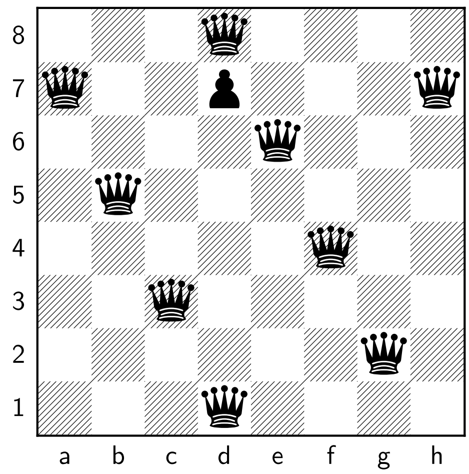

Figure 2: Nine mutually non-attacking queens with one pawn on a standard chessboard.

How much further can we go? If we don't care how many pawns we place on the board, we can place $\frac{n^2}{4}$ queens if n is even and $\frac{(n+1)^2}{4}$ queens if n is odd. To see this, take an $n \times n$ board and divide it into two-column strips (with a one-column strip at the end if n is odd) and divide each strip into two-row block (with one-row blocks at the bottom if n is odd) as shown in Figure 3. Now consider the blocks. If we put two queens in any of these blocks, they will be in adjacent squares and will attack each other, regardless of how many pawns are on the board. So, the maximm number of queens we can put on the board equals the number of blocks, which is $\frac{n^2}{4}$ queens if n is even and $\frac{(n+1)^2}{4}$ queens if n is odd. To see that we can place that many queens, on the first, third, etc. rows place a queen in the first, third, etc. squares and then place pawns in every empty square, as indicated in Figure 3.

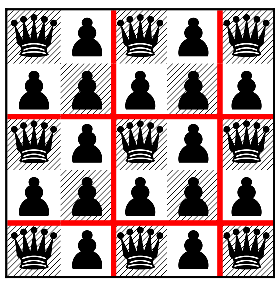

Figure 3: A 5×5 chessboard divided into blocks, each of which can hold at most one queen if no queens can attack other queens.

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Can we place the maximum number of queens with fewer pawns? If n is odd, the answer is “no”. Consider Figure 3 again. When n is odd, the partition produces a 1×1 block in the last row and column. To get the maximum number of queens, each block must have a queen, so the 1×1 corner must have a queen. This forces the placement of all the other queens. There is only one way to place the queens, and the squares without queens are each between two queens and therefore require pawns. We now consider the case where n is even.

Proposition 1. *For $n = 4k + 2$ with $k \geqslant 1$, it is possible to place $\frac{n^2}{4}$ queens and $\frac{n^2}{4} - 3$ pawns on an $n \times n$ board so that no queens attack each other.*

Proof Sketch: We present a pattern with $\frac{n^2}{4}$ queens and $\frac{n^2}{4}$ pawns. Label the rows and columns $0, 1 \ldots n-1$ as shown in Figure 4. In rows with labels of the form $4k+1$ (i.e. rows 1, 5, 9,etc.) in the squares whose row number exceeds their column number, place queens in even-numbered columns and pawns in odd-numbered columns. In rows with labels of the form $4k+3$ (i.e. rows 3, 7, etc.) , in the squares whose row number exceeds their column number, place pawns in even-numbered columns and queens in odd numbered columns. To obtain the rest of the pattern, reflect across the main diagonal and change the piece type, so for each position (x, y), the piece in (y, x) is a queen if and only if the piece in (x, y) is a pawn.

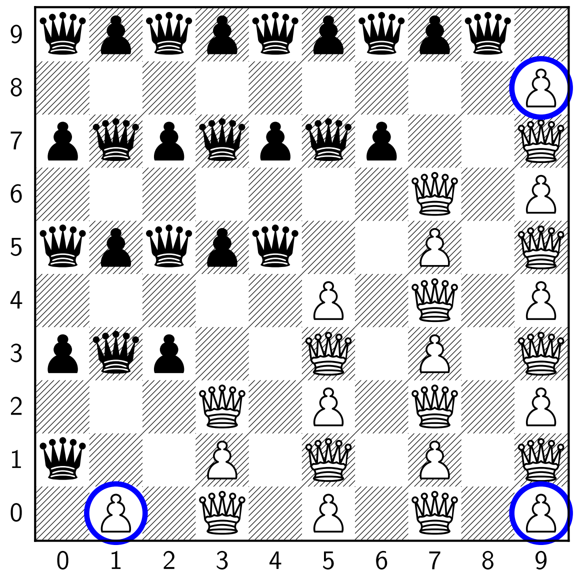

Figure 4: “Corner” pattern.

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We can show that the board has $\frac{n^2}{4}$ queens and $\frac{n^2}{4}$ pawns and that none of the queens attack each other. Finally, we note that we can remove the pawns in positions $(0,1),(0,n-1)$, and $(n-2,n-1)$ and the queens will still not attack each other. ■

When n is a multiple of 4, we can do slightly better.

Proposition 2. *For $n = 4k$ with $k \geqslant 1$, it is possible to place $\frac{n^2}{4}$ queens and $\frac{n^2}{4} - 4$ pawns on an $n \times n$ board so that no queens attack each other.*

Proof Sketch: Again we present a pattern (as illustrated in Figure 5) that produces the desired results. Given a $4k \times 4k$ board, label the rows and columns $-2k \dots 2k-1$. In the upper-right quadrant (i.e. the positions with both coordinates nonnegative), place queens and pawns in the "corner" pattern of the previous Proposition, except do not remove any pawns like we did previously. Rotate the board a quarter-turn, relabel rows and columns, and place queens and pawns in the new upper-right quadrant in the corner pattern. Repeat the previous sentence two more times.

We now have a "dartboard" pattern, symmetric with respect to quarter-turn rotation, with $\frac{n^2}{4}$ queens and $\frac{n^2}{4}$ pawns. We can check that none of the queens attack any of the other queens. We conclude by noting that four pawns in the outer ring formed by the first and last rows and the first and last columns (circled in Figure 5) can be removed. ■

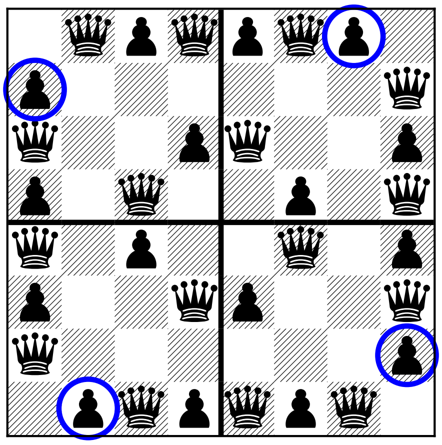

Figure 5: "Dartboard" pattern

Can we do better? Are there patterns on even-order boards with the maximum number of queens but fewer pawns? Computer experiments for small board sizes indicate that the answer is "no", but we don't have a general proof.

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Stretching to Rectangles

We can extend the results to rectangular boards of size $m \times n$ where m is not necessarily equal to n. Dividing the board into blocks as we did for square boards, we get that the maximum number of non-attacking queens we can place on such boards is $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$, where $\lceil x \rceil$ is the smallest integer greater than or equal to x. For example, on a 4×7 board we can place at most $\lceil \frac{4}{2} \rceil \lceil \frac{7}{2} \rceil = \lceil 2 \rceil \lceil 3.5 \rceil = (2)(4) = 8$ queens, as illustrated in Figure 6. If both m and n are odd, there is only one way to place the queens and every other square requires a pawn.

So we make at least one of the dimensions even.

Proposition 3. *If m or n is even, we can place $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ queens and $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil - 2$ pawns on an $m \times n$ board so that no two queens attack each other.*

Proof sketch: Suppose without loss of generality that m is even. Label the rows $0, 1, \ldots, m-1$ and the columns $0, 1, \ldots, n-1$. In columns with labels of the form $4k$, place queens in the even-numbered rows and pawns in the odd-numbered rows. In columns with labels of the form$4k+2$, place pawns in the even-numbered rows and queens in the odd-numbered rows. We get a striped pattern as illustrated in Figure 6. We can check that none of the queens attack each other and that there are the right number of queens. We conclude by eliminating a pawn in the first and last non-empty columns, as indicated in Figure 6.

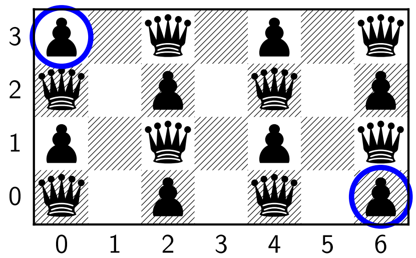

Figure 6: "Stripe" pattern

If m or n is 2, that is the best possible result: A 2-row board without pawns can hold at most 2 queens, one per row. Each pawn increases the number of available "rows" by at most 1. So the total number of queens on such a board is at most 2 more than the number of pawns. The number of pawns needed to get $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ queens on a 2-row (or 2-column) board is $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil - 2$.

If both m and n are even and larger than 2, we can do better, using portions of the "corner" pattern.

Proposition 4. *If $m > 2$ and $n > 2$ are even, then we can place $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ queens and $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil - 3$ pawns on an $m \times n$ board so that no two queens attack each other.*

Proof Sketch: Suppose without loss of generality that $m < n$. Take an $n \times n$ board and place queens and pawns according to the procedure for the corner pattern, without the final pawn removal. If m is a multiple of 4, remove rows $m, \ldots, n-1$, and note that the pawns at positions $(0,1), (m-1,0)$ and either $(0, n-1)$ or $(m-1, n-1)$ can be removed. If m is not a multiple of 4, remove columns $m, \ldots, n-1$, note that pawns at $(0,1), (0, m-1)$, and either $(n-1, m-1)$ or $(n-1, 0)$ can be removed, and then transpose rows and columns. ■

Can we do better? The results of computer experimentation with low values of m and n indicate that the answer is "no", but, again, we have no general proof.

Other Directions

This paper leaves many questions open for future research. Here are a few:

1. A common thing to do with the classic n-queens problem is to count the number of solutions for particular values of n. Except when m and n are both odd, we haven't done that with the maximum queens problem. Given the maximum number of queens and the minimum necessary number of pawns, how many ways can those pieces be arranged on an $m \times n$ board (with m or n even) so that none of the queens attack each other?

2. Clearly, with enough pawns, we can place q mutually non-attacking queens for $0 \leqslant q \leqslant \lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$. How many pawns are needed?

3. What results can we get on other types of board, such as cylindrical boards, Mobius strips, and toruses? Observe that the maximum number of queens is the same as the maximum number of kings you can put on the board so that no kings are on adjacent squares: If no two pieces are on adjacent squares, we can separate them with pawns. The maximum number of nonadjacent kings is referred to as the "kings independence number" and it is known for many board types [5]. For example, the kings independence number on an $m \times n$ torus is $\min\{\lfloor \frac{m \lfloor \frac{n}{2} \rfloor}{2} \rfloor, \lfloor \frac{n \lfloor \frac{m}{2} \rfloor}{2} \rfloor\}$ [5, Theorem 11.1] (where $\lfloor x \rfloor$ is the largest integer less than or equal to x) and that number is also the maximum number of non-attacking queens we can put on that board with sufficiently many pawns. So, the interesting question is "How many pawns do we need?"

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