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Informations
The *Recreational Mathematics Magazine* is electronic and semiannual.
The issues are published in the exact moments of the equinox.
The magazine has the following sections (not mandatory in all issues):

Articles
Games and Puzzles
Problems
MathMagic
Mathematics and Arts
Math and Fun with Algorithms
Reviews
News

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Contents

Games and Puzzles

The Tetraball Puzzle

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Abstract: *In this paper, the* Tetraball Puzzle, *a spatial puzzle involving tetrahedral arrangements, is presented and discussed.*

Key-words: Spatial puzzles, tetrahedral arrangements, packing problems.

1 The puzzle

A *Tetraball* is made of 4 equal size balls each of which touches the others (Figure 1). This tetrahedral arrangement of balls is the smallest non-trivial case of a tetrahedral stack of balls, which, if there are n balls per edge, has the nth tetrahedral number,

$$T(n) = \frac{n(n+1)(n+2)}{6}$$

balls in all. Figure 2 illustrates the case $T(4) = 20$ balls.

Figure 1: Tetraball.

The spheres of any such arrangement can be colored in 4 colors so as to satisfy the usual coloring condition that any two spheres that touch must be colored differently.

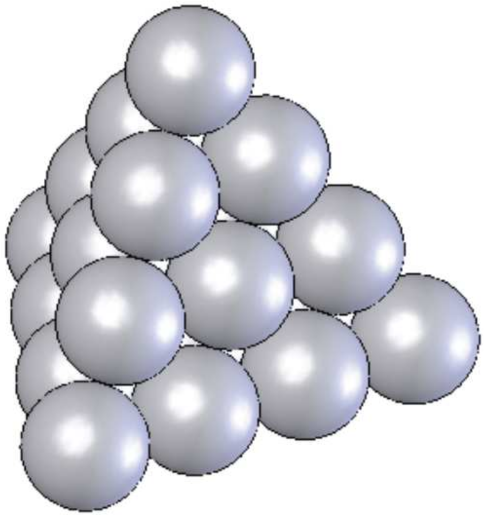

Figure 2: $T(4)$.

The *Tetraball Puzzle* asks to assemble five such 4-colored 2-stacks, or "tetraballs" into a 4-stack so as to satisfy this coloring condition.

The attentive reader will have noticed that we have not uniquely defined our problem, because the tetraball has two enantiomorphous (mirror image) forms, so there are really three problems according to how many of each form there are, namely $5-0$, in which the tetraballs are all the same, $1-4$ in which one is different to the other 4, and $3-2$ in which 3 are different from the other 2. All three problems are uniquely solvable and not extremely hard, although our quick-fire treatment will make them seem much easier than they really are.

2 Discussion

To work! When the puzzle is solved, it will contain one central tetraball and 4 corner ones. We can suppose that the central one is as in Figure 3 with the bottom sphere yellow. The topmost sphere of the upper tetraball is either yellow (Figure 3) or another color (Figure 4). For each possibility there are two cases called:

UNCHANGED
The top sphere of the top piece is the same color as the bottom sphere of the central piece (Figure 3). This implies that the colors of the remaining spheres are completely determined, since each already touches spheres of 3 other colors and must be colored in the fourth, and we find that the two pieces are enantiomorphs of each other. A circled vertex in a geometrical diagram (Figure 8) indicates that the piece at that vertex is unchanged.

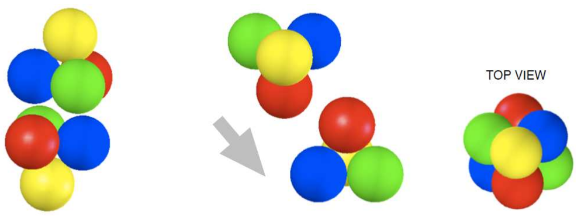

Figure 3: Unchanged.

EXCHANGED

In this second case the bottom sphere of the central piece is yellow (say), while the top sphere of the top piece is another color, say blue. The coloring of the top piece is unique (given this information), and we find the two pieces are identical rather than enantiomorphic. The exchanged coloring is obtained from the unchanged one by interchanging the colors blue and yellow for that piece. We indicate this exchange of colors on a piece at vertex A by a double arrow on the appropriate half of an edge.

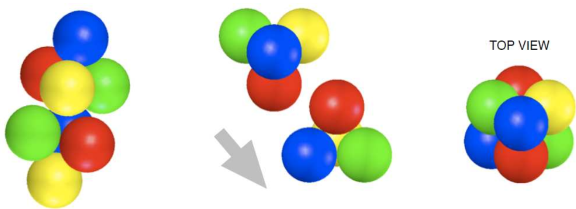

Figure 4: Exchanged.

Now the four balls along any edge belong to two pieces. So when this exchange happens to the first and second spheres along the edge, the third sphere must change, otherwise it would be the same color as the second. The only possibility now exists that the other is exchanged. So in our sketch, the other half of that edge must have a double arrow.

We call this process "trading up" the edge AB. Now there are only two possibilities, either the pieces at CD are unchanged or both exchanged as the pieces in edge AB, i.e., "trading up". We have now proved that, up to symmetry, the tetraball puzzle has just three solutions, classified by the number of edges we have traded up (since we often use square diagrams, we will denote "diagonals" and "boundaries").

We call them 0 *up*, 1 *up* or 2 *up* as follows (Figures 5, 6 and 7):

0 *up* – The 4 vertex pieces are the same as each other and opposite to the central piece so the splitting is $\frac{1}{4}$. In the orbifold notation it is $*322$ [1], the full tetrahedral group, of order 24. All edges have just 2 colors.

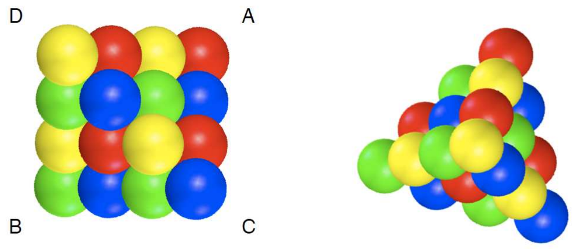

Figure 5: 0 *up*.

1 *up* – The central piece is 1 of 3 that are opposite to the other 2, making the splitting be 3/2. The symmetry group is $*22$ of order 4, the same as that of a rectangular table. The 4 "boundary" edges have been increased to 3 colors and the 2 "diagonal" edges remain 2-colored.

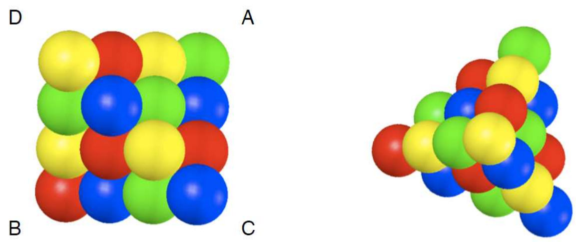

Figure 6: 1 *up*.

2 *up* – All five pieces are the same. (splitting 5/0). The symmetry group is $2*2$ of order 8, the same as that of a tennis ball. The 4 boundary edges are now upped to 4 colors, leaving the two diagonals 2-colored. Some of these properties can be stated uniformly: t *up* – maximum number of colors per edge $= 2 + t$; number of vertex pieces identical to the central piece $= 2t$. The pieces therefore split as $(2t + 1) : (4 - 2t)$.

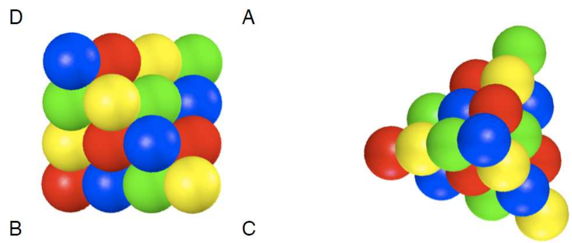

Figure 7: 2 *up*.

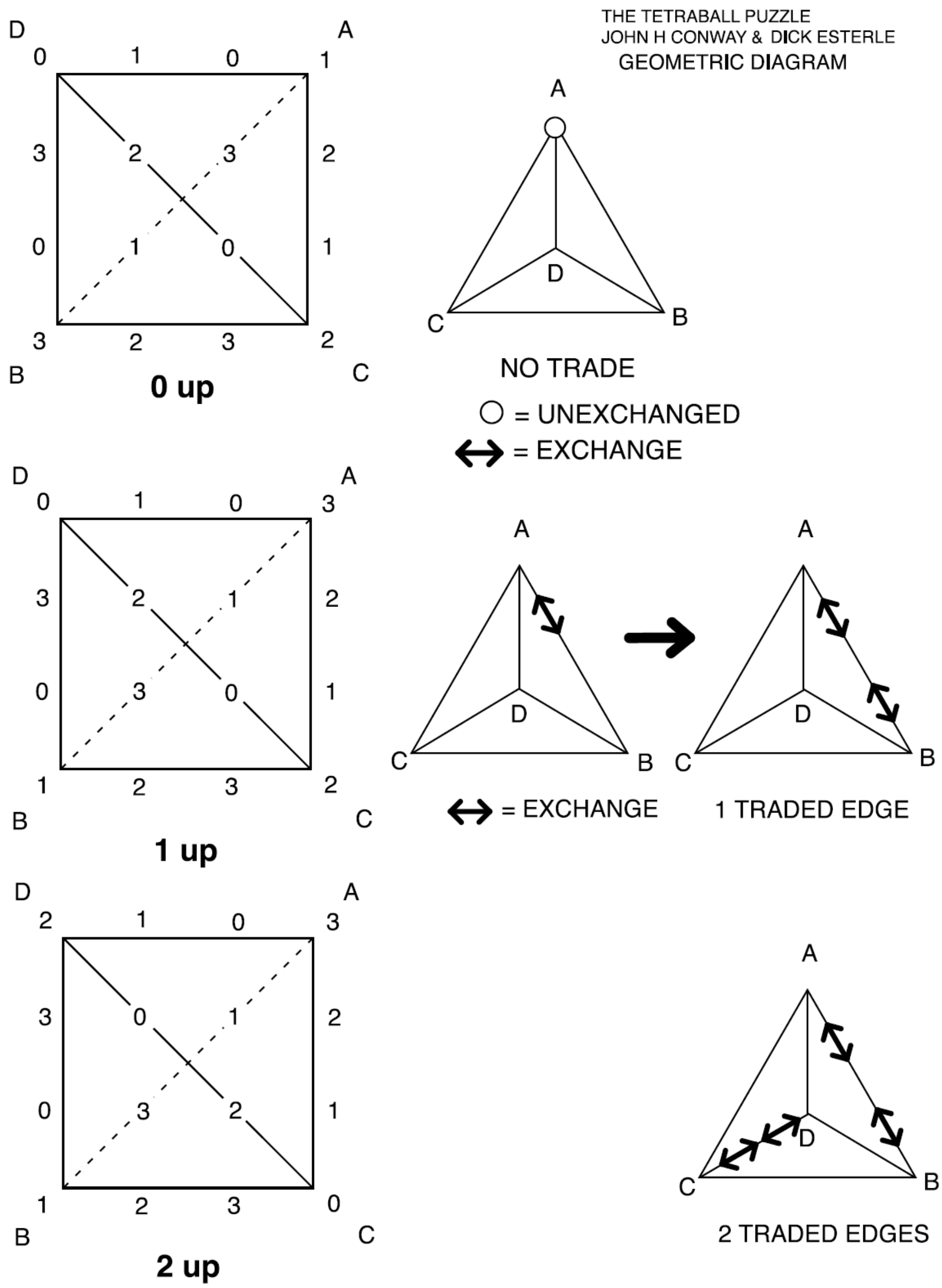

Figure 8: Geometric diagram.

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[1] Conway, J., Burgiel, H., Goodman-Strauss, C. *The Symmetries of Things*, A K Peters, 2008.

Games and Puzzles

Some early topological puzzles Part 2*

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An earlier version of this was presented at the Fourth *Gathering for Gardner*, Atlanta, 19 Feb 2000. Because there were too many pictures, it was not included in the proceedings of that meeting [22]. I have scanned in the 24 pictures that I showed as OHP slides and seven extra pictures and corresponding material. I have now inserted these in this version of this work, with the picture name after each Figure reference.

The most important development in the field is that Dario Uri has systematically examined Part Two of the Pacioli MS which I had found difficult [19]. He has discovered that the following puzzles occur. The Alliance or Victoria Puzzle, Cap. c, cii & ciii. Solomon's Seal, Cap. ci. The Cherries Puzzle, Cap. ciiii & cx (the latter being with a tube). The Chinese Wallet, Cap. cxxxii. As mentioned below, I had already noticed Cap. c & ci, and a bit later I noticed ciiii, but had set the transcription aside to work on later. In addition, Dario has found that Cap. cvii is the Chinese Rings! Further, Cap. cxiiii is the problem of joining three castles to three wells by paths that do not cross; Cap. cxvii is a trick of removing a ring from a loop between a person's thumbs – cf the top of Figure 12: Schwenter410; Cap. cxxi is the trick where strings are just looped inside a ball so they can pulled away; Cap. cxxix is the problem of making a support from three knives. As far as we know, these are the earliest appearance of all these puzzles. There are about 140 problems in this Part! Unfortunately, though Pacioli refers to diagrams, the only diagram in the MS is for Cap. c. Perhaps Dario will find the missing drawings in the library at Bologna. Dario has found that Cap. cxxix refers to Pacioli being shown the problem on 1 Apr 1509, so this MS should be dated as 1510?

I have just added much further material on two topological patterns: The Star of David and The Borromean Rings; The Möbius Strip; which go back rather further than I knew.

*First part: David Singmaster. "Some early topological puzzles: Part 1", *Recreational Mathematics Magazine* 3, 9–38, 2015.

†School of Computing, Information Systems and Mathematics South Bank University; London, SE1 0AA, UK; Last amended on 25 Dec 2014.

Abstract: *For some time, I considered the 1723-1725 edition of Ozanam's* Récréations mathématiques et physiques *as the first book to cover topological puzzles in detail and I only knew of a few earlier examples. Ozanam certainly gives many more examples than any previous book. In the last few years, I have discovered some early sources which show several topological puzzles as being considerably older than I previously knew. Here I show them.*

Key-words: Topological puzzles, The Cherries Puzzle, Six-Piece Burrs

6 The Cherries Puzzle

This comes in two common forms (Figure 32). Both are given in Ozanam [11]. The classic cherries form is [[11], vol. IV, prob. 33, p. 436 & fig. 39, plate 12 (14)]: *On peut passer des queues de Cerises dans un papier...*

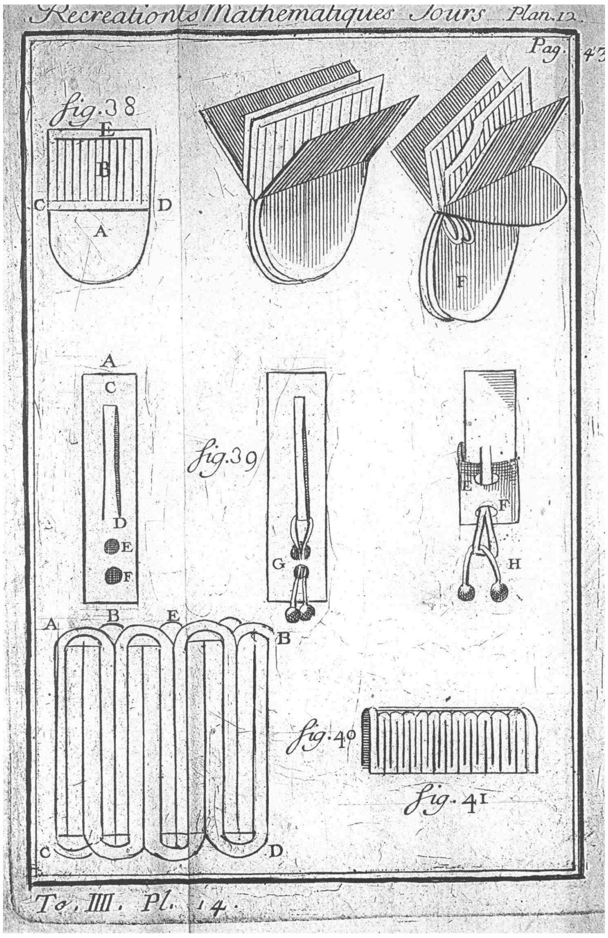


Figure 32: Ozanam 33.

Here I have not found anything earlier than Witgeest [[21], prob. 19, pp. 162-163], who has a delightful picture with realistic cherries (Figure 33).

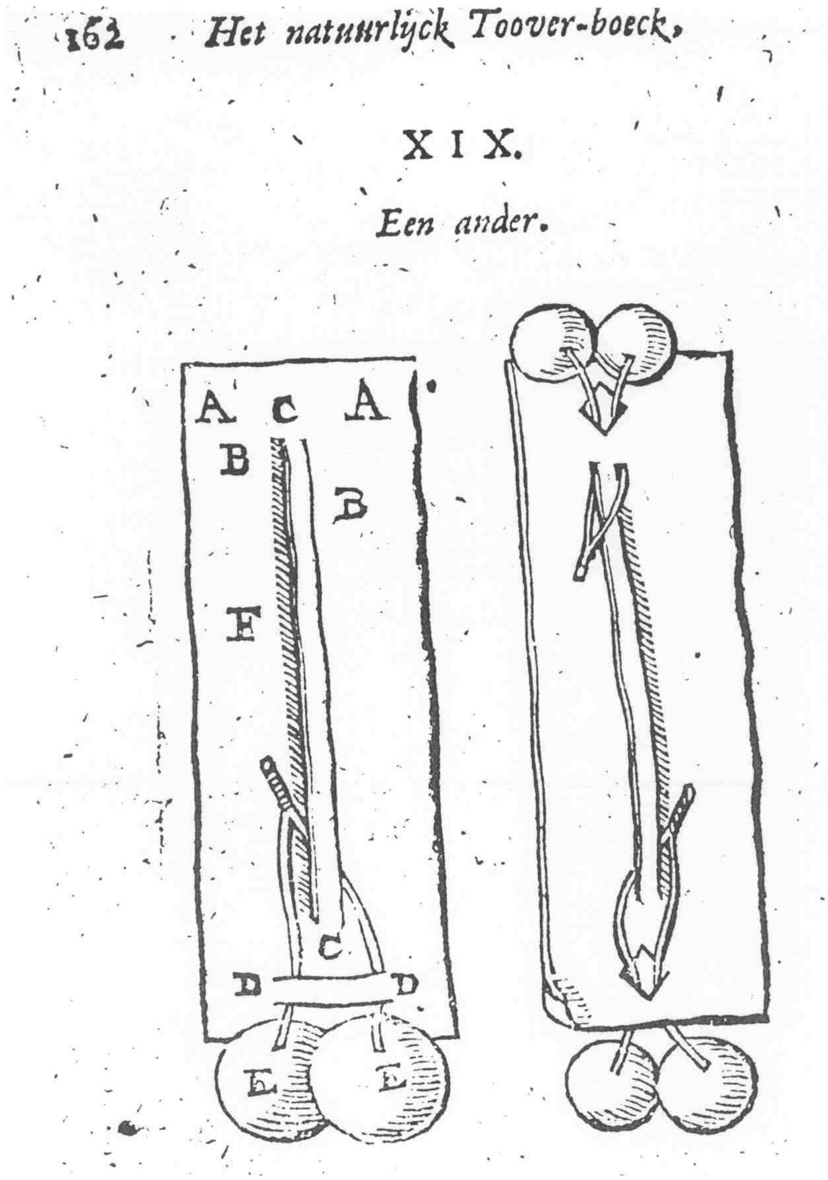
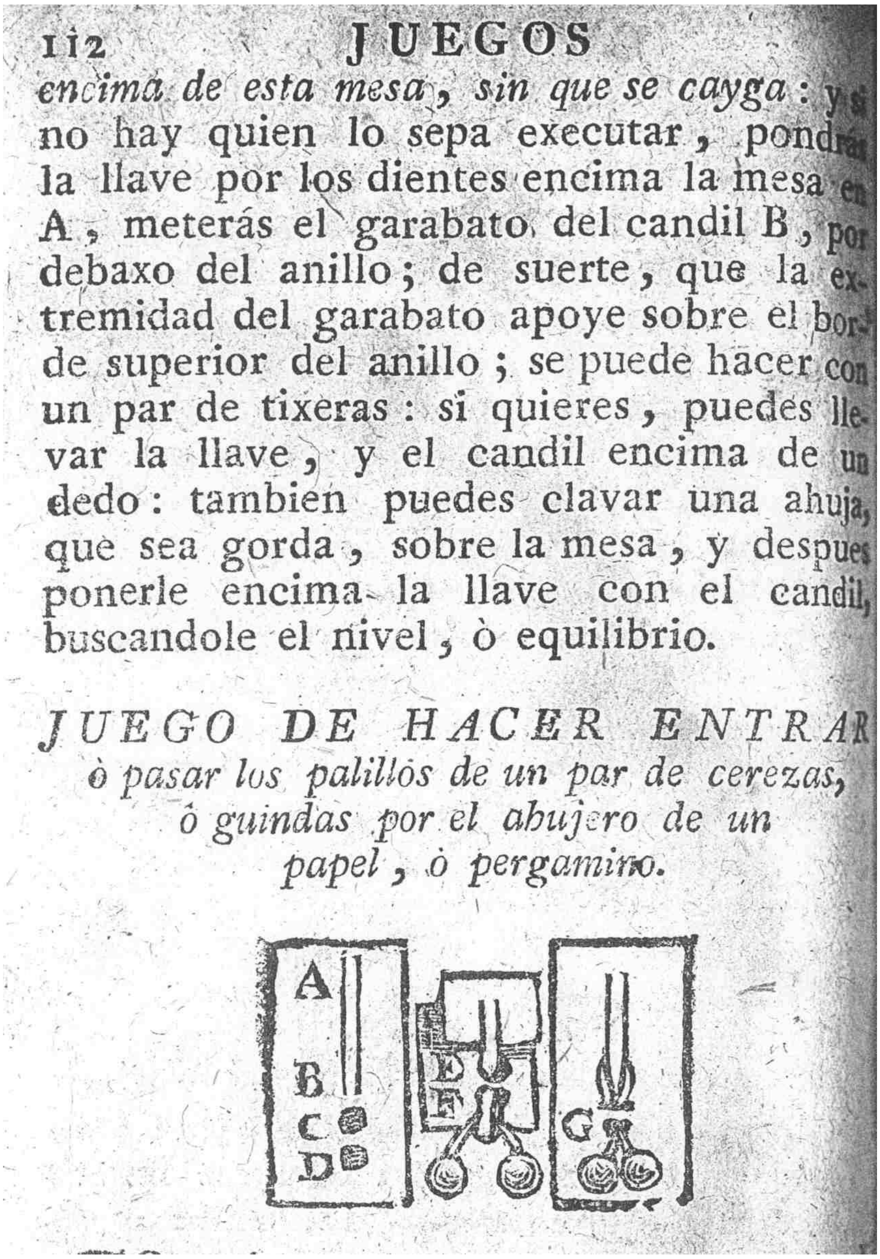


Figure 33: Witgeest 162.

Another pleasant picture (Figure 34) comes from Minguét, the first magic book in Spanish, from 1733 [[9], pp. 112-113].


112 JUEGOS

encima de esta mesa, sin que se cayga: y si no hay quien lo sepa executar, pondrás la llave por los dientes encima la mesa en A, meterás el garabato del candil B, por debaxo del anillo; de suerte, que la extremidad del garabato apoye sobre el borde superior del anillo; se puede hacer con un par de tixeras: si quieres, puedes llevar la llave, y el candil encima de un dedo: tambien puedes clavar una ahuja, que sea gorda, sobre la mesa, y despues ponerle encima la llave con el candil, buscandole el nivel, ò equilibrio.

JUEGO DE HACER ENTRAR ò pasar los palillos de un par de cerezas, ò guindas por el ahujero de un papel, ò pergamino.

Figure 34: Minguét 112.

The second version has a folded piece of paper or leather hanging from a card or tube which has two parallel cuts along much of it (Figure 35). The paper has large ends and there is a ring on the thin part which cannot go over the ends. The solution is essentially the same as for the Cherries Puzzle. We have already seen this in Ozanam, Prob31 [[11], vol. IV, prob. 30, p. 434 & fig. 36, plate 11 (13)].

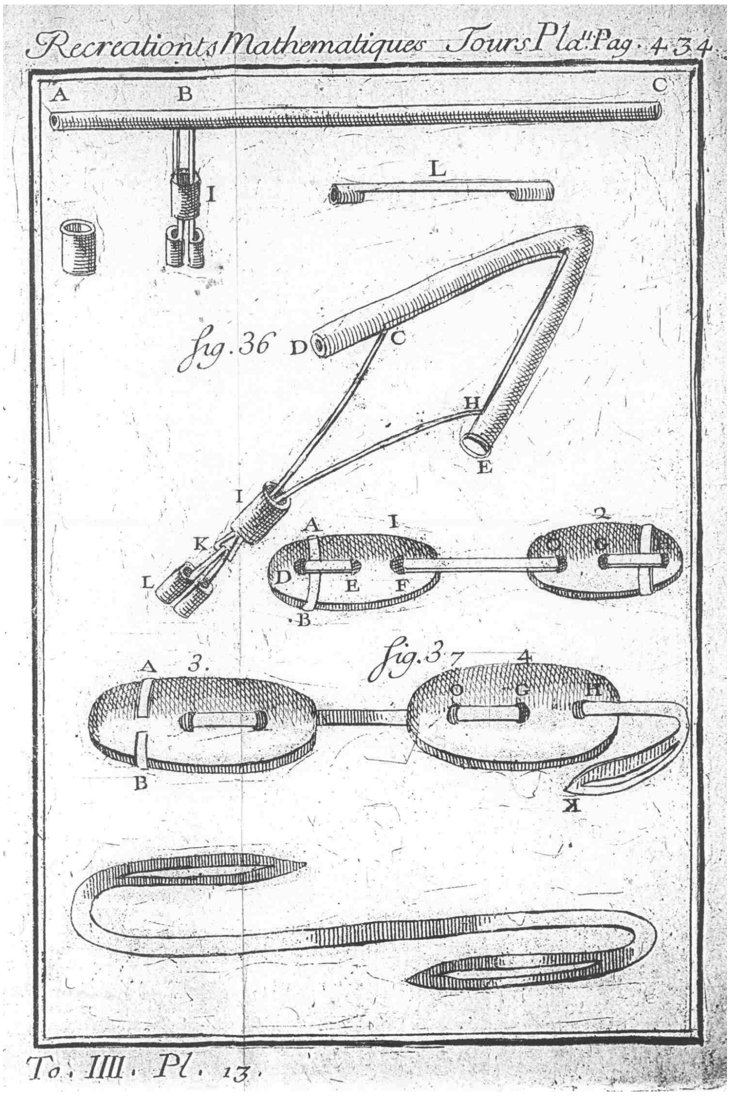


Figure 35: Ozanam 31.

Here (Figure 36) we find a version in Schwenter [[15], part 10, exercise 30, p. 411].

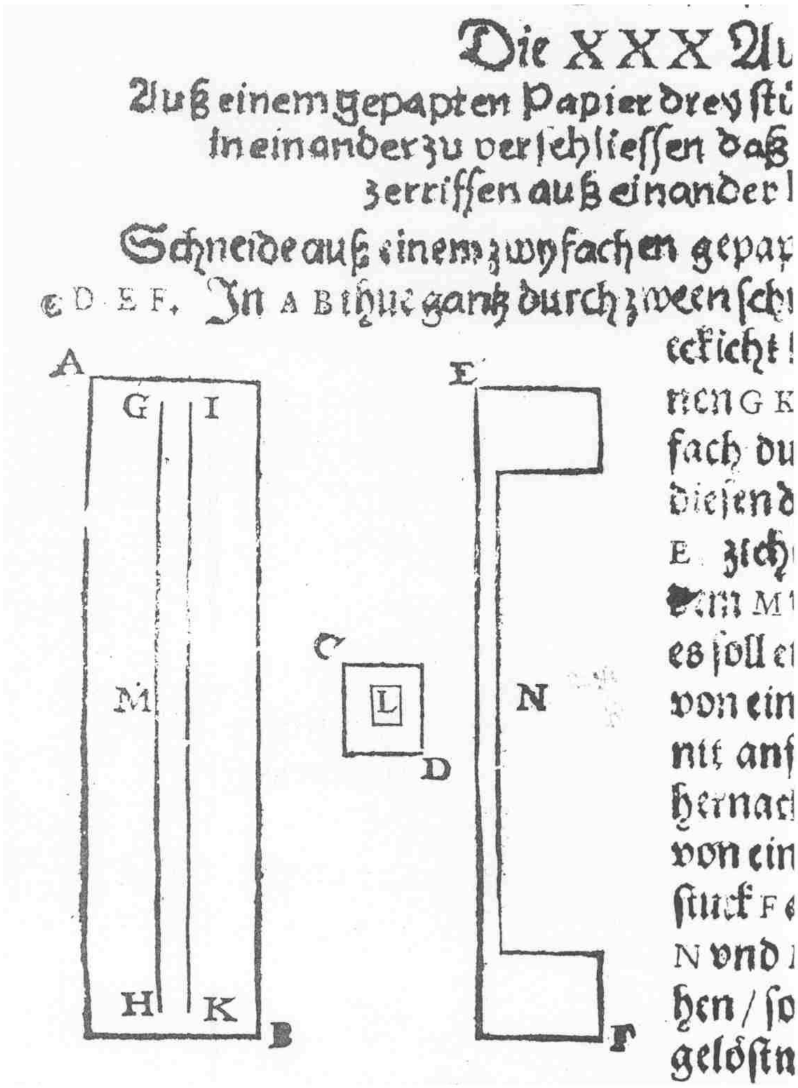


Figure 36: Schwenter 411.

And a quite different version (Figure 37) in Witgeest [[20], prob. 18, pp. 160-161].

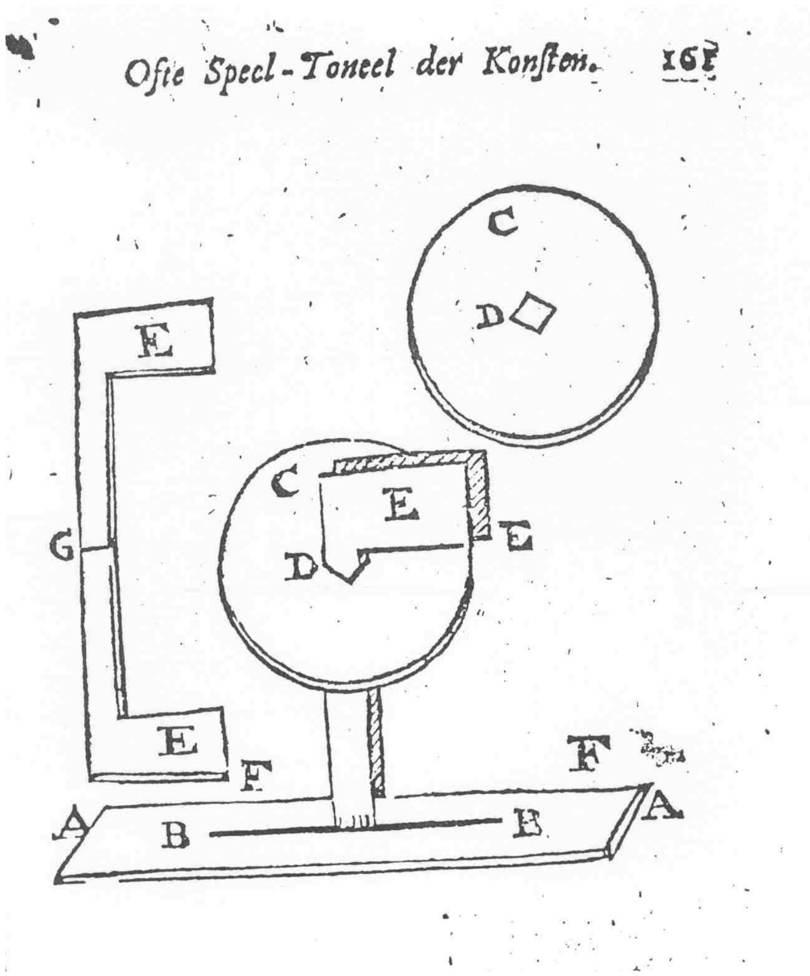


Figure 37: Witgeest 161.

7 Six-Piece Burrs

This is not really topological, but I have found a much earlier version than previously known from an unexpected area. For some time, the example called *Die kleine Teufelsklaue* (the little Devil's claw, item 147) in the 1801 catalogue of Bestelmeier was the earliest known. About twenty years ago, I turned up the 1790 catalogue of the predecessor firm of Catel which has the same figures – see item 16 in the Figure 38.

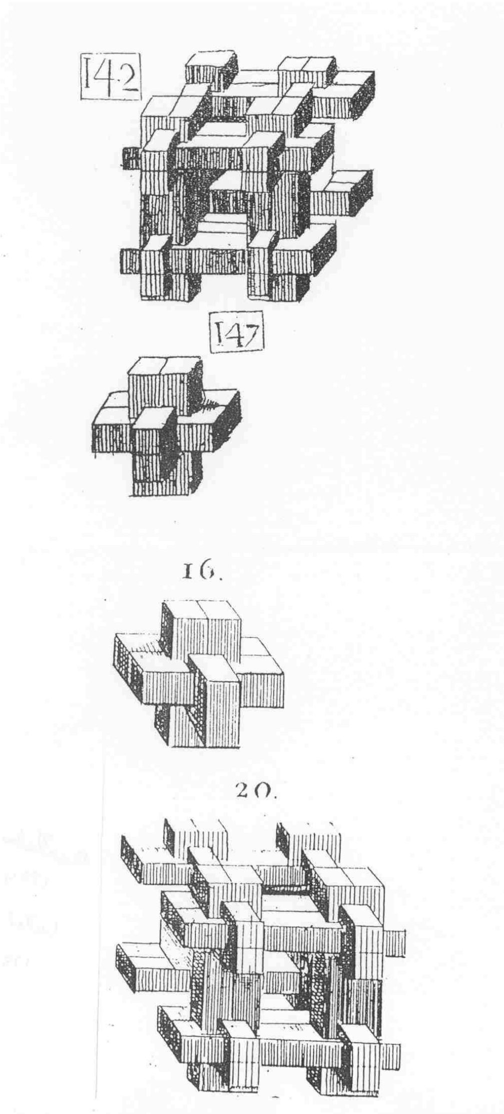


Figure 38: Bestelmeier-Catel.

Dieter Gebhardt and Jerry Slocum then did a lot of hunting through German libraries and found the original parts of the Bestelmeier catalogue and early price lists of Catel, so we know the dates can be revised backward to 1794 and 1785 respectively [17]. However, the earliest known diagram (Figure 39) of the pieces was in *The Magician's Own Book* of 1857, where it is called *The Chinese Cross* [[8], prob. 1: *The Chinese Cross*, pp. 266-267 & 291].

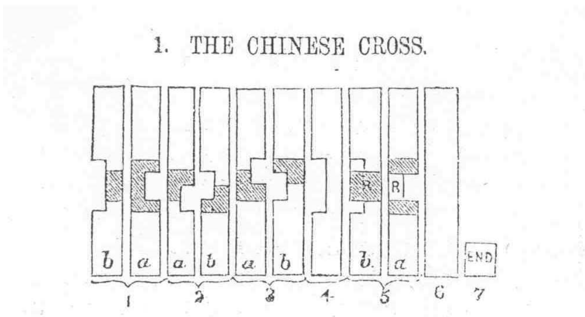
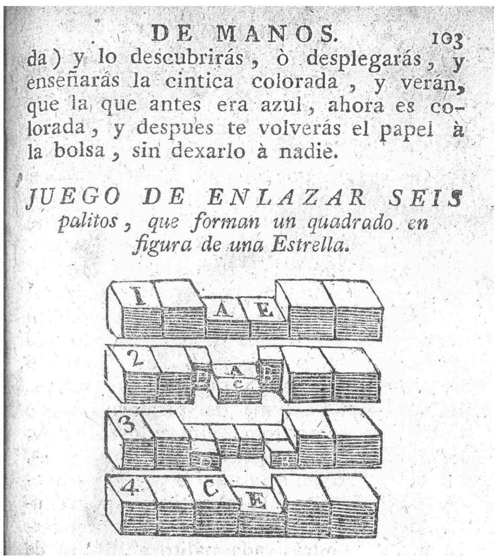


Figure 39: MOB 266.

Surprisingly, I found the clearer diagram (Figure 40) in Minguét of 1733 [[9], pp. 103-105], where it is just called a star. Piece 3 must be duplicated and there is a plain key piece. This is not the same burr as the previous one.



DE MANOS. 103

da) y lo descubrirás, ò desplegarás, y enseñarás la cintica colorada, y verán, que la que antes era azul, ahora es colorada, y despues te volverás el papel à la bolsa, sin dexarlo à nadie.

JUEGO DE ENLAZAR SEIS palitos, que forman un quadrado en figura de una Estrella.

Figure 40: Minguét 103.

8 Conclusions

Although recent work has pushed back our knowledge of these problems, sometimes by several hundred years, it seems clear that we really don't know the origins of them. None of the sources cited are massively original; they are simply compilations of well known examples of their time.

9 Appendix: Chinese Rings

This is so well known that I won't try to explain it. There are several Oriental stories about its origins, but I have not yet seen any definite evidence to support these stories. One of the earlier versions is in Ozanam [[11], vol. IV; unnumbered figure on plate 14 (16)], which has already been seen in Figure 14, [11], prob. 40 (Figure 41). Interestingly, there is no text corresponding to this picture.

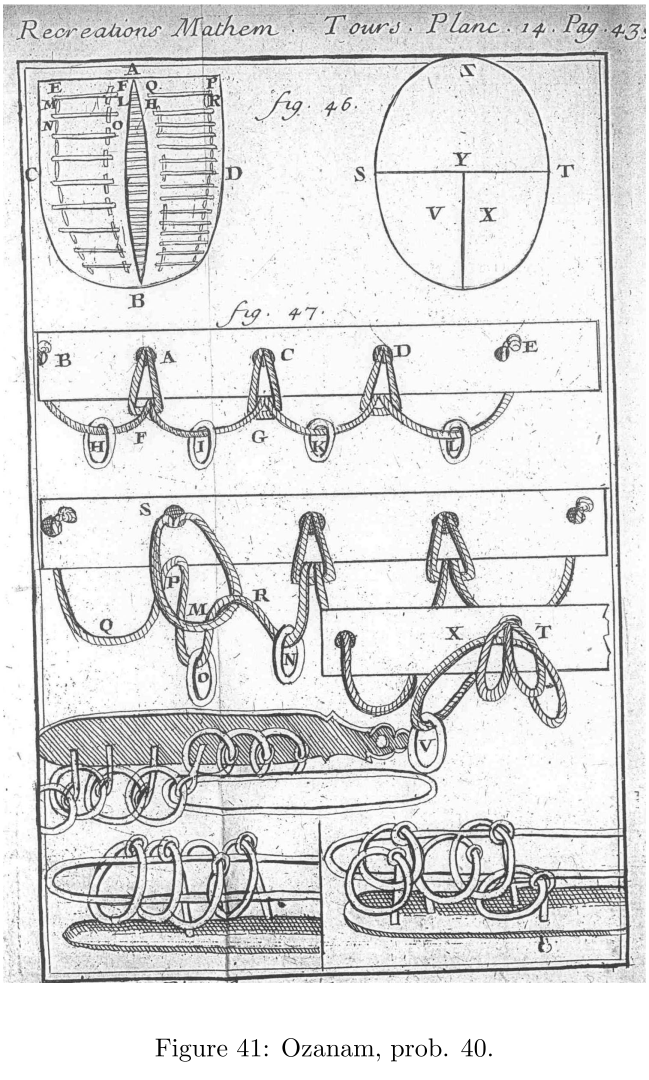
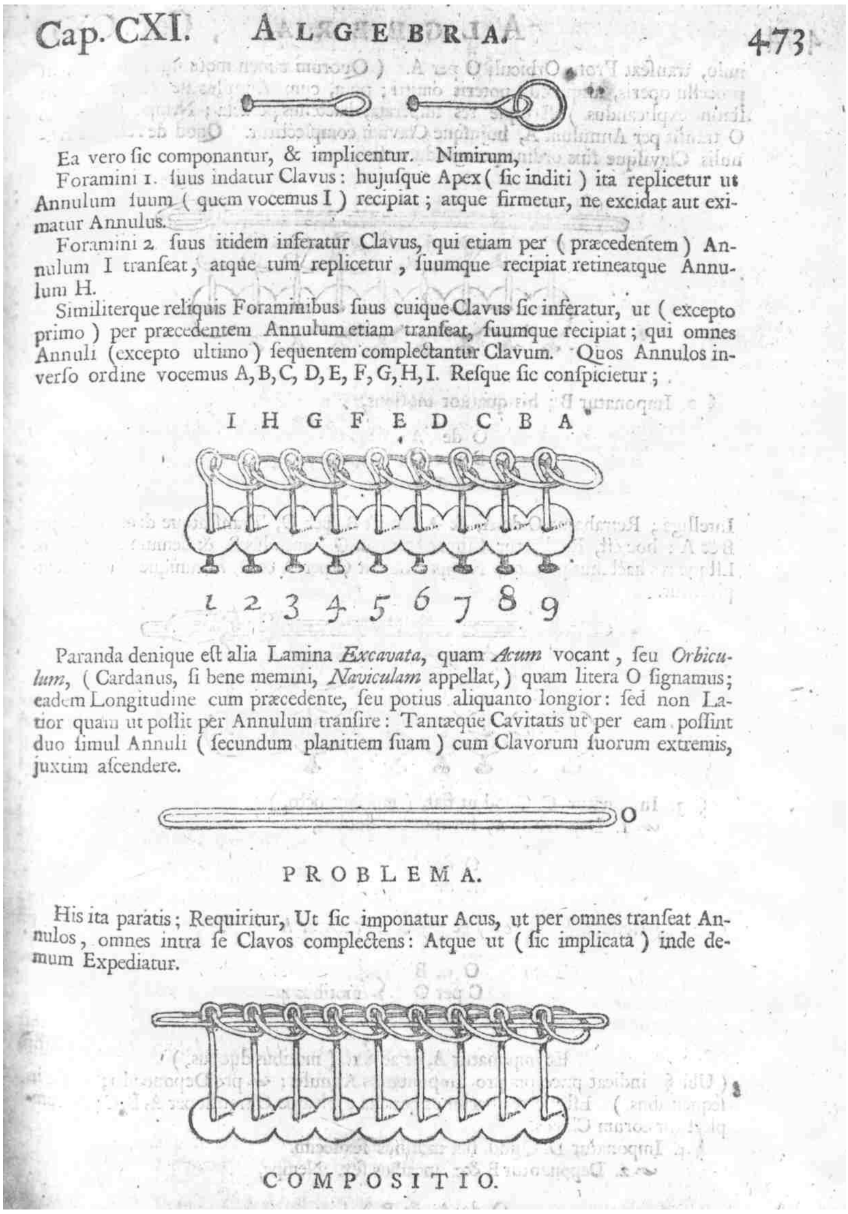


Figure 41: Ozanam, prob. 40.

The first person to analyse the problem was John Wallis [*De Algebra Tractatus*, 1685, Not Yet Seen, but = *Opera Math.*, Oxford, 1693, vol. II, chap. CXI, *De Complicatus Annulis*, pp. 472-478] and he gives several illustrations (Figure 42).


Cap. CXI. ALGEBRA. 473

Ea vero ſic componantur, & implicentur. Nimirum,

Foramini 1. ſuus indatur Clavus: hujuſque Apex (ſic inditi) ita replicetur ut Annulum ſuum (quem vocemus I) recipiat; atque firmetur, ne excidat aut eximatur Annulus.

Foramini 2. ſuus itidem inferatur Clavus, qui etiam per (præcedentem) Annulum I tranſeat, atque tum replicetur, ſuumque recipiat retineatque Annulum H.

Similiterque reliquis Foraminibus ſuus cuique Clavus ſic inferatur, ut (excepto primo) per præcedentem Annulum etiam tranſeat, ſuumque recipiat; qui omnes Annuli (excepto ultimo) ſequentem complectantur Clavum. Quos Annulos inverſo ordine vocemus A, B, C, D, E, F, G, H, I. Reſque ſic conſpicietur;

I H G F E D C B A

1 2 3 4 5 6 7 8 9

Paranda denique eſt alia Lamina *Excavata*, quam *Acum* vocant, ſeu *Orbiculum*, (Cardanus, ſi bene memini, *Naviculam* appellat,) quam litera O ſignamus; eadem Longitudine cum præcedente, ſeu potius aliquanto longior: ſed non Latior quam ut poſſit per Annulum tranſire: Tantæque Cavitatis ut per eam poſſint duo ſimul Annuli (ſecundum planitiem ſuam) cum Clavorum ſuorum extremis, juxtim aſcendere.

O

PROBLEMA.

His ita paratis; Requiritur, Ut ſic imponatur Acus, ut per omnes tranſeat Annulos, omnes intra ſe Clavos complectens: Atque ut (ſic implicata) inde demum Expediatur.

COMPOSITIO.

Figure 42: Wallis 473.

Until Dario Uri's recent discovery of the Chinese Rings in Pacioli, c1500 [19], the earliest known version of the puzzle was given by Cardan in 1550 [3] [Girolamo Cardano (= Hieronymus Cardanus); *De Subtilitate Libri XXI*; J. Petreium, Nuremberg, 1550, and many later printings and editions; *Liber XV*; *Instrumentum ludicrum*, pp. 294-295]. This is a very cryptic description - see the Figure 43. Note Cardan's diagram of a single ring! In the 1663 *Opera Omnia*, the ring is stretched and labelled "Navicula" (little boat).

294

LIBER XV.

De incerti generis aut inutilibus ſubtilitatibus.

Cur alterius medio tabulæ percuſſæ minimè franguntur

EIVSMODI ſunt, uelut cur ſi ſuper tabulam ictus incidat, non adeò frangitur ſuppoſita tabula, ut in cuneis figendis: certè quia ictus per totam ſuppoſitã tabulam diſtribuitur, unde minor euadit in ſingulis partibus; nec cogitur pars deſcendere alia relicta, quod maximè fracturæ initium eſt, eo etiam intermedio temperatur ictus, aer quoqꝫ cum impulſu careat, uiolenter ingredi non poteſt ſecũdam tabulam ſubiectam, quare nec diuidere.

Vitrea uaſa igni expoſita ſurculo ſuppoſito, cur non rumpantur.

Simile illud, cur uaſa uitrea igni expoſita ſuperimpoſito ſurculo non rumpantur: quoniam lignum non incaleſcit, cogitqꝫ ut uitri partes miſceantur: frangitur aũt quia aqueum humidũ prius in flatũ conuertitur.

Inſtrumentum ludicrum.

Verum nullius uſus eſt inſtrumentum ex ſeptem annulis. Bractea ferrea digitum lata, palmi longitudine tenuis, in qua ſeptem foramina rotunda, anguſta æquisqꝫ ſpacijs, ſecundum longitudinẽ diſpoſita, ſeptem excipiunt uirgulas tenues, altitudine unciæ fermè, mobiles in imo, & in ſuprema parte circumflexas, ut annulos digiti magnitudine incluſos retineant, ipſæ uerò uirgæ à ſequenti annulo infra flexuram continentur. Ob idqꝫ omnes annuli præter primum ab antecedente, ne exiliant liberè extra anteriorem uirgam prohibentur: ferrea omnia & ferrea etiam nauicula, cuius ſpeciem ad unguem in margine reddidimus, longa lataqꝫ pro magnitudine ſubiectæ laminæ. Hoc inſtrumento ludus excogitatus miræ ſubtilitatis. Primus ſecundusqꝫ annulus per inane A ſpaciũ immittitur, inde nauicula per eoſdem annulos, poſt illorum primus per inane A demittitur, poſt quem tertius annulus per nauiculæ uacuam mediam partem, ut primi duo ſurſum trahitur: illicqꝫ nauicula intruditur: tum etiam primo ſurſum deducto, iam tres circumambiunt nauiculam ipſam, demittes igitur duos primos exempta prius nauicula, ita illa ſoli tertio incluſa manebit, inde quartum ſuperinducere licebit, ut omnis hæc induſtria tribus præceptis contineatur. Primum, quòd annulus ſurſum trahendus dimittendusue unum tantum habeat ante ſe cui nauicula includatur. Secundum, ut dum demittis unà ſemper primos duos demittas, & unum trahas uel unum demittendo duos primos trahas. Tertiũ, ut quocunqꝫ ſurſum tracto uel demiſſo, omnes qui ante ſunt ſurſum trahere neceſſe

eſt

Figure 43: Cardan 294.

Wei Zhang produced a new edition (Figure 44) of a Chinese work at the time of the *Fourth Gathering for Gardner* and I have the following slides in my folder, probably made shortly after receiving the book.

Figure 44: Zhang' Back Cover.

Ch'ung-En Yü. *Ingenious Ring Puzzle Book* (Figure 45). In Chinese: Shanghai Culture Publishing Co., Shanghai, 1958. English translation by Yenna Wu, published by Puzzles – Jerry Slocum, Beverly Hills, Calif., 1981. On p. 6, it states that the puzzle was well known in the Sung Dynasty (960-1279). [There is a recent version, edited into simplified Chinese (with some English captions, etc.) by Lian Huan Jiu, with some commentary by Wei Zhang, giving the author's name as Yu Chong En, published by China Children's Publishing House, Beijing, 1999.]

Figure 45: Zhang' Plate 4.

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[9] Pablo Minguét è (or é) Yról (or Irol). *Engaños à Ojos Vistas, y Diversion de Trabajos Mundanos, Fundada en Licitos Juegos de Manos, que contiene todas las diferencias de los Cubiletes, y otras habilidades muy curiosas, demostradas con diferentes Láminas, para que los pueda hacer facilmente qualquier entretenido*, Pedro Joseph Alonso y Padilla, Madrid, 1733 [6].

[10] H.P. L'Orange, P. J. Nordhagen. *Mosaics*, Dreyers Forlag, Oslo, 1958 [7].

[1] This bibliography also points to the first part of this article – David Singmaster. "Some early topological puzzles: Part 1", *Recreational Mathematics Magazine* 3, 9–38, 2015. However, here it is widely commented by the author. The reader should have that in consideration when reading both parts.

[2] Original work published 1907.

[3] 2nd ed., 1557; 5th ed., 1581, ??NYS. Included in Vol. III of the *Opera Omnia*, Joannis Antonius Huguetan, Marcus Antoniius Ravaud, Lyon, 1663, and often reprinted, e.g. in 1967.

[4] 2nd ed., amended, 2002.

[5] The authorship is a matter of debate. Henry Llewellyn Williams Jr. is generally credited with it, probably assisted by Mr. Dick and John Wyman. There was a UK book of the same title but quite different content and there were several books which used large amounts of this book. My thanks to Jerry Slocum for providing a copy of this.

[6] Frontispiece + 12 + 218 pp. This had a number of editions and printings: 1755, c1760, 1766, 1820, 1822, 1847, 1888, 1888 and there was a 1981 facsimile of an 1864 ed. The early history of this book is confused. Pp. 1-25 is a fairly direct translation of the 1725 Ozanam, vol. IV, pp. 393-406. A number of other pictures and texts also are taken from Ozanam.

[7] English version, Methuen, 1966.

[11] Jacques Ozanam. *Recreations Mathematiques et Physiques, qui contiennent Plusieurs Problêmes d'Arithmétique, de Géométrie, de Musique, d'Optique, de Gnomonique, de Cosmographie, de Mécanique, de Pyrotechnie, & de Physique. Avec un Traité des Horloges Elementaires.* [8], Claude Jombert, Paris, 1723 [9].

[12] J. Prévost, *La Première Partie des Subtiles et Plaisantes Inventions, Contenant Plusieurs Jeux de Récréation*, Antoine Bastide, Lyons, 1584; ??NYS. [10].

[13] Margaret Rule. *Floor Mosaics in Roman Britain*, Macmillan Leisure Books for Sussex Archaeological Trust, 1974.

[14] Tudy Sammartini. *The Decorative Floors of Venice*, English ed., Merrell Publishers, London, 2000 [11].

[15] Daniel Schwenter (1585-1636). *Deliciæ Physico-Mathematicae. Oder Mathemat- und Philosophische Erquickstunden, Darinnen Sechshundert Drey und Sechsig, Schöne, Liebliche und Annehmliche Kunststücklein, Auffgaben und Fragen, auf; der Rechenkunst, Landtmessen, Perspctiv, Naturkündigung und andern Wissenschafften genom?n, begriffen seindt, Wiesolche uf der andern seiten dieses blats ordentlich nacheinander verzeichnet worden: Allen Kunstliebenden zu Ehren, Nutz, Ergössung des*

[8]Par feu [misprinted Parfeu in Vol. 1] M. Ozanam, de l'Académie Royale des Sciences, & Professeur en Mathematique. Nouvelle edition, Revûë, corrigée & augmentée. Vol. 4 has different title page: *Recreations Mathematiques et Physiques, ou l'on traite Des Phosphores Naturels & Artificiels, & des Lampes Perpetuelles. Dissertation Physique & Chimique. Avec l'Explication des Tours de Gibeciere, de Gobelets, & autres récréatifs & divertissans.* Nouvelle edition, Revûë, corrigée & augmentée.

[9]The bibliography of this book is a little complicated. I have prepared a more detailed 7 pp. version covering the 19 (or 20) French and 10 English editions, from 1694 to 1854, as well as 15 related versions – as part of my *The Bibliography of Some Recreational Mathematics Books.* The book first appeared in 1694, but the material of interest to us first appears in the new edition of 1723-1725 which extended the work form two vols to four vols and printed with varying dates. Various sources identify the editor as "one Grandin". It was reprinted in 1735, 1737?, 1741, 1750/1749, 1770. The text and plates of the 1725 and 1735 eds. seem identical, though some of the accessory material – lists of corrections and of plates – has been omitted and other has been rearranged. I have seen two versions of the 1735 - one has the plates inserted in the text, the other has them at the end as ordinary pages, while my 1725 has them at the end on folding pages. Most of the 1725 plates are identical to the 1696 plates, but there were a number of additions and reorderings. The 1725 plates have their 1696 plate numbers and 1725 page references at the top with new, more sequential, plate numbers at the bottom. The 1725 text sidenotes refer to the plate numbers at the top, while the 1735 and later sidenotes refer to the bottom numbers (However some of the new illustrations in vol. 4 are not described in the text and this makes me wonder if there was an earlier version with these new plates??). I will give the 1725 top plate numbers, followed by the bottom numbers in ( ) – e.g. plate 12 (14). The 1741 and the 1750/1749 eds. are essentially identical to the 1735 ed.

[10]Translated by Sharon King as *Clever and Pleasant Inventions Part One Containing Numerous Games of Recreations and Feats of Agility, by Which One May Discover the Trickery of Jugglers and Charlatans.* Hermetic Press, Seattle, 1998. [No Second Part ever appeared. This is apparently the first book primarily devoted to conjuring. Only five copies of the original are known. There was a facsimile in 1987. My thanks to Bill Kalush for bringing this work to my attention.

[11]Vianello Libri, 1999.

Gemüths und sonderbahren Wolgefallen am tag gegeben Durch M. Danielem Schwenterum., Jeremiæ Dümleis, Nuremberg, 1636 [12].

[16] J. Shearman. *The Early Italian Pictures in the Collection of Her Majesty The Queen*, CUP, 1983.

[17] Jerry Slocum & Dieter Gebhardt. *Puzzles from Catel's Cabinet and Bestelmeier's Magazine 1785 to 1823*, English translations of excerpts from the German Catel-Katalog and Bestelmeier-Katalog (Intro. by David Singmaster), *History of Puzzles Series*, The Slocum Puzzle Foundation, PO Box 1635, Beverly Hills, California, 90213, USA, 1997.

[18] John [Johann (or Hanss) Jacob] Wecker. *Eighteen Books of the Secrets of Art & Nature Being the Summe and Substance of Naturall Philosophy, Methodically Digested...* (as *De Secretis Libri XVII*), P. Perna, Basel, 1582, ??NYS [13].

[19] Luca Pacioli (or Paciuolo) (??-c1509). *De Viribus Quantitatis*, c1500, Italian MS in Codex 250, Biblioteca Universitaria di Bologna, one bibliography dates it as 1498 [14].

[20] Simon Witgeest. *Het Nieuw Toneel der Konsten, Bestaande uyt Sesderley Stukken: het eerste, handelt van alderley aardige Speeltjes en Klugjes: het tweede, van de Verligt-konst in 't Verwen en Schilderen: het derde, van het Etzen en Plaat-shijden: het vierde, van de Glas-konst: het vijfde, heest eenige aardige remedien tegen alderley Ziekten: het sesde, is van de Vuur-werken. Uyt verscheyde Autheuren by een vergadert, door S. Witgeest, Middel-borger.*, Jan ten Hoorn, Amsterdam, 1679 [15].

[21] Simon Witgeest. *Het Natuurlyk Tover-Boek, Of 't Nieuw Speel-Toneel Der Konsten. Verhandelende over de agt hondert natuurlijke Tover-Konsten. so uyt de de Gogel-tas, als Kaartspelen, Mathematische Konsten, en meer andered diergelijke aerdigheden, die tot vermaek, en tijtkorting verstrecken. Mitsgaders een Tractaet van alderley Waterverwen, en verligteryen; Als oock Een verhandelinge van veelderley Blanketsels. Om verscheyde welruykende Wateren, Poederen en Balsemen, als ook kostelijke beeydselen,*

[12] The text is in elaborate Gothic type with additional curlicues so that it is not always easy to tell what letter is intended! Probably edited for the press by Georg Philip Harsdörffer. Extended to three volumes by Harsdörffer in 1651 & 1653, with vol. 1 being a reprint of the 1636 vol. The 3 vol. version was reprinted in 1677 and 1692. There is a modern facsimile of the 3 vol. version edited by Jörg Jochen Berns, Keip Verlag, Frankfurt Am Main, 1991. Schott described this as a German translation of van Etten/Leurechon, but this is quite wrong.

[13] Now much augmented and inlarged by Dr. R. Read, Simon Miller, London, 1660, 1661. Reproduced by Robert Stockwell, London, nd [c1988].

[14] This has never been published, but Part 1: *Delle forze numerali cioe di Arithmetica* is extensively described in: A. Agostini; Il "De viribus quantitatis" di Luca Pacioli; Periodico di Matematiche (4) 4, 165-192, 1924. All references are to the problem numbers in this part, as given by Agostini, unless specified otherwise. There is a microfilm available in various places. In 1998, an Italian edition by Augusto Marinoni was published, but I haven't been able to get the publisher to reply. See `http://www.uriland.it/matematica/DeViribus/Presentazione.html`

[15] Facsimile with epilogue by John Landwehr, A. W. Sijthoff's Uitgeversmaatschappij N. V., Leiden, 1967 (present from Bill Kalush). There were many later editions, but Nanco Bordewijk has examined these and discovered that the 3rd ed. of 1686 (I can't recall if he saw the 1682 ed.) was so extensively revised and extended as to constitute a new book, and it has the different title given in the following entry (Other sources indicate these revisions are already in the 2nd ed. of 1682). Landwehr has written a bibliographical article on this book.

om het Aensicht, Hals en Handen, wit en sagt te maecken, door Simon Witgeest, Jan ten Hoorn, Amsterdam, 1686 [16].

[22] D. Wolfe & T. Rogers. *Puzzlers' Tribute, a Feast for the Mind*, A. K. Peters, Natick, Massachusetts, 2002.

[16] This is a much expanded and retitled 3rd edition of Witgeest's 1679 work. The new material is stated to already be in the 2nd ed. of 1682. There are many later editions in Dutch and it was translated into German as *Naturliches Zauber-Buch oder neuer Spiel-Platz der Künste*, Hoffmanns sel Wittw & Engelbert Streck, Nürnberg, 1702, with many later editions, mostly in Nürnberg, all apparently based on the 1682 ed.

Problems

A Random Logic Puzzle

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Abstract: *This paper introduces and solves a challenging logic puzzle inspired by George Boolos' "Hardest Logic Puzzle Ever". The analysis hinges on a characterization of questions in terms of the relevant knowledge which may be gleaned from their answers.*

Key-words: Logic puzzle, binary trees, information theoretic bounds.

1 The Hardest Logic Puzzle Ever

In [1] George Boolos presents the following intriguing logic puzzle (together with a solution). Boolos attributes the puzzle to Raymond Smullyan and dubs it "The Hardest Logic Puzzle Ever".

(HLPE) Three gods A, B, and C are called, in some order, True, False, and Random. True always speaks truly, False always speaks falsely, but whether Random speaks truly or falsely is a completely random matter. Your task is to determine the identities of A, B, and C by asking three yes-no questions; each question must be put to exactly one god. The gods understand English, but will answer all questions in their own language, in which the words for "yes" and "no" are "da" and "ja", in some order. You do not know which word means which.

Boolos' method of solution reveals that the puzzle contains uncoupled layers of complexity that may be tackled independently. Once some god is known not to be Random, the truth value of statement S may be determined from that god's answer to "Does 'da' mean true iff (you are True iff S)?" Accordingly, in the HLPE-inspired puzzle I offer here, 'da' and 'ja' are eliminated in favor of simple 'yes' and 'no' answers, and only one species of nonrandom god is used. The puzzle focuses on the problem of separating truthful gods from random gods. This is similar to and yet, as we will see, distinctly different from the HLPE 'layer' handled by finding a god who does not answer randomly. Furthermore, to ratchet up the difficulty and inject an algorithmic analysis flavoring we will not specify an allotment of questions, but rather make the determination of an optimal strategy the heart of the puzzle.

2 The Random Logic Puzzle

Here's the random logic puzzle.

(RLP) Two of four gods are called True, and the other two are called Random. Each of the two called True always answer questions truly. The responses of the two called Random are generated randomly. Your task is to separate the Trues from the Randoms by asking as few yes-no questions as necessary. Each question must be addressed to exactly one god.

Although some clarification is in order, those puzzle enthusiasts who prefer to attack problems without even a whiff of a hint may wish to set this paper aside and take a crack at the RLP now.

3 Representing Solution Strategies

Boolos' solution to the HLPE makes essential use of the freedom to choose the second question's addressee on the basis of the answer to the first question. Indeed any solution to the HLPE must exploit this freedom. So a solution strategy is inherently more flexible (and complicated) than a simple list of questions. A strategy for the RLP may be modelled as a (finite) rooted binary tree with a question and an addressee specified at each internal node, and a partition of the gods into two pairs specified at each leaf. The root represents the opening question, each internal right child represents the question to be posed if the parent question has been answered affirmatively, and each internal left child represents the question to be posed if the parent question has been answered negatively. A strategy may be called successful if at each leaf there is no assignment of names to gods which is consistent with all the answers leading from the root to the leaf, but inconsistent with the partition given at the leaf. The RLP's task is to establish the minimum possible height of a successful strategy.

Figure 1 illustrates an unsuccessful strategy which might fittingly be called the naive strategy. The four gods have been labelled A, B, C and D arbitrarily. Notice for example that the leaf at the end of a YES, YES answer sequence is consistent with A and C as Randoms and B and D as Trues. Yet this naming scheme is inconsistent with the partition $\{\{\mathrm{A},\mathrm{B}\},\{\mathrm{C},\mathrm{D}\}\}$ at the leaf.

4 Clarifications

In [3] Rabern and Rabern propose a solution to the HLPE using only two yes-no questions. They argue that the question 'Are you going to answer this question with the word that means *no* in your language?' can not be answered truthfully

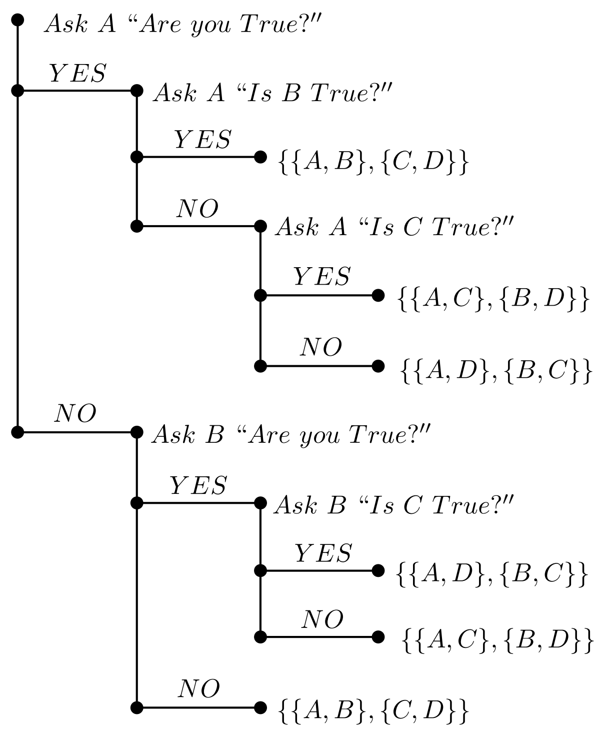

Figure 1: An unsuccessful strategy

by True, and thus, if he is the addressee of this question his head must explode revealing his identity. Notice that our model of a strategy implicitly precludes such questions. Admissible questions can not have any outcome other than a yes or no answer. (From here on this requirement on yes-no questions will be in force not just in our treatment of the RLP, but also in our continuing discussion of the HLPE.) Furthermore, it should be understood that

> Each time one of the Randoms is asked a question, the response is determined at random independently of any prior responses from that god or the other Random.
>
> Each god knows his own identity as well as the identities of the other gods.
>
> Once the gods have been separated into two pairs it is not required to specify which pair is which, and
>
> The gods are distinguishable in a manner that permits questions to reference a specific god or subset of the gods other than the addressee, and enables the questioner (and the gods themselves) to know at all times which god was addressed in any previous question, and which god(s) were referenced. (This assumption was already applied in our use of the labels A, B, C and D for the four gods.)

Those readers who haven't taken a crack at the puzzle yet are encouraged to do so now.

5 Information Theoretic Bounds

Since there's no limit to the number of questions which can be contrived, proving the optimality of a strategy may appear to be a daunting task. In the case of the HLPE, however, a simple information theoretic observation proves the optimality of any correct three question strategy. In that puzzle there are six possible assignments of names to gods that must be discriminated amongst. The answers to two yes-no questions (two bits of information) can discriminate among at most four cases. (cf. [2]) So what's the information theoretic lower bound on the number of questions required by the RLP? Since there are only three ways to partition a set of four elements into two pairs, only two bits of information are needed. Here, however, the simple information theoretic lower bound can not be achieved. We'll need a more careful accounting of what can and can not be learned from arbitrary yes-no questions.

6 The Tree of Knowledge

Let's track information by annotating each node in a strategy tree with the set of assignments of names to gods which are consistent with the answers that lead to the node. We'll call this annotated tree the strategy's tree of knowledge. (In effect we're characterizing each question by the relevant knowledge which may be gleaned from its answer. This characterization transforms the unbounded class of allowed questions into a finite, and thus readily analyzed, set.) Notice that for a successful strategy tree, the annotation at each leaf must be a subset (not necessarily proper) of the complementary pair of name assignments which are consistent with the partition specified at the leaf.

Figure 2 shows the tree of knowledge for the unsuccessful naive strategy of Figure 1. The one leaf with a "correct" partition is marked with a ✓, the others with ✗. Let's adopt a shorthand for assignments of names to gods. A string with two 'T's and two 'R's represents an assignment which gives the name True to A (resp. B, C, D) if the first (resp. second, third, fourth) character in the string is 'T', and gives the name Random to A (resp. B, C, D) if the first (resp. second, third, fourth) character is 'R'. Let $U = \{\mathrm{TTRR}, \mathrm{TRTR}, \mathrm{TRRT}, \mathrm{RTTR}, \mathrm{RTRT}, \mathrm{RRTT}\}$ be the collection of all possible assignments. This collection U decomposes into three mutually disjoint pairs of complementary assignments $\{\mathrm{TTRR}, \mathrm{RRTT}\}$, $\{\mathrm{TRTR}, \mathrm{RTRT}\}$ and $\{\mathrm{TRRT}, \mathrm{RTTR}\}$ each of which corresponds to a possible separation of the Randoms from the Trues. For convenience introduce the following notation. Let $\mathcal{N}$ be the set of all nodes in a strategy's knowledge tree. At each node $P \in \mathcal{N}$, $k(P)$ equals the subset of U stored at P. At each internal node P, $k_l(P) = k(P\text{'s left child})$ and $k_r(P) = k(P\text{'s right child})$. So in particular the tree of knowledge for any strategy satisfies $k(\text{root}) = U$. Also we clearly have $k_l(P) \subseteq k(P)$ and $k_r(P) \subseteq k(P)$ at every internal node P. Our assumption that any admissible question must result in a yes or no answer yields $k_l(P) \cup k_r(P) = k(P)$ for every internal node P. Now let's partition each collection $k(P)$ into $k^R(P)$ and $k^T(P)$ where $k^R(P)$ contains the assignments in

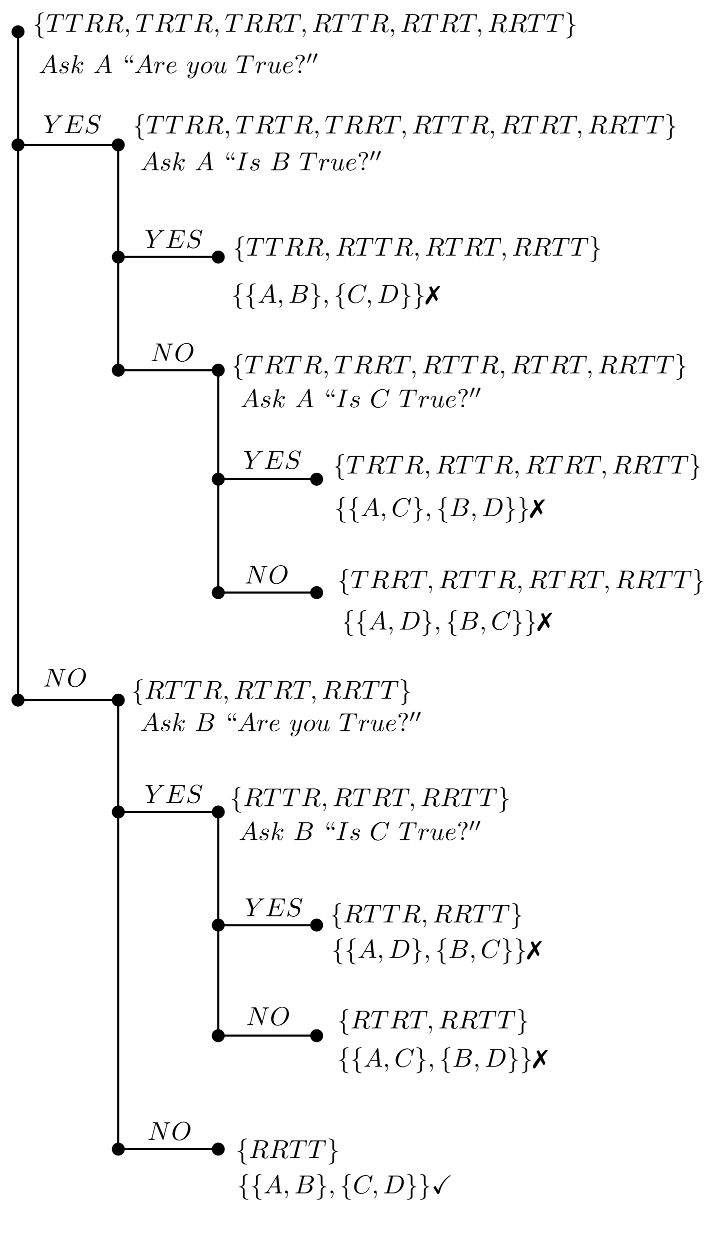

Figure 2: Naive strategy's tree of knowledge

$k(P)$ which give the name Random to the addressee of the question at P, and $k^T(P) = k(P) \setminus k^R(P)$ contains the assignments in $k(P)$ which give the name True to the addressee at P. Since each Random may answer yes or no to any question in any context, $k^R(P) \subseteq k_l(P)$ and $k^R(P) \subseteq k_r(P)$ at each internal node P.

7 Improving Lower bounds

The observations and notational conventions of the previous section prepare us to prove

The Progress Speed Limit Lemma.
For each $P \in \mathcal{N}$, $max\{|k_l(P)|, |k_r(P)|\} \geqslant |k(P)| - 1$.

In other words, there is never any question whose answer is guaranteed to enable us to eliminate more than one possible naming scheme from consideration.

Proof. The collection U itself only has three assignments that give the name True to any one god. So regardless of the addressee of the question at P, $|k^T(P)|$ must always be less than or equal to 3. The inclusion $k^T(P) \subseteq k(P) = k_l(P) \cup k_r(P)$ implies that $k^T(P) \setminus k_l(P)$ and $k^T(P) \setminus k_r(P)$ are disjoint. So $|(k^T(P) \setminus k_l(P)) \cup (k^T(P) \setminus k_r(P))| \leqslant |k^T(P)| \leqslant 3$, and at least one of the two sets $k^T(P) \setminus k_l(P)$ and $k^T(P) \setminus k_r(P)$ has size less than or equal to 1. Suppose $|k^T(P) \setminus k_l(P)| \leqslant 1$. Then since $k^R(P) \subseteq k_l(P)$ we know $k(P) \setminus k_l(P) = k^T(P) \setminus k_l(P)$. Thus $|k(P) \setminus k_l(P)| \leqslant 1$. So $|k_l(P)| \geq |k(P)| - 1$. Similarly, if $|k^T(P) \setminus k_r(P)| \leqslant 1$, then $|k_r(P)| \geqslant |k(P)| - 1$. □

As noted earlier, $|k(\text{root})| = |U| = 6$. Since there are only two assignments of names to gods compatible with each separation of the gods into pairs, for every leaf in any successful strategy tree $|k(\text{leaf})| \leqslant 2$. It thus immediately follows from the Progress Speed Limit Lemma that an optimal RLP strategy tree has a height of at least four. In other words, there is no successful strategy that never requires the use of more than three questions. Though this lower bound improves upon the simple information theoretic lower bound of two, it still fails to be tight. The following additional observation will show that a successful RLP strategy tree must in fact have a height of at least five.

Lemma. *If $|k(P)| = 3$ for some node P in a successful RLP strategy tree, then P's children can't both be leaves.*

Proof. Given $|k(P)| = 3$, $|k_l(P)|$ and $|k_r(P)|$ can't both be less than or equal to 1 since $k_l(P) \cup k_r(P) = k(P)$. Similarly, $|k_l(P)| = 2$, $|k_r(P)| = 0$ and vice versa are impossible. Furthermore $k_l(P)$ and $k_r(P)$ cannot both consist of complementary pairs. Since distinct complementary pairs are disjoint, if both $k_l(P)$ and $k_r(P)$ were complementary pairs, then $|k(P)| = |k_l(P) \cup k_r(P)|$ would be 2 or 4. Finally we need to consider (and dismiss) the possibility that one of the sets $k_l(P)$ and $k_r(P)$ is a complementary pair and the other a singleton. Suppose, for instance, $k_l(P) = \{\text{TRRT}, \text{RTTR}\}$ and $k_r(P) = \{\text{WXYZ}\}$, where WXYZ represents some element of U. If the addressee of the question at P is B or C, then TRRT$\in k^R(P) \subseteq k_r(P)$ forces WXYZ=TRRT, which is impossible. Similarly, if the addressee of the question at P is A or D, then RTTR$\in k^R(P) \subseteq k_r(P)$ forces WXYZ=RTTR, which is also impossible. The other permutations are perfectly analogous. □

From the preceding arguments and observations a successful strategy tree must have some node P with $|k(P)| = 3$. This node must be at a depth of at least 3 from the root. This node must also have some descendant with depth greater by at least 2. So the height of a successful RLP strategy tree must be at least 5. This time we've finally honed in on the answer. All that remains is to exhibit a successful strategy with a tree of height exactly 5. (This result may be labelled a theorem by any reader who feels the use of lemmas so demands.)

8 An Optimal Strategy

Though the construction of an optimal strategy leaves us with plenty of latitude, it will be convenient to exploit the notation and structures we've already introduced. Although the decomposition $k(P) = k^R(P) \cup k^T(P)$ depends on the addressee of the question at P, note that $k(P)$ itself is determined by the part of the strategy tree strictly above node P. So node P may be constructed from $k(P)$. (This is even vacuously true at the root.) If there is some X from {A, B, C, D} such that X is assigned the name True by more than one element of $k(P)$ then clearly there is some Y from {A, B, C, D}\{X} such that X and Y are both assigned True by one element, say α, of $k(P)$, while X and not Y are assigned True by another element, say β, of $k(P)$. In this case let node P be an internal node with the question "Is Y named True?" addressed to X. Otherwise, let P be a leaf. Notice that when P is made into an internal node $k_r(P)$ must contain α and not β, while $k_l(P)$ must contain β and not α. It follows both that $max\{|k_l(P)|, |k_r(P)|\} \leqslant |k(P)| - 1$ and that $k_l(P)$ and $k_r(P)$ are both nonempty. Thus any internal node P (for which $|k(P)|$ is clearly greater than or equal to 2) has depth at most four. So our construction yields a tree with depth at most five, as claimed. For any leaf P there is no X in {A, B, C, D} such that X is assigned the name True by more than one element of $k(P)$. This implies that the nonempty set $k(P)$ is either a singleton or a complementary pair. Either way the leaf P can be assigned a partition of {A, B, C, D} into two pairs which is consistent with $k(P)$. This completes the construction of a successful optimal strategy.

9 Exercises

Clearly the RLP may be viewed as a special case of the parametrized family of similar puzzles with n Trues and m Randoms. Our original statement of the RLP presumes the existence of a successful strategy. The interested reader is invited to show that this presumption is justified whenever $n \geqslant m$, but does not hold when $m > n > 0$. The final exercise left to the reader is to demonstrate the claim made earlier that any solution to the HLPE must exploit the freedom to choose the second question's addressee on the basis of the answer to the first question.

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[4] T.S. Roberts. "Some thoughts about the hardest logic puzzle ever", *Journal of Philosophical Logic* (30), 2001.

A Recursive Solution to Bicolor Towers of Hanoi Problem

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Dedicated to my father to whom it would never reach.

Abstract: *In this paper, we present a recursive algorithm to solve bicolor towers of Hanoi problem.*

Key-words: Algorithms, recursion, towers of Hanoi.

1 Introduction

The bicolor towers of Hanoi problem is a variation of traditional towers of Hanoi [1] problem. It was offered to grade 3-6 students at *2ème Championnat de France des Jeux Mathématiques et Logiques* held in July 1988 [2]. Suppose there are three pegs, A, B and Via. Suppose there are two sets of disks α and β, where $\alpha = \{ \alpha_i \mid 1 \leqslant \mathrm{i} \leqslant \mathrm{n} \}$ and $\beta = \{ \beta_i \mid 1 \leqslant \mathrm{i} \leqslant \mathrm{n} \}$ such that color of every disk in α is white and color of every disk in β is black. Also, for every i, radius of α_i is i and radius of β_i is i. Suppose there are placed a finite number of n disks of alternating colors on pegs A and B with decreasing size from bottom to top. Peg Via is an auxillary peg. The goal of the problem is to make the towers of pegs A and B monochrome. The biggest disks at the bottom of the pegs A and B are required to swap positions. The rules of the problem are the following:

1. Only one disk can be moved at any time.
2. At no time can a larger disk be placed on a smaller disk. Same size disks can be placed over one another.

Figures 1a and 1b show the initial and final configuration of bicolor towers of Hanoi problem for $n = 4$. Out of the bicolor towers of Hanoi problem, we derive a new problem, which is same as the bicolor towers of Hanoi problem except that in this variation, we compel the output configuration to maintain the base disks of pegs A and B in their original place as was in its original configuration. We name this problem as easy bicolor towers of Hanoi problem. Figures 2a and 2b show the initial and final configuration of easy bicolor towers of Hanoi problem for $n = 4$. Hereafter, we refer the bicolor towers of Hanoi problem as the bicolor problem and the easy bicolor towers of Hanoi problem as the easy bicolor problem.

(a) (b)

Figure 1: (a) Initial configuration of bicolor towers of Hanoi problem ($n = 4$). (b) Final configuration of bicolor towers of Hanoi problem (n=4).

(a) (b)

Figure 2: (a) Initial configuration of easy bicolor towers of Hanoi problem (n=4). (b) Final configuration of easy bicolor towers of Hanoi problem (n=4).

2 Double Towers of Hanoi Problem

We will study another variation to traditional towers of Hanoi problem which is double towers of Hanoi problem. Suppose there are three pegs, A, B and Via. Suppose there are two sets of disks α and β, where $\alpha = \{ \alpha_i \mid 1 \leqslant \mathrm{i} \leqslant \mathrm{n} \}$ and $\beta = \{ \beta_i \mid 1 \leqslant \mathrm{i} \leqslant \mathrm{n} \}$ such that color of every disk in α is white and color of every disk in β is black. Also, for every i, radius of α_i is i and radius of β_i is i. Suppose there are placed a finite number of *2n* disks on peg A with non-increasing size from bottom to top such that for every same size pair, the black disk is always placed below the white disk. Peg Via is an auxillary peg. The goal of the problem is to move all the disks from peg A to peg B. In the output configuration on peg B, for every same size disk pair, the black disk is supposed to get placed below the white disk. The rules of the problem remains the same. Figures 3a and 3b show the initial and final configuration of double towers of Hanoi problem for n=4. Algorithm 1 solves the double towers of Hanoi problem with a small fault in the output configuration. The fault in the output configuration is that the position of the bottommost same size disk pair gets swapped, that is, the white disk get placed below the black disk as shown in Figures 4a and 4b.

To overcome this fault, we present enhanced double towers of Hanoi algorithm which calls the double towers of Hanoi routine twice to overcome this fault. The correctness of algorithm 1 (with the fault) is straightforward whereas the correctness of algorithm 2 is shown in Figures 5a and 5b.

(a) (b)

Figure 3: (a) Initial configuration of double towers of Hanoi problem (n=4). (b) Final configuration of double towers of Hanoi problem (n=4).

(a) (b)

Figure 4: (a) Initial configuration of double towers of Hanoi problem (n=4). (b) Final configuration of double towers of Hanoi problem with a fault due to algorithm 1 (n=4).

Algorithm 1 Algorithm for Double Towers of Hanoi Problem

```
procedure DoubleTowersOfHanoi(A, B, Via, n)
    if n == 1 then
        print: Move from A to B
        print: Move from A to B
    else
        DoubleTowersOfHanoi(A, Via, B, n − 1)
        print: Move from A to B
        print: Move from A to B
        DoubleTowersOfHanoi(Via, B, A, n − 1)
    end if
end procedure
```

The recurrence relation of algorithm 1 is

$$A_1(n) = \begin{cases} 2 & n = 1 \\ 2.A_1(n-1) + 2 & n > 1 \end{cases}$$

Algorithm 2 Enhanced Algorithm for Double Towers of Hanoi Problem

procedure EnhancedDoubleTowersOfHanoi(A, B, Via, n)
  DoubleTowersOfHanoi(A, Via, B, n) ▷ step 1
  DoubleTowersOfHanoi(Via, B, A, n) ▷ step 2
end procedure

The recurrence relation of algorithm 2 is

$$\boxed{A_2(n) = \ 2.A_1(n) \qquad n \geqslant 1}$$

(a) (b)

Figure 5: (a) Configuration after step 1 of algorithm 2 (n=4). (b) Configuration after step 2 of algorithm 2 (n=4).

3 The Merge Problem

In this section, we study the merge problem. Suppose there are three pegs, A, B and Via. Suppose there are two sets of disks α and β, where $\alpha = \{\ \alpha_i \mid 1 \leqslant \mathrm{i} \leqslant \mathrm{n}\ \}$ and $\beta = \{\ \beta_i \mid 1 \leqslant \mathrm{i} \leqslant \mathrm{n}\ \}$ such that color of every disk in α is white and color of every disk in β is black. Also, for every i, radius of α_i is i and radius of β_i is i. Suppose there are placed a finite number of n disks on peg A from set β with decreasing size from bottom to top and there are placed a finite number of n disks on peg B from set α with decreasing size from bottom to top. Peg Via is an auxillary peg. The goal of the problem is to place all the *2n* disks on peg A with non-increasing size from bottom to top such that for every same size pair, the black disk is always placed below the white disk. The rules of the problem remains the same.

Figures 6a and 6b show the initial and final configuration of the merge problem for n=4. Algorithm 3 solves the merge problem.

(a) (b)

Figure 6: (a) Initial configuration of the merge problem (n=4). (b) Final configuration of the merge problem (n=4).

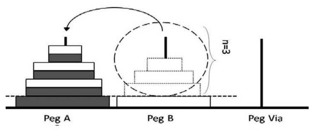
Algorithm 3 Algorithm for Merge Problem

```
procedure MergeProblem(A, B, Via, n)
    if n == 1 then
        print: Move from B to A
    else
        MergeProblem(A, B, Via, n − 1)                ▷ step 1
        DoubleTowersOfHanoi(A, Via, B, n − 1)         ▷ step 2
        print: Move from B to A                       ▷ step 3
        DoubleTowersOfHanoi(Via, A, B, n − 1)         ▷ step 4
    end if
end procedure
```

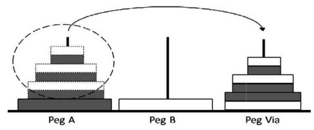
The recurrence relation of algorithm 3 is

$$A_3(n) = \begin{cases} 1 & n = 1 \\ A_3(n-1) + 2.A_1(n-1) + 1 & n > 1 \end{cases}$$

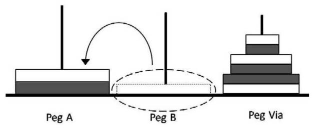
The correctness of algorithm 3 is shown in Figures 7a, 7b, 7c and 7d.



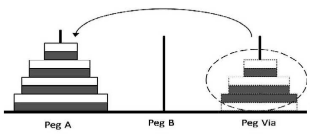
(a)



(b)



(c)

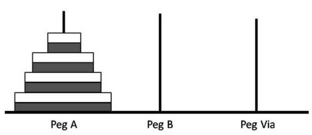


(d)

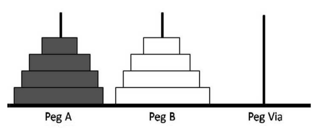
Figure 7: (a) Configuration after step 1 of algorithm 3 (n=4). (b) Configuration after step 2 of algorithm 3 (n=4). (c) Configuration after step 3 of algorithm 3 (n=4). (d) Configuration after step 4 of algorithm 4 (n=4).

4 The Split Problem

In this section, we study the split problem. Suppose there are three pegs, A, B and Via. Suppose there are two sets of disks α and β, where $\alpha = \{ \alpha_i \mid 1 \leqslant \mathrm{i} \leqslant \mathrm{n} \}$ and $\beta = \{ \beta_i \mid 1 \leqslant \mathrm{i} \leqslant \mathrm{n} \}$ such that color of every disk in α is white and color of every disk in β is black. Also, for every i, radius of α_i is i and radius of β_i is i. Suppose there are placed a finite number of $2n$ disks on peg A with non-increasing size from bottom to top such that for every same size pair, the black disk is always placed below the white disk. Peg Via is an auxillary peg. The goal of the problem is to place n disks on peg A from set β with decreasing size from bottom to top and n disks on peg B from set α with decreasing size from bottom to top. The rules of the problem remains the same. Figures 8a and 8b show the initial and final configuration of the split problem for n=4. Algorithm 4 solves the split problem. The correctness of algorithm 4 is shown in Figures 9a, 9b, 9c and 9d.



(a)



(b)

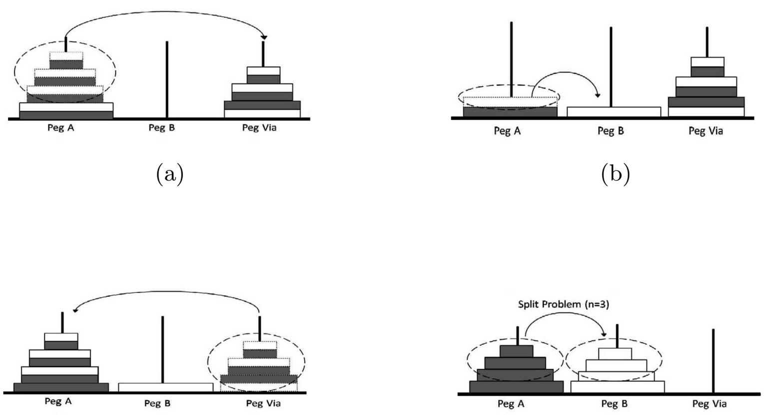
Figure 8: (a) Initial configuration of the split problem (n=4). (b) Final configuration of the split problem (n=4).

Algorithm 4 Algorithm for Split Problem

```
procedure SplitProblem(A, B, Via, n)
    if n == 1 then
        print: Move from A to B
    else
        DoubleTowersOfHanoi(A, Via, B, n − 1)        ▷ step 1
        print: Move from A to B                      ▷ step 2
        DoubleTowersOfHanoi(Via, A, B, n − 1)        ▷ step 3
        SplitProblem(A, B, Via, n − 1)               ▷ step 4
    end if
end procedure
```

The recurrence relation of algorithm 4 is

$$A_4(n) = \begin{cases} 1 & n = 1 \\ A_4(n-1) + 2.A_1(n-1) + 1 & n > 1 \end{cases}$$



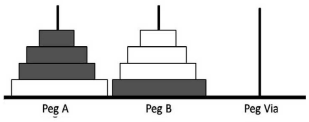
(a) (b) (c) (d)

Figure 9: (a) Configuration after step 1 of algorithm 4 (n=4). (b) Configuration after step 2 of algorithm 4 (n=4). (c) Configuration after step 3 of algorithm 4 (n=4). (d) Configuration after step 4 of algorithm 4 (n=4).

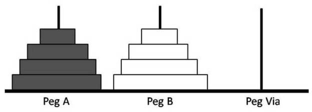
5 The Swap Base Disk Problem

In this section, we study the swap base disk problem. Suppose there are three pegs, A, B and Via. Suppose there are two sets of disks α and β, where $\alpha = \{ \alpha_i \mid 1 \leqslant \mathrm{i} \leqslant \mathrm{n} \}$ and $\beta = \{ \beta_i \mid 1 \leqslant \mathrm{i} \leqslant \mathrm{n} \}$ such that color of every disk in α is white and color of every disk in β is black. Also, for every i, radius of α_i is i and radius of β_i is i. Suppose there are placed a finite number of n disks on peg

A from set $\beta^* = \{ \beta_i \mid 1 \leqslant \text{i} \leqslant \text{n-1} \} \cup \{ \alpha_n \}$ with decreasing size from bottom to top and there are placed a finite number of n disks on peg B from set $\alpha^* = \{ \alpha_i \mid 1 \leqslant \text{i} \leqslant \text{n-1} \} \cup \{ \beta_n \}$ with decreasing size from bottom to top. Peg Via is an auxillary peg. The goal of the problem is to make the towers on peg A and peg B monochrome by swapping the base disks of towers on peg A and peg B. The rules of the problem remains the same. Figures 10a and 10b show the initial and final configuration of the swap base disk problem. Algorithm 5 solves the swap base disk problem. The correctness of algorithm 5 is shown in Figures 11a, 11b, 11c, 11d,11e,11f, 11g and 11h.



(a)



(b)

Figure 10: (a) Initial configuration of swap base disk problem (n=4). (b) Final configuration of swap base disk problem (n=4).

Algorithm 5 Algorithm for Swap Base Disk Problem

procedure SwapBaseDiskProblem(A, B, Via, n)
  if $n == 1$ **then**
    print: Move from A to Via
    print: Move from B to A
    print: Move from Via to B
  else
    MergeProblem(A, B, Via, $n-1$) ▷ step 1
    print: Move from B to Via ▷ step 2
    DoubleTowersOfHanoi(A, Via, B, $n-1$) ▷ step 3
    print: Move from A to B ▷ step 4
    DoubleTowersOfHanoi(Via, B, A, $n-1$) ▷ step 5
    print: Move from Via to A ▷ step 6
    EnhancedDoubleTowersOfHanoi(B, A, Via, $n-1$) ▷ step 7
    SplitProblem(A, B, Via, $n-1$) ▷ step 8
  end if
end procedure

The recurrence relation of algorithm 5 is

$$A_5(n) = \begin{cases} 3 & n = 1 \\ A_3(n-1) + A_4(n-1) + A_2(n-1) + 2.A_1(n-1) + 3 & n > 1 \end{cases}$$

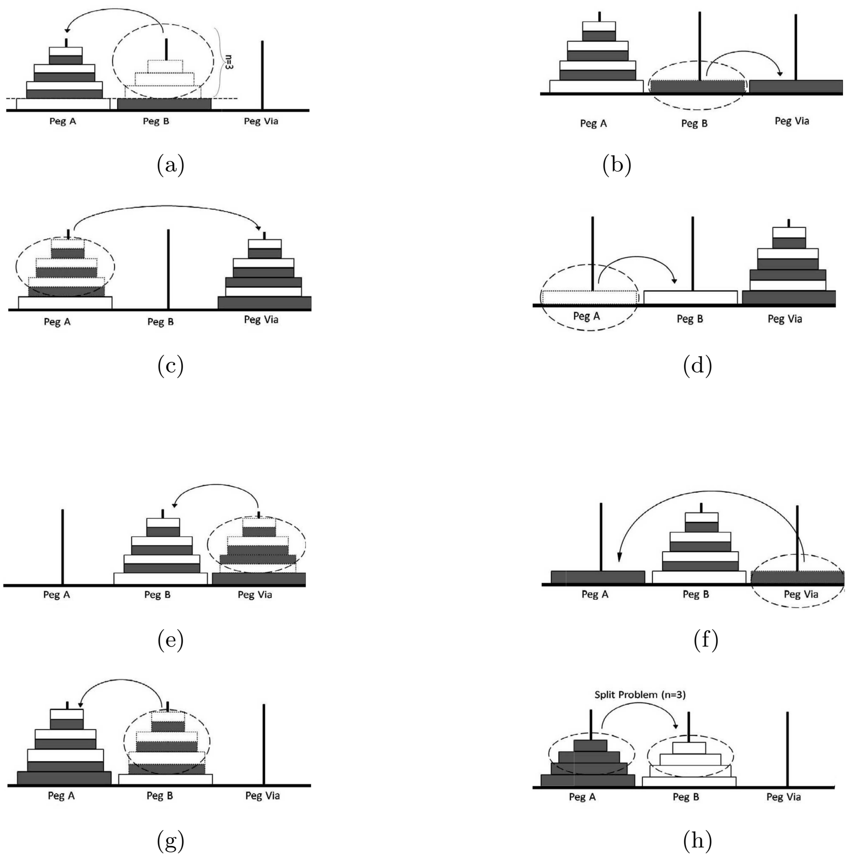


Figure 11: (a) Configuration after step 1 of algorithm 5 (n=4). (b) Configuration after step 2 of algorithm 5 (n=4). (c) Configuration after step 3 of algorithm 5 (n=4). (d) Configuration after step 4 of algorithm 5 (n=4). (e) Configuration after step 5 of algorithm 5 (n=4). (f) Configuration after step 6 of algorithm 5 (n=4). (g) Configuration after step 7 of algorithm 5 (n=4). (h) Configuration after step 8 of algorithm 5 (n=4).

6 The Easy Bicolor Towers of Hanoi

In this section, we present the algorithm for solving the easy bicolor problem. Algorithm 6 solves the easy bicolor problem. The correctness of algorithm 6 is shown in Figure 12.

Algorithm 6 Algorithm for Easy Bicolor Problem

```
procedure EasyBicolorProblem(A, B, Via, n)
    if n == 1 then
▷ do nothing
    else
        BicolorProblem(A, B, Via, n − 1)                ▷ step 1
    end if
end procedure
```

The recurrence relation of algorithm 6 is

$$A_6(n) = \begin{cases} 0 & n = 1 \\ A_7(n-1) & n > 1 \end{cases}$$

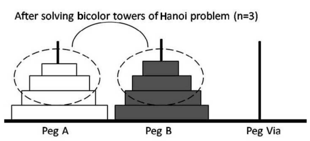


Figure 12: Configuration after step 1 of algorithm 6 (n=4).

7 The Bicolor Towers of Hanoi

In this section, we present the algorithm for solving the bicolor problem. Algorithm 7 solves the bicolor problem. The correctness of algorithm 7 is shown in Figures 13a and 13b.

Algorithm 7 Algorithm for Bicolor Problem

```
procedure BicolorProblem(A, B, Via, n)
    if n == 1 then
        print: Move from A to Via
        print: Move from B to A
        print: Move from Via to B
    else
        EasyBicolorProblem(A, B, Via, n − 1)          ▷ step 1
        SwapBaseDiskProblem(A, B, Via, n)             ▷ step 2
    end if
end procedure
```

The recurrence relation of algorithm 7 is

$$A_7(n) = \begin{cases} 3 & n = 1 \\ A_6(n-1) + A_5(n) & n > 1 \end{cases}$$

(a) (b)

Figure 13: (a) Configuration after step 1 of algorithm 7 (n=4). (b) Configuration after step 2 of algorithm 7 (n=4).

8 Conclusion

We have studied the recursive solution to bicolor towers of Hanoi problem for size n.

Appendix A

A Recursive Solution to the Traditional Towers of Hanoi Problem

Algorithm 8 Algorithm for traditional Towers of Hanoi Problem

```
procedure TowersOfHanoi(A, B, Via, n)
    if n == 1 then
        print: Move from A to B
    else
        TowersOfHanoi(A, Via, B, n − 1)
        print: Move from A to B
        TowersOfHanoi(Via, B, A, n − 1)
    end if
end procedure
```

The recurrence relation of algorithm 8 is

$$\boxed{A_8(n) = \begin{cases} 1 & n = 1 \\ 2.A_8(n-1) + 1 & n > 1 \end{cases}}$$

References

[1] Rosen, K., H. *Discrete Mathematics & and Its Applications: With Combinatorics and Graph Theory*, McGraw-Hill Offices, 2010.

[2] `http://www.cut-the-knot.org/recurrence/BiColorHanoi.shtml`

Ludus CIUHCT

Mathematics and Arts

3D Anamorphic Sculpture and the S-Cylindrical Mirror

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Abstract: *Stand in just the right place in front of a painting and see an image. Stand in any other position, and the image distorts to become unrecognizable. That is an anamorphic image. The ancient Greeks used such images. Artists of the Renaissance explored the concept in paintings and murals. A great leap forward was made in 17th century France when artists made drawings that had to be viewed with the aid of a cylindrical mirror. In this work we show how anamorphism appears in sculpture, creating distorted 3D scenes that can be viewed in normal proportions in a mirror mounted within the piece. We show examples that use many kinds of cylindrical mirrors but also irregular and S-shaped wave mirrors.*

Key-words: Mirrors, anamorphosis, 3D anamorphic sculpture, vertical S-cylindrical mirror, *Alice in Wonderland*.

1 Mirrors

A mirror is a flat, smooth surface that reflects an image. And over time, mirrors have been endowed with magical, and even supernatural powers; depending on the circumstance a mirror can evoke love, terror, greed, jealousy, or horror.

Literature abounds with examples: the Greek mythology of Narcissus; the Grimm Brothers' *Snow White and the Seven Dwarfs*; Lewis Carroll's *Through the Looking Glass*; William Shakespeare's *Richard II*; Sylvia Plath's poem "Mirror"; Bram Stoker's *Dracula*; Alfred, Lord Tennyson's poem "The Lady of Shalott"; the Mirror of Erised and two-way mirrors in the *Harry Potter* novels; the Manuel of the Planes for *Dungeons & Dragons*; and H. P. Lovecraft and Henry S. Whitehead's "The Trap".

Mirrors are such a part of our everyday life that we take them for granted. We get up in the morning, wash, dress, and before we leave the house, take one last look in the mirror. Yet what we see is not a true representation of who we are. The flat mirror leaves shape and dimensions unaltered, but shows a reversed left-to-right image of us, which is not how the outside world sees us.

Mirrors that are not flat can change, in a variety of ways, the dimensions, the shape, and even the orientation of an object. Ask anyone who has stood in front of a carnival mirror what they had experienced, and they will respond that they saw distorted, undulating images – images that changed as they moved in close towards and then backed away from the mirror.

Much like the reaction someone would have had with a carnival mirror, when I was a very young child, I saw the most astonishing mirror reflection. Seeing this reflection startled me, and its effect has influenced how I looked at mirrors and reflections to this day. The experience happened when I was around 12 years old, and it took place at a formal family dinner.

For this special dinner, the very best linen, china, crystal, and silverware were brought out. And as I sat waiting for the adults to be seated, my soupspoon caught my gaze (Figure 1).

Figure 1: Soupspoon.

I moved closer to take a better look, and what to my surprise, there was something floating in the bowl of the spoon. To my astonishment, the object was the dinning room chandelier. My eyes were fixed on the reflection in the spoon, which seemed all too real, yet when I tried to grab the floating shape between my fingers, the fingers passed through the image as if it were a ghost. When the dinner was over, I was called to help with the kitchen clean up. And with the spoon reflection fresh in my mind, I looked at the sterling silver coffee urn and large service pieces were looked at in a new way: as the dishes were being dried, I started to move in close and then back away from the curved mirrored pieces. I was amazed by the most delightful reflections, which changed as I moved around the kitchen. There looking back at me were reflections that merged and then changed from a one-eyed Cyclops, to a bird beaked faces without eyes, to a short-bodied, and then a tall-bodied figure.

Today, there are two childhood experiences that have influenced my artwork. The first was soupspoon observation I just mentioned. The second was a discovery of a book about the anamorphic arts. As I look back, I recall being mesmerized by the images in that book, an eighteenth-century French anamorphic art book with pictures that were flat, very distorted, and only recognizable when the images were viewed in a cylindrical mirror.

2 Anamorphosis

The anamorphic arts have held a special fascination for both the artists and viewers, with possibly one of the earliest examples being found in the prehistoric Lascaux caves (Figure 2).

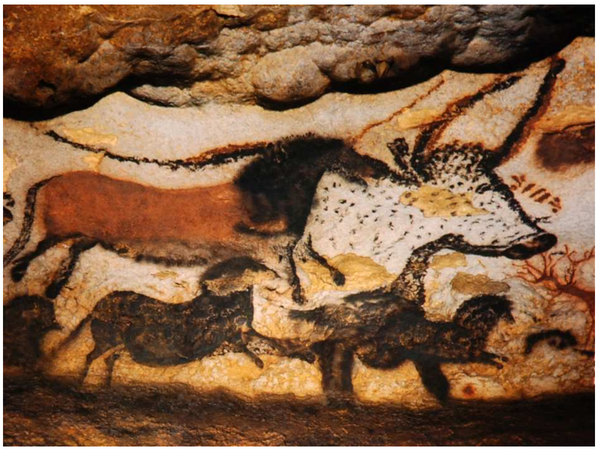

Figure 2: Lascaux caves.

The cave paintings have oblique angles and distorted images and viewing angles, all of which might have been created with the intuitive use of anamorphic perspective.

Then there are the examples of the calculated use of anamorphic concepts, which can be seen in the mathematical proportions of the Greek parthenon (Figures 3 and 4) and the varied letter sizes on the Roman triumphal arches. But it was the European Renaissance that brought the greatest advancement in visual perception with the invention of perspective (Figure 5).

Figure 3: Greek parthenon.

The artists and scientists of the Renaissance thoroughly researched the one-point, two-point, and three-point perspective concepts. For the first time, artists had a tool that let them create, on a flat surface, an image that had depth and a 3D realism.

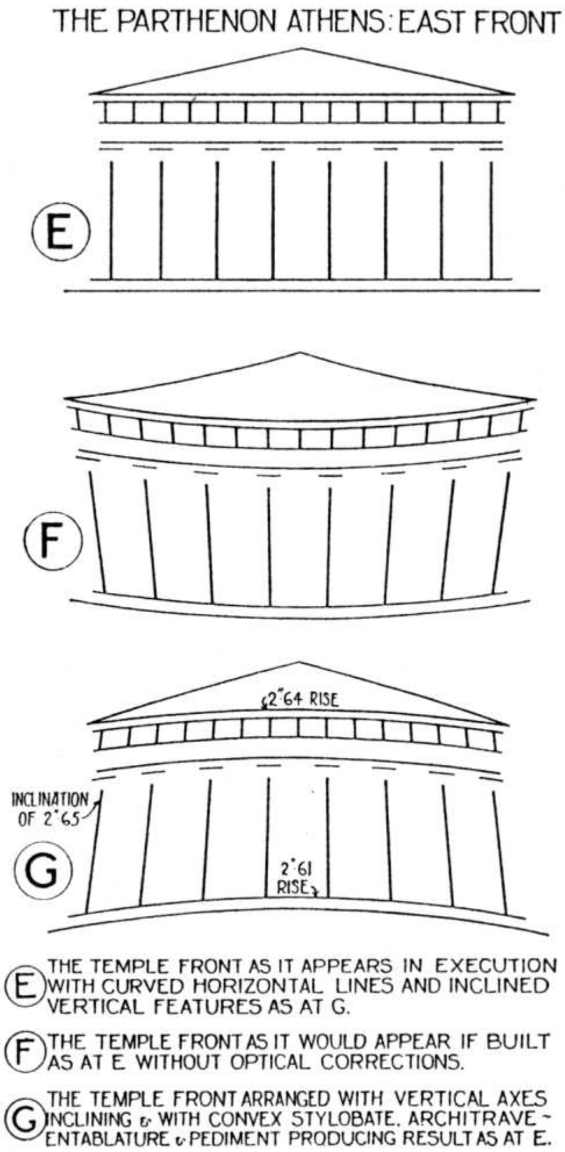

Figure 4: Parthenon drawing.

When first viewed, these images were considered magical. And the magic continued with the addition of anamorphosis, which is just the opposite of perspective. Anamorphosis is a distortion of the shapes and dimensions of an image, making it impossible for an observer to recognize an image when standing in front of it. There are two types of anamorphosis: oblique and catoptric (mirror). Both types of anamorphosis are distorted images on a flat plane; to view a recognizable image requires that the observer must stand in a particular spot and look at the distorted image obliquely or view a corrected reflection in a curved mirror.

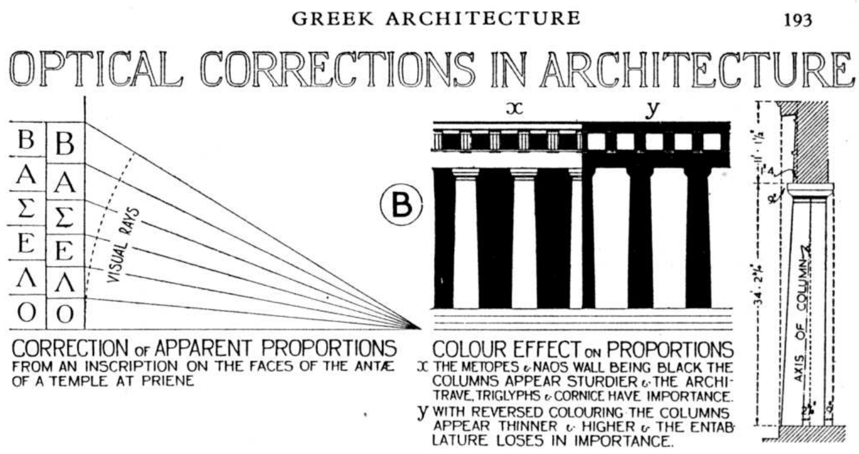

Figure 5: Optical corrections.

Piero della Francesca and Leonardo da Vinci are considered to be among the first Renaissance artists to use geometric perspective and anamorphosis. One of the earliest examples of anamorphosis works is seen in da Vinci's notebook drawing from *Codice Atlanticus*, c.1483-1518, a drawing of a distorted child's face that must be viewed obliquely to see the realistic image of a face and eyes (Figure 6).

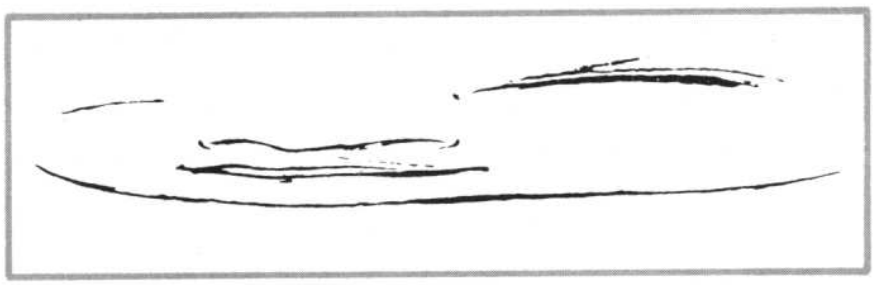

Figure 6: Distorted drawing.

Hans Holbein, the Younger's painting, *The Ambassador*, 1533, is often cited for the use of anamorphic perspective. In this work, Holbein took great care to present near-perfect perspective, but resting in the lower area of the canvas is an odd, diagonally placed shape. The odd shape is an anamorphic skull that can be viewed only from a position to the right of the painting (Figures 7 and 8).

Figure 7: *The Ambassadors.*

Figure 8: Skull.

The eighteenth century saw the introduction of the cylindrical mirror, which was augmented by the cone and pyramid mirrors (Figure 9). And continuing into the twentieth and twenty-first centuries, there have been a number of creative additions to the anamorphic arts.

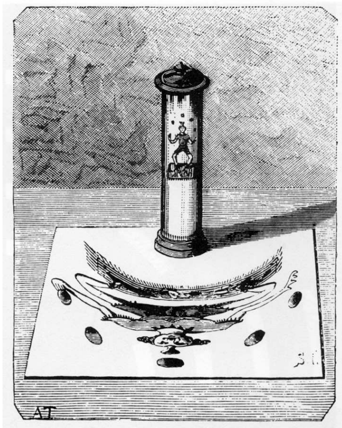

Figure 9: Mirror.

3 3D Anamorphic Sculpture

There are several twentieth-century additions to the anamorphic arts that can be seen in my own artwork. I researched the idea for a 3D anamorphic sculpture was questioned and researched in 1986 and completed my first 3D anamorphic sculpture, *Self-Portrait*, in 1988 (Figure 10).

Figure 10: *Self-Portrait*.

Self-Portrait merged seventeenth-century mirrored anamorphosis with sculpture: instead of using a distorted flat image, I used clay to create a distorted fan-shaped relief. Initially, I used a conventional flat geometric anamorphic technique, but because of the solid nature of the clay, the work required a new approach, which involved sculpting directly in front of the mirror. When the sculpture was placed in front of a 180-degree mirror the distorted relief reflected a 3D image.

The working method I used initially used to create *Self-Portrait* included a 12-inch flat glass mirror, a 6-inch cylindrical Mylar mirror, an 8×12-inch black and white self-portrait photograph with a 1-inch inked grid drawn to the borders of the photograph, a curved fan shaped 14×24-inch grid that was drawn on a sheet of paper, a 14×24-inch inked fan shaped grid on a sheet of clear plastic, and oil-based clay. I numbered the grid units were numbered across the horizontal border of the self-portrait photograph and then alphabetically labeled each grid unit on the vertical border. The curved grid was identically labeled on the horizontal and vertical borders. Then I placed the self-portrait photograph next to the curved grid and translated the photograph into a curved image by drawing each square of the self-portrait photograph grid with pencil in gray values on the paper's curved grid units. From time to time, during the drawing process, I put the drawing in front of the cylindrical Mylar mirror to check on my progress (Figure 11).

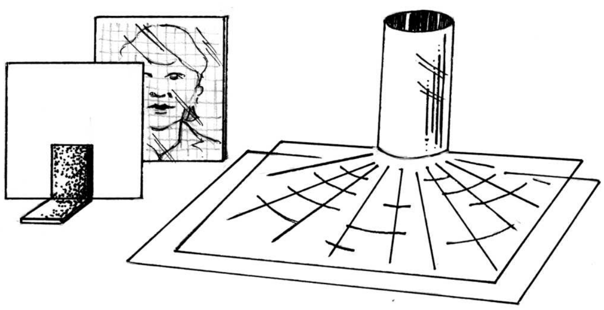

Figure 11: Method.

Once the distorted curved-grid drawing was completed, I placed the clear plastic inked fan grid was placed on top (the plastic was used to insure that the pencil drawing was not damaged by the clay), making sure that the overlapping grids matched, and then placed both the drawing and the plastic in front of the Mylar mirror, which I used to make a final check for accuracy; the drawing had the depth and width of a seventeenth-century anamorphic drawing. The next step was to achieve the height that was needed to create the piece's 3D feature. Starting at the outside edges of the drawing and working towards the Mylar mirror, I spread a thin layer of clay on top of the drawing. Adding the clay did started to give the piece height, but at the same time, the drawing under the plastic grid was being eradicated and the facial features were gone.

The course I took to solve the sculpting problem involved making a few changes to the set-up. I placed the self-portrait photograph so that it faced the flat mirror, with the mirror facing out towards the viewer. I then placed the photograph and mirror next to the cylindrical Mylar mirror and the distorted fan-drawn grid. While standing in front of the mirror, and while looking at the flat mirror, I then applied clay to the flat distorted grid, and at the same time, I looked into the cylindrical Mylar mirror to check on the progress of the sculpting. Working the clay in this manner required me to create the figure upside down and reversed.

4 A New Mirror: The Vertical S-Cylindrical Mirror

I created a much more complicated 3D anamorphic sculpture, *Brothers*, in 1990. For this new piece, instead of using a cylinder, cone, or pyramid mirror, I created a new mirror by slicing a slicing a cylindrical mirror in half, then shifting and joining the opposite edges together, which created an “S” or wave, cylindrical mirror (Figure 12).

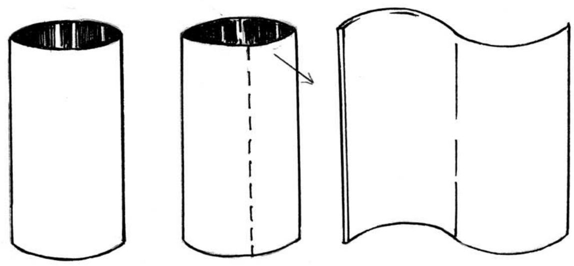

Figure 12: S-cylindrical mirror.

Because of the design of the mirror’s design, I used two distorted anamorphic sculptures for *Brothers*, distorting each sculpture in a uniquely different way than the other 13). The first sculpture is a high-relief fan shaped man sitting in an overstuffed chair with his feet up on an ottoman; this figure is placed in front of the S-cylindrical mirror’s convex area and reflects a traditional anamorphic image. The second sculpture is a standing figure that appears wider than normal, but when the sculpture is placed in the concave area of the S-cylindrical mirror, the figure’s reflection appears slimmed down and normal. What the viewer experiences is a parabolic reflection, which is much like a hologram for an off-the-surface and floating-in-space reflection [1].

[1] `http://www.karenmortillaro.com/videos.php`

Figure 13: *Brothers.*

5 Alice in Wonderland

Lewis Carroll's *Alice in Wonderland* was written during the Victorian era, an era that produced extraordinary advancements in the areas of mathematics, science, literature, art, politics, and exploration. In writing his children's book, Carroll took the creative energy of his time and, instead of writing a story that was dull and moralizing, he wrote a book that was magic for a young reader, a book that is filled with fun, puns, nonsense logic, math, science, and popular culture.

Figure 14: Alice table.

Alice in Wonderland is a classic story that has been illustrated by many fine and accomplished artists over the past one 150 years; but there had never been an attempt to illustrate the story in sculpture. Such a task seemed to me to be quite an interesting and worthwhile undertaking.

Figure 15: One more Alice table.

Using Lewis Carroll's masterpiece, along with the richness of the Victorian period, I found it difficult to create just one piece of sculpture that would fully illustrate Alice's dream journey. So, I decided to create, in 3D anamorphic sculpture, one table-like sculpture for each chapter; when all twelve Alice tables are brought together, they will create one united sculpture. Each table-size sculpture is first sculpted in clay, then cast in bronze, and includes a stainless steel S-cylindrical mirror. The S-cylindrical mirror is ideal for this project because it allows for the figures on one side of the mirror to be sculpted realistically, while on the opposite side of the mirror are distorted and unrecognizable. The mirror is symbolic of the parallel worlds that Alice might have experienced in her dream state; the world of reality is on one side of the mirror; and the world of illusion is on the mirror's opposite side (Figures 14, 15 and 16).

Figure 16: Even one more Alice table.

Having only a minimal background in the areas of mathematics, physics, and the neurosciences, I have created all of the 3D anamorphic sculptures by making intuitive calculations by hand. To date, six of the twelve sculptures of the Alice Series have been completed. The working method I have used since 1988 to create these sculptures involves the following steps: (1) I visualize the sculpture is seen in my mind's eye, (2) I make notebook sketches and notations, (3) I look directly into the S-cylindrical mirror while hand sculpting in front of the curved mirror, and (4) I cast the finished clay sculpture is cast in bronze, with the addition of a stainless steel mirror. It should be noted that 29 years ago, when I first started to create the 3D anamorphic sculptures, many of the technological tools, such as the PC computer and CNC milling machines, were in their infancy or they were not available to me. But within the past few years, today's PC computers and computer software have become greatly advanced. In addition, 3D scanning and 3D printing not only are readily available but also have become much more cost effective and user friendly. Using these new technological tools will lead to design and fabrication possibilities that cannot be achieved by hand, and I am looking forward to using these tools to assist with the completion of the Alice Series, but also with future sculptures.

6 Acknowledgment

I would be remiss if I failed to acknowledge two individuals, Lewis Carroll and Martin Gardner; each of these gentlemen has had a major influence on my *Alice Series*. Their influence can be seen in my notebooks, which are kept in my

studio, close to two other books; Carroll's *Alice in Wonderland* [2] and Martin Gardner's *The Annotated Alice* [1]. My notebooks contain copious sketches and notes that are visual concept interpretations of Carroll's story. Gardner's The Annotated Alice, a brilliant analysis of Carroll's story, is an invaluable resource that also adds to the notebook studies. Both men wrote children's literature and were very involved with mathematics. Carroll taught and wrote about math, and Gardner, although not a mathematician, was a prolific writer on the subject. Both men enjoyed puzzles, magic, and games. Both men were also very socially shy, shunning all public recognition, but last year there was a world wide celebration of Gardner's 100-year birthday, and this year there will be a worldwide 150th-anniversary celebration of the publication of Lewis Carroll's *Alice in Wonderland*. To both of these creative individuals, I thank you.

For more information on Lewis Carroll, please check the web for the Lewis Carroll Society; there are societies in the UK, USA, Canada, Brazil, and Japan. Information on Martin Gardner can be found on the Martin Gardner website: `martin-gardner.org`.

References

[1] Gardner, M. *The Annotated Alice*, Clarkson N. Potter, 1960.

[2] Carroll, L. *Alice's Adventures in Wonderland*, 1865.