Recreational
Mathematics
Magazine

Ludus

CIUHCT

Informations

The *Recreational Mathematics Magazine* is electronic and semiannual.
The issues are published in the exact moments of the equinox.
The magazine has the following sections (not mandatory in all issues):

Articles
Games and Puzzles
Problems
MathMagic
Mathematics and Arts
Math and Fun with Algorithms
Reviews
News

Contents

CALCULATING THE DAY OF THE WEEK
NULL-DAYS ALGORITHM

Xiang-Sheng Wang
Southeast Missouri State University
xswang@semo.edu

Abstract: *In this paper, we propose a new algorithm of calculating the day of the week for any given century, year, month and day in Gregorian calendar. We provide two simple formulas to convert the century and the year into two integers. Then we introduce a list of null-days to transform the month and the day into another integer. Adding these three integers together and calculating the sum's residue modulo 7 gives a number between 0 and 6, which corresponds to Sunday until Saturday. Slight modification is needed if we have a leap year and the given month is either January or February. Our null-days algorithm is simple enough to be done by mental calculation, and the list of null-days has memorable patterns.*

Key-words: Day of the week, mental calculation, null-days algorithm.

1 Introduction

It is interesting to see someone calculating the day of the week in mind. In this paper, we will propose a simple algorithm to fulfill this task with very basic mathematics and very few to be memorized.

For any given Gregorian calendar date, we can isolate the century item (c), year item (y), month item (m) and day item (d), respectively. Here, m denotes the number of months in the year, i.e., $m = 1$ for January, $m = 2$ for February, and so on. Early in 1887, Lewis Carroll [1] devised a method to convert the items c, y, m, d into a number w (week), which modulo 7 gives the day of the week. That is, $w \equiv 1$ denotes Monday, $w \equiv 2$ stands for Tuesday, and so on. The most complicated parts in Carroll's method are the translations of year item and month item, which involve a division of 12 and additions of large numbers. To avoid addition of large numbers, one may assign certain numbers for the months, which is referred to as table method. The disadvantage of table method is that the months table is so irregular that it is easy to forget. John

H. Conway [2] introduced the famous doomsday rule to resolve this short back. Conway's doomsday consists of several memorable dates such as Valentine's Day, Boxing Day and the dates where $m = d = 4, 6, 8, 10, 12$.

In this paper, we will modify Conway's doomsday rule and make a list of so called null-days which have certain memorable patterns. In addition, we provide two simple formulas to convert the century item and year item into two numbers. Coupling these two steps, we will be able to do mental calculation of the day of the week for any given date in Gregorian calendar.

2   The null-days algorithm

First, we introduce a list of null-days:

$\quad$ 1/1;$\quad$ 3/5, 5/7, 7/9, 9/3;$\quad$ 2/12, 12/10, 10/8, 8/6, 6/4, 4/2;$\quad$ 11/12.

It is easy to verify that except for leap years, the above null-days in the same year share the same day of the week. The null-days are memorable in the sense that $1/1$ is just the first day of the year; the dates $3/5, 5/7, 7/9, 9/3$ are simply rotating the odd numbers $3, 5, 7, 9$; the dates $2/12, 12/10, 10/8, 8/6, 6/4, 4/2$ are rotating the even numbers $12, 10, 8, 6, 4, 2$. The only exceptional null-day is $11/12$, which could be memorized in the way that 11 (November) is equivalent with $1 + 1 = 2$ (February). We denote by $w_0(m, d)$ the difference of the date m/d from the corresponding null-day with the same month value m.

Next, we propose two simple formulas which convert the century and year items into two integers. Recall that c denotes the century item and $y \in [0, 99]$ denotes the year item. We multiple the residue of c modulo 4 by -2 and denote the result by w_1. It is readily seen that

$$w_1(c) = \begin{cases} 0, & c = 4k; \\ -2, & c = 4k + 1; \\ -4, & c = 4k + 2; \\ -6, & c = 4k + 3. \end{cases} \tag{1}$$

Our second formula is given as

$$w_2(y) = \left\lfloor \frac{5y}{4} \right\rfloor, \tag{2}$$

where $\lfloor x \rfloor$ denotes the largest integer no more than x. Let $y = 10y_1 + y_0$ with $y_0, y_1 \in [0, 9]$ being the ones and tens digits of y respectively. The above formula can be modified as

$$w_2(y) = \left\lfloor \frac{25y_1}{2} + \frac{5y_0}{4} \right\rfloor \equiv y_0 - y_1 + \left\lfloor \frac{y_0}{4} - \frac{y_1}{2} \right\rfloor \mod 7. \tag{3}$$

Finally, we compute $w(m, d, c, y) = w_0(m, d) + w_1(c) + w_2(y)$ modulo 7 to obtain the day of the week for any given date (m, d, c, y). If the given date is in January or February of a leap year, then we should subtract the sum by 1, namely, $w(m, d, c, y) = w_0(m, d) + w_1(c) + w_2(y) - 1$ modulo 7.

3 Examples

We use two examples to illustrate our null-days algorithm.

Example 1. *For the date March 26, 2014, we observe $m = 3$, $d = 26$, $c = 20$, $y = 14$. Moreover, $y_0 = 4$ and $y_1 = 1$. First, we obtain from the null-day $3/5$ that $w_0 = 21 \equiv 0 \mod 7$. Next, we have $w_1 = 0$ since $c \equiv 0 \mod 4$. Furthermore, $w_2 = 4 - 1 + \lfloor 4/4 - 1/2 \rfloor = 3$. Adding the above three numbers gives $w = w_0 + w_1 + w_2 = 3$, which implies that March 26, 2014 is a Wednesday.*

Example 2. *For the date February 10, 1984, we observe $m = 2$, $d = 10$, $c = 19$, $y = 84$. Moreover, $y_0 = 4$ and $y_1 = 8$. First, we obtain from the null-day $2/12$ that $w_0 = -2 \equiv 5 \mod 7$. Next, we have $w_1 = -6$ since $c \equiv 3 \mod 4$. Furthermore, $w_2 = 4 - 8 + \lfloor 4/4 - 8/2 \rfloor = -7 \equiv 0 \mod 7$. Since 1984 is a leap year and $m = 2$, we add the above three numbers and subtract the sum by one to obtain $w = w_0 + w_1 + w_2 - 1 = -2 \equiv 5 \mod 7$, which implies that February 10, 1984 is a Friday.*

References

[1] Lewis Carroll. "To Find the Day of the Week for Any Given Date", *Nature*, 35, p. 517-517, 1887.

[2] John Horton Conway. "Tomorrow is the Day After Doomsday", *Eureka*, 36, p. 28-31, 1973.

Some early topological puzzles
Part 1

David Singmaster
London South Bank University
zingmast@gmail.com

An earlier version of this was presented at the Fourth *Gathering for Gardner*, Atlanta, 19 Feb 2000. Because there were too many pictures, it was not included in the proceedings of that meeting [16]. I have scanned in the 24 pictures that I showed as OHP slides and seven extra pictures and corresponding material. I have now inserted these in this version of this work, with the picture name after each Figure reference.

The most important development in the field is that Dario Uri has systematically examined Part Two of the Pacioli MS which I had found difficult [17]. He has discovered that the following puzzles occur. The Alliance or Victoria Puzzle, Cap. C, CII & CIII. Solomon's Seal, Cap. CI. The Cherries Puzzle, Cap. CIIII & CX (the latter being with a tube). The Chinese Wallet, Cap. CXXXII. As mentioned below, I had already noticed Cap. C CI, and a bit later I noticed CIIII, but had set the transcription aside to work on later. In addition, Dario has found that Cap. CVII is the Chinese Rings! Further, Cap. CXIIII is the problem of joining three castles to three wells by paths that do not cross; Cap. CXVII is a trick of removing a ring from a loop between a person's thumbs – cf the top of Figure 12: Schwenter410; Cap. CXXI is the trick where strings are just looped inside a ball so they can pulled away; Cap. CXXIX is the problem of making a support from three knives. As far as we know, these are the earliest appearance of all these puzzles. There are about 140 problems in this Part! Unfortunately, though Pacioli refers to diagrams, the only diagram in the MS is for Cap. C. Perhaps Dario will find the missing drawings in the library at Bologna. Dario has found that Cap. CXXIX refers to Pacioli being shown the problem on 1 Apr 1509, so this MS should be dated as 1510?

I have just added much further material on two topological patterns: The Star of David and The Borromean Rings; The Möbius Strip; which go back rather further than I knew.

Abstract: *For some time, I considered the 1723-1725 edition of Ozanam's* Récréations mathématiques et physiques *as the first book to cover topological puzzles in detail and I only knew of a few earlier examples. Ozanam certainly gives many more examples than any previous book. In the last few years, I have discovered some early sources which show several topological puzzles as being considerably older than I previously knew. Here I show them.*

Key-words: African beads puzzle, Chinese wallet, flick-flack, Jacob's ladder, Möbius strip, Solomon's seal, star of David, topological puzzles, Victoria puzzle.

1   The Star of David

The hexagram pattern of two overlapping equilateral triangles, Mogen David, is known from antiquity, both from Hebrew and other cultures, including Turkish, Hindu and Japanese! - see the Wikipedia entry for numerous references back to about 0 CE. It was sometimes called Solomon's Shield or Seal. Early examples do not show the triangles interlocking, as in this example from the Leningrad Codex of 1008 (Figure 1).

Figure 1: From the Leningrad Codex of 1008.

Interlocking does go back to the 12C - as seen here in an 1141 mosaic at Santi Maria e Donato, Murano, Venice ([11] p.49) (Figure 2) and a 1641 mosaic at San Salvadore, Venice (ibid., p. 85) (Figure 3).

Figure 2: An 1141 mosaic at Santi Maria e Donato, Murano, Venice.

Figure 3: 1641 mosaic at San Salvatore, Venice.

At various points, this was extended in several different directions.

<u>First</u>: With interlocking squares. c4C mosaic from Lenhay Green, Sherborne, Dorset ([7], plate 44, p. 55) (Figure 4).

Figure 4: c4C mosaic from Lenhay Green, Sherborne, Dorset.

Second: Into a knot pattern of two interlocked loops, also known as "Solomon's Knot" or "Seal". Floor mosaic from Fishbourne, c200 ([10], p. 26) (Figure 5).

Figure 5: Floor mosaic from Fishbourne, c200.

And [11], p. 51, from an 1141 mosaic at Santi Maria e Donato (Figure 6).

Figure 6: From an 1141 mosaic at Santi Maria e Donato.

Second (A): This is often further embellished by a third interlaced loop, e.g. from c315 Christian mosaics at Aquileia ([8], pl. 41) (Figure 7).

Figure 7: c315 Christian mosaics at Aquileia.

Third: To have three interlocking triangles, in the form known as the "Borromean Rings".

Figure 8: Photo of my example (Borromean Rings).

In the Figure 8 each loop is made from six right-angle Tangle pieces which make models of the cyclohexane molecule isomers - these are in the "chair" configuration. In this pattern of three rings, no two are actually linked, but all three are. It is part of the coat of arms of the Borromeo family, who are counts of the area north of Milan since the 15C. However a considerably earlier version occurs as "Odin's Triangle" or "Walknot" in Viking culture.

Figure 9: 9C Larbro Picture Stone from the island of Gotland.

The 9C Larbro Picture Stone from the island of Gotland is now in the Historiska Museum, Stockholm (Figure 9). A similar carving is in the Viking Ships Museum, Oslo.

Stemma della famiglia Borromeo
Armoiries de la famille Borromeo
Coat of arms of the family Borromeo
Wappen der Familie Borromeo
Wapen van de Familie Borromeo

Figure 10: Borromeo Crest. *Albero Genealogico Storico Biografico della nobile Famiglia Borromeo* (1903). This is available at: `www.verbanensia.org`. This says it is copied from a manuscript of the archivist Pietro Canetta, with a footnote: *Il Bandello, p. 243, vol. VIII*. I suspect that this refers to a publication of the MS. This simply says that the three rings represent the three houses of Sforza, Visconti and Borromeo which are joined by marriages. (Thanks to Dario Uri [email of 17 Jul 2001] for this source).

The Golfo Borromeo and the Borromean Islands are in Lago Maggiore, off the town of Stresa, NW of Milan. In the 16 and 17C, the Counts of Borromeo built a baroque palace and gardens on the main island, Isola Bella (or Isola Borromeo). The Borromean rings can be seen in many places in the palace and gardens, including the sides of the flower pots! Although the Rings have been described as a symbol of the Trinity, I don't know how they came to be part of the Borromean crest, though the guide book describes some of the other features of the crest. (Thanks to Alan and Philippa Collins for the information and loan of the guide book.) Perhaps the most famous member of the family was San Carlo Borromeo (1538-1584), Archbishop of Milan and a leader of the Counter-Reformation, but he does not seem to have used the rings in his crest.

Hornung, plates 10, 20, 24 & 39 are examples of Borromean rings, shown in the Figure 11. These designs have no descriptions and the only dating is in the Publisher's Note which says such designs were common in the 12C-17C.
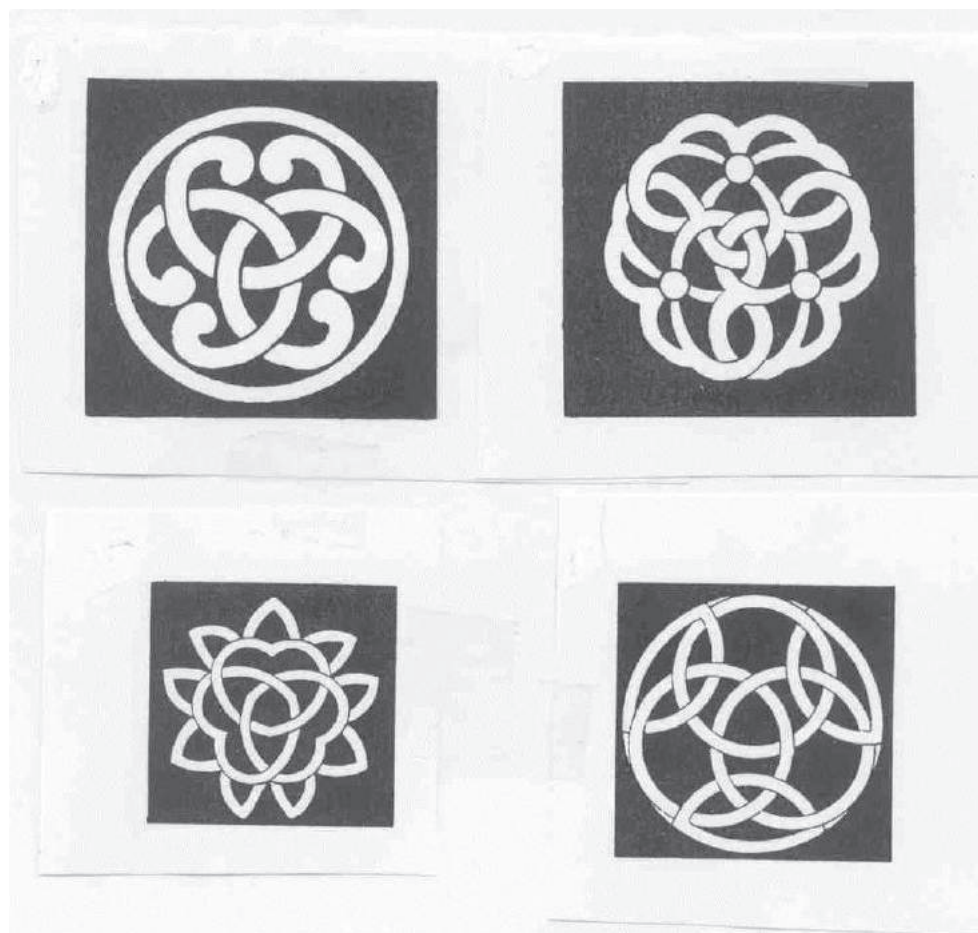

Figure 11: Japanese patterns from Hornung.

The pattern continues to intrigue - see the Wikipedia entry for more examples. The 20C British mathematical sculptor John Robinson has made several versions with interlocked triangles, squares and rhombi. This example, called "Intuition", is outside the Isaac Newton Institute in Cambridge, England (Figure 12).

Figure 12: Intuition.

In 2006, the pattern has been adopted as the logo of the International Mathematical Union (Figure 13).

Figure 13: IMU Logo.

2   The Möbius Strip

This is well known, but I was surprised to recently discover its age (Figures 14 and 15).
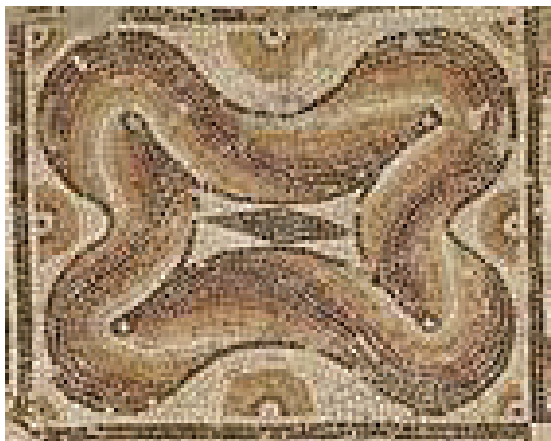

Figure 14: Mosaic of Orpheus from c3C in the Museum of Pagan Art, Arles.

Figure 15: Early 3C Mosaic from Sentium (now Sassoferrato, Umbria).

This is now in the Glyptothek (Figure 15), Munich, discovered by Charles Seife in 2002.

There are a number of practical uses for the Möbius strip, but the most unusual is as a non-inductive electrical resistor, patented by Richard L. Davis in 1966 - US Patent 32674906 A (Figure 16).
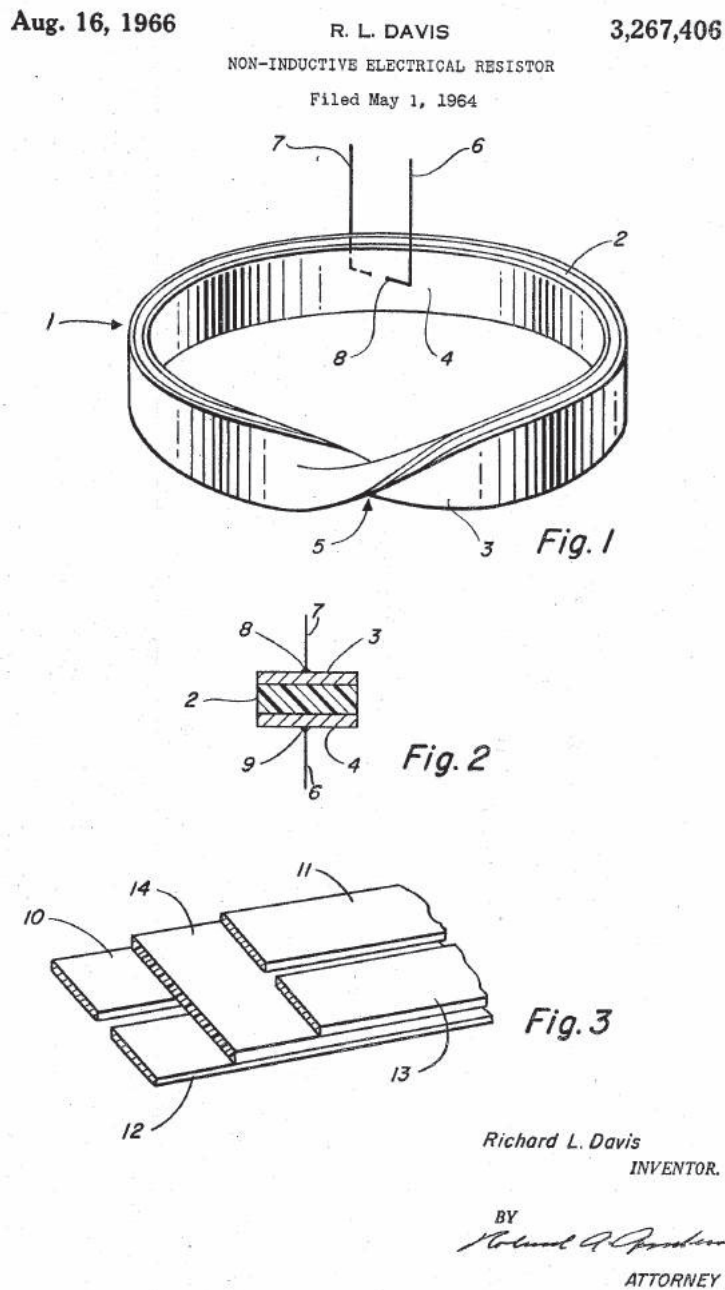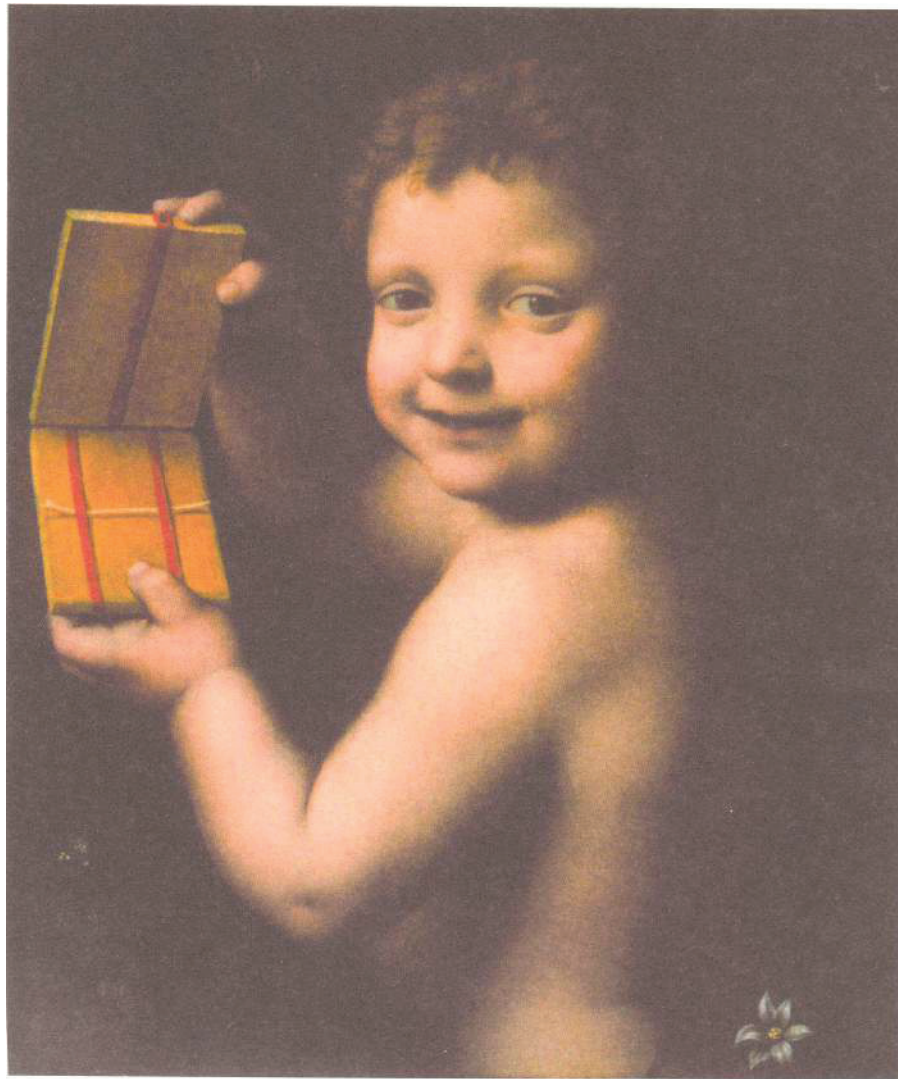

Figure 16: Davis' patent.

3 The Chinese Wallet or Flick-Flack or Jacob's Ladder

In May 1998, Peter Hajek reported seeing a painting at Hampton Court showing this (Figure 17). Some research in art history books turned up a picture of it and a reference to another depiction, probably earlier.

Bernardino Luini's Boy with a Puzzle, *ca. 1520, oil on wood, 15" × 13", Peterborough, Elton Hall, the Proby Collection*

Figure 17: Luini Colour.

This is a painting (Figure 17) by the fairly well known Lombard follower of Leonardo, Bernardino Luini, born c1470 and last known in 1533. It is in the Proby Collection at Elton Hall, a stately home at Peterborough, Cambridgeshire. It is 15" by 13" (38 cm by 33 cm). The picture is variously cited as "A Boy with a Toy" or "Cherub with a Game of Patience". There is no indication of its date, but the middle of Luini's working life is c1520. A description says the tapes holding the boards together are apparently holding a straw, but doesn't seem to recognize the object. The painting was exhibited in London in 1898 and a contemporary review in a German journal calls it a *Taschenspielerstückchen*, a little juggler's trick – but recall that juggler was long a synonym for magician – with two boards which allow one to vanish the straw. This version appeared in an article by Volker Huber [5].

In 2005, I saw: Tancred Borenius & J. V. Hodgson; A Catalogue of the Pictures at Elton Hall in Huntingdonshire in the possession of Colonel Douglas James Proby; The Medici Society, London, 1924. Plate 5 is a B&W photo: Boy with a Puzzle. The facing p. 11 describes the picture. This says Luini died in 1532 and the picture is 17" by 13"" (43 x 35 cm). It is stated to have been in the Arundel Collection, but there is no evidence for this. It was owned by Sir William Hamilton and William Beckford. It is mentioned by Bernard Berenson ([2], p. 251) and is the frontispiece of Luca Beltrami ([1], pp. 563-564). The same child appears in another Luini painting, Christ and the Baptist as Children. The authors identify the "straw" as a fishbone and say it will disappear.

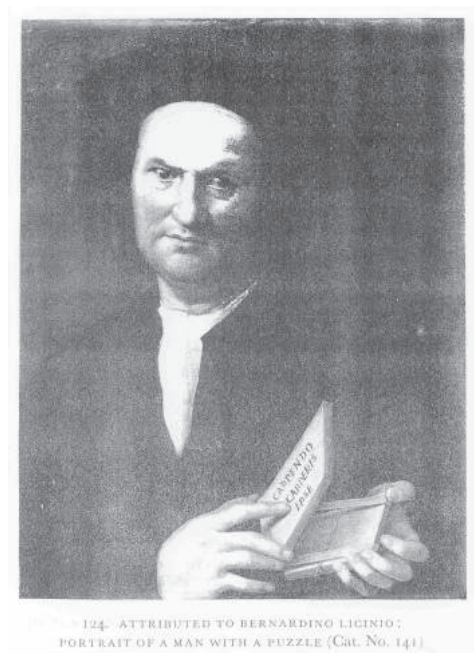124. ATTRIBUTED TO BERNARDINO LICINIO; PORTRAIT OF A MAN WITH A PUZZLE (Cat. No. 141)

Figure 18: Licinio Painting.

This is the picture (Figure 18) that Peter Hajek saw in the Picture Gallery at

Hampton Court Palace - but I saw it at Windsor Castle in about 2012. It is attributed to Bernardino Licinio, a Venetian painter born about 1485 and last known in 1549. It is 22" by 18" (56 cm by 45 cm). It is described and illustrated in John Shearman, [13], where it is called "Portrait of a Man with a Puzzle". It is very similar to another painting known to be by Licinio and dated 1524, so this is probably c1524 and hence a bit later than the Luini. Shearman cites the Luini painting and Pauli's notice of it. The description says the binding tapes are red, as in the Luini, and both show something like a straw being trapped in the wallet, which suggest some connection between the two pictures, though it may just be that this toy was then being produced in or imported to North Italy and was customarily made with red tapes. On the toy is an inscription: *Carpendo Carperis Ipse* (roughly: Snapping snaps the snapper), but Shearman says it definitely appears to be an addition, though its paint is not noticeably newer than the rest of the painting. Shearman says the toy comprises "three or more rectangles...", though both paintings clearly show just two pieces.
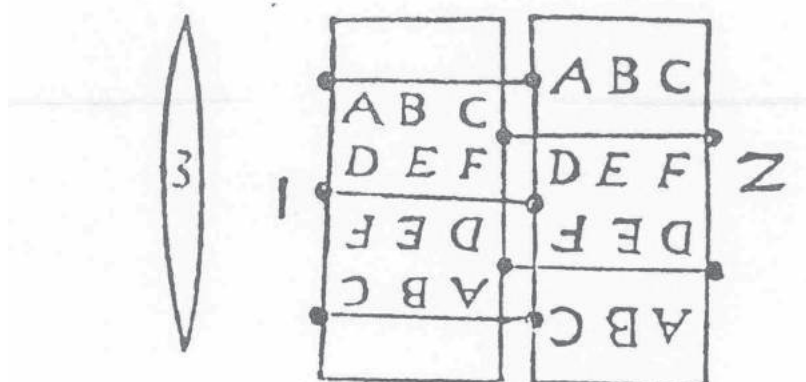

Figure 19: Prevost 139.

A little later in the 16C, we have the first known published version of the "Chinese Wallet" ([9], p. 139). This is the first book devoted to magic and conjuring. It is rather rare – only five copies are known – and a 1987 facsimile is out of

print. Fortunately, it has been recently translated into English as *Clever and Pleasant Inventions - Part One - Containing Numerous Games of Recreations and Feats of Agility, by Which One May Discover the Trickery of Jugglers and Charlatans*, and published in 1998. My thanks to Bill Kalush for bringing this to my attention. The figure is taken from the new edition (Figure 19).

Another recently received source is Schwenter's book of 1636, which has a version ([12], part 15, exercise 27, pp. 551-552) (Figure 20). I read this as *Ein Einmaul*, but Huber's article, above, reads it as *Ein Ginmaul* [yawning mouth].
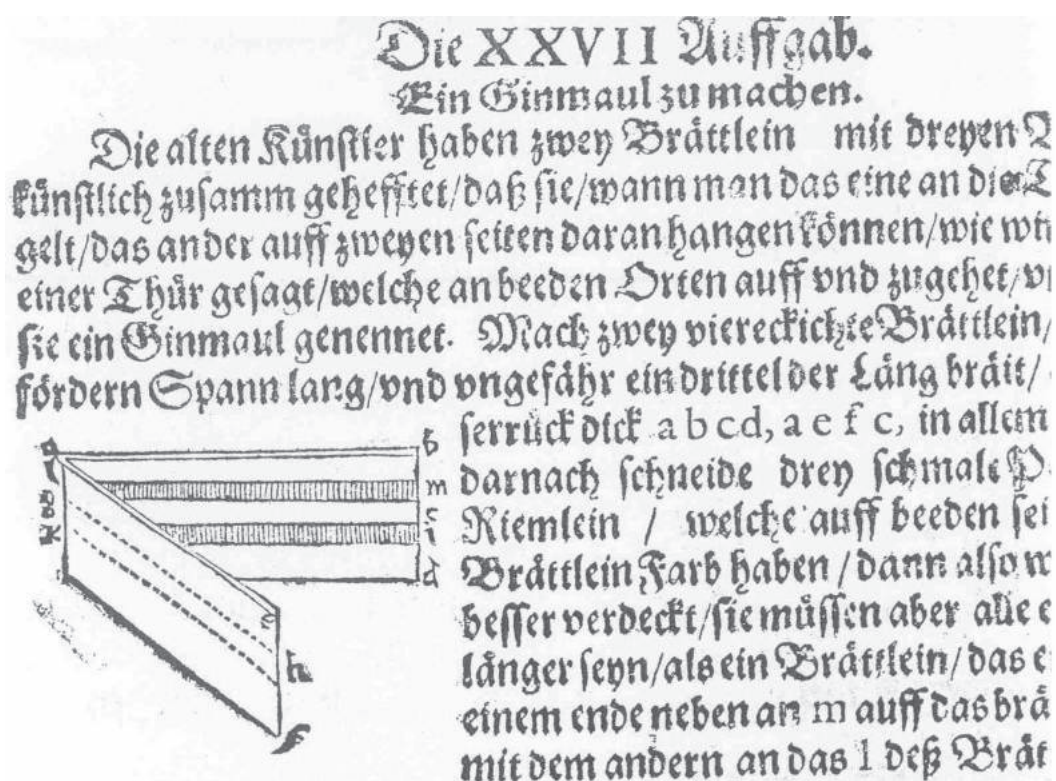

Figure 20: Schwenter 551.

And another early source is Witgeest (Figure 21). However, most of the interesting material is not in the first edition of 1679 of which there is a recent facsimile. The new material apparently occurs in the 2nd ed. of 1682 and this is so extensively revised and retitled as to constitute a new book. I only have some photocopies from the 3rd ed. of 1686, sent by Jerry Slocum. There were many further editions, both in Dutch and in German. Much of the new material seems to be derived from Schwenter, but here it is clearly different. ([15], prob. 66, pp. 49-50).
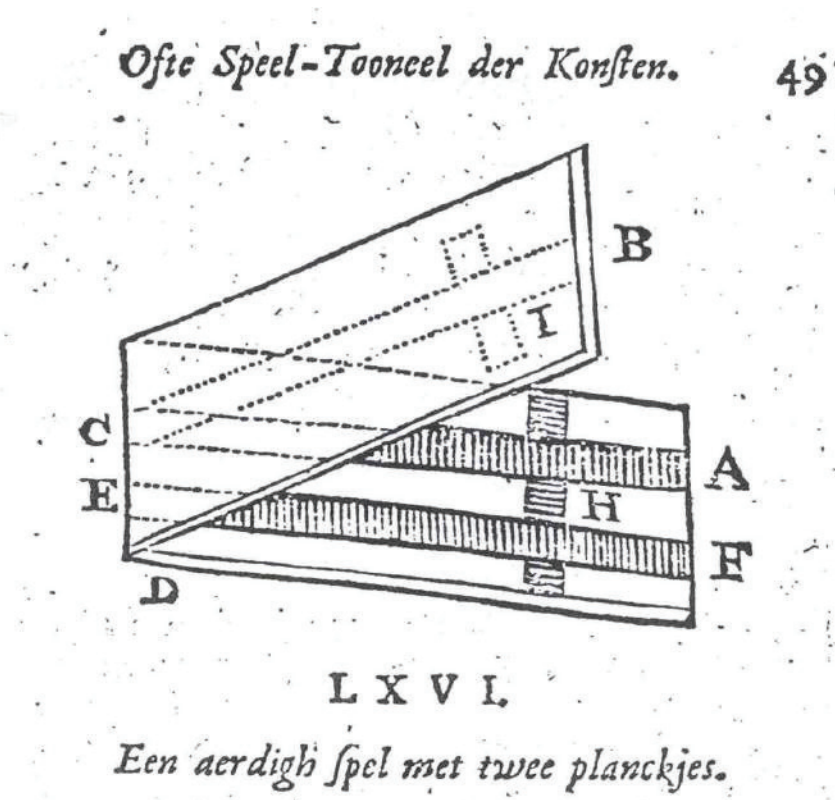
Figure 21: Witgeest 49.

I don't know when the toy advanced to having more boards and becoming the "Jacob's Ladder". Ozanam does not give any version of the toy. I distinctly recall having seen a painting of a gentlemen holding one with more parts but my note of it is buried somewhere. Later thought indicates that version had Victoria and Albert on the two sides. The first example I have record of is a c1850 example of a "Hand-operated game of changing pictures" illustrated in ([4], plate 6 on p. 24) (Figure 22).
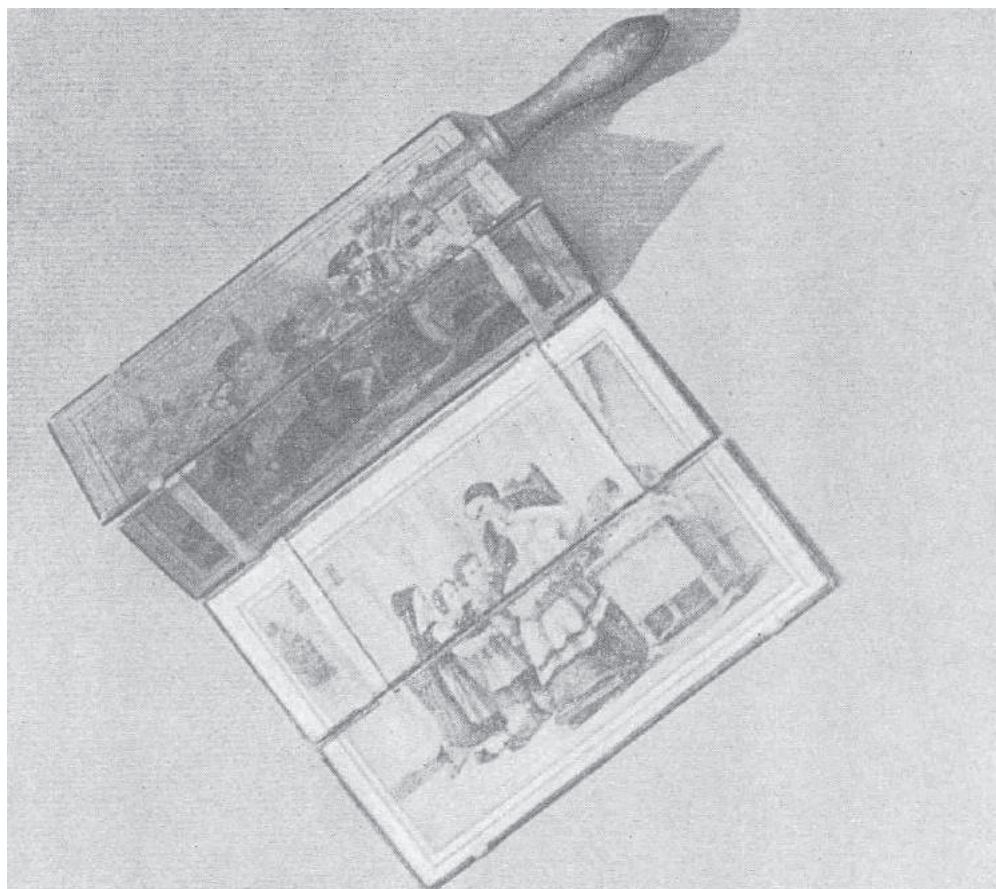
6. Hand-operated game of changing pictures, c. 1850

Figure 22: Daiken 24.

4   The Alliance or Victoria Puzzle

This does occur in Ozanam [[6], vol. IV, prob. 31, p. 435 & fig. 37, plater 11 (13)] and I show it first as most later versions look much like this (Figure 23).

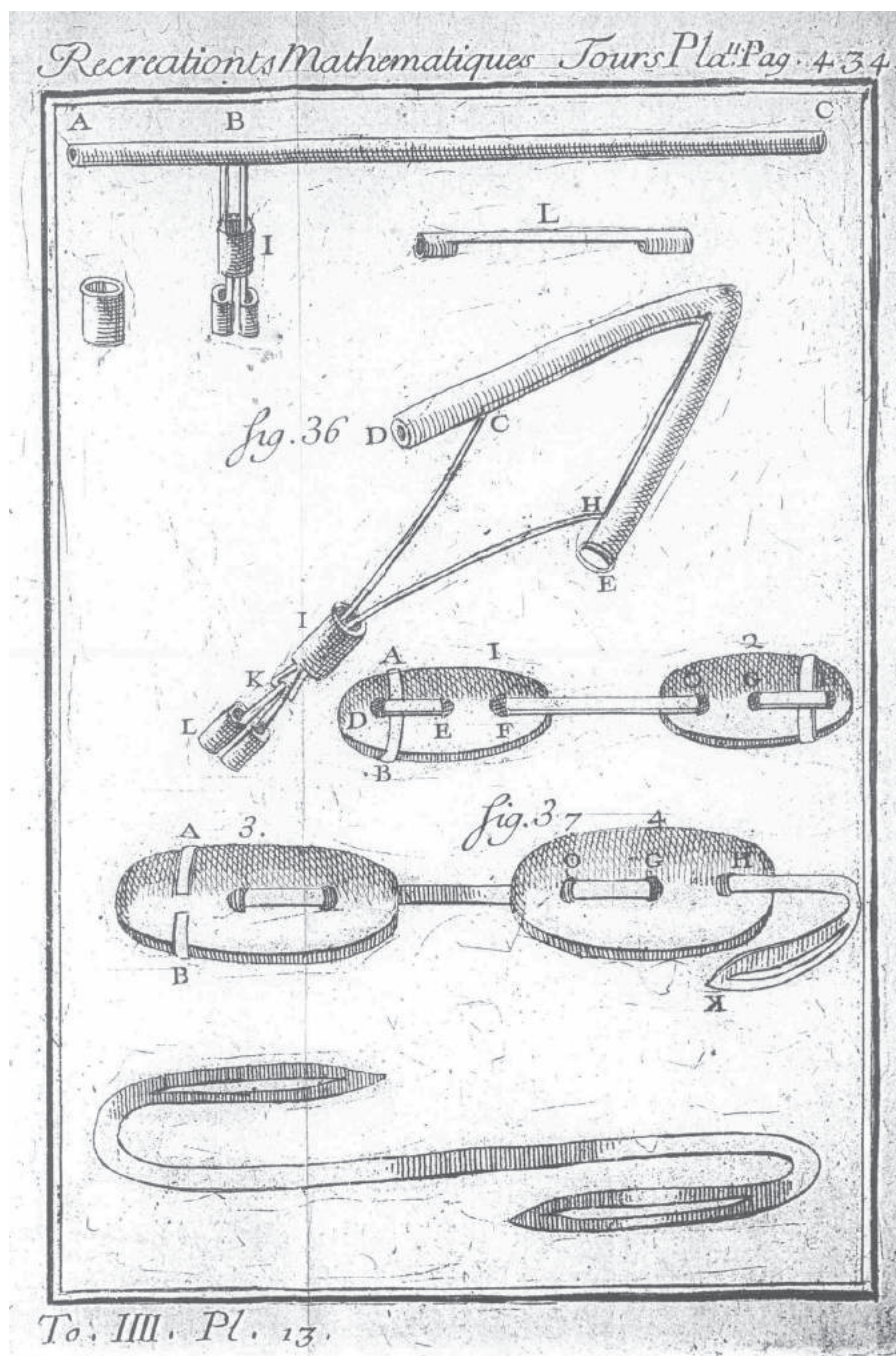Figure 23: Ozanam Prob 31.

In 1557, Cardan shows both two hole and three hole versions but Cardan's Latin is generally cryptic and I didn't really recognise what the pictures were for until I saw them repeated in the 1660 English version of Wecker [[14], p. 338 (This is in Book XVIII and I don't know if it appeared in the 1582 ed.)] (Figure 24).
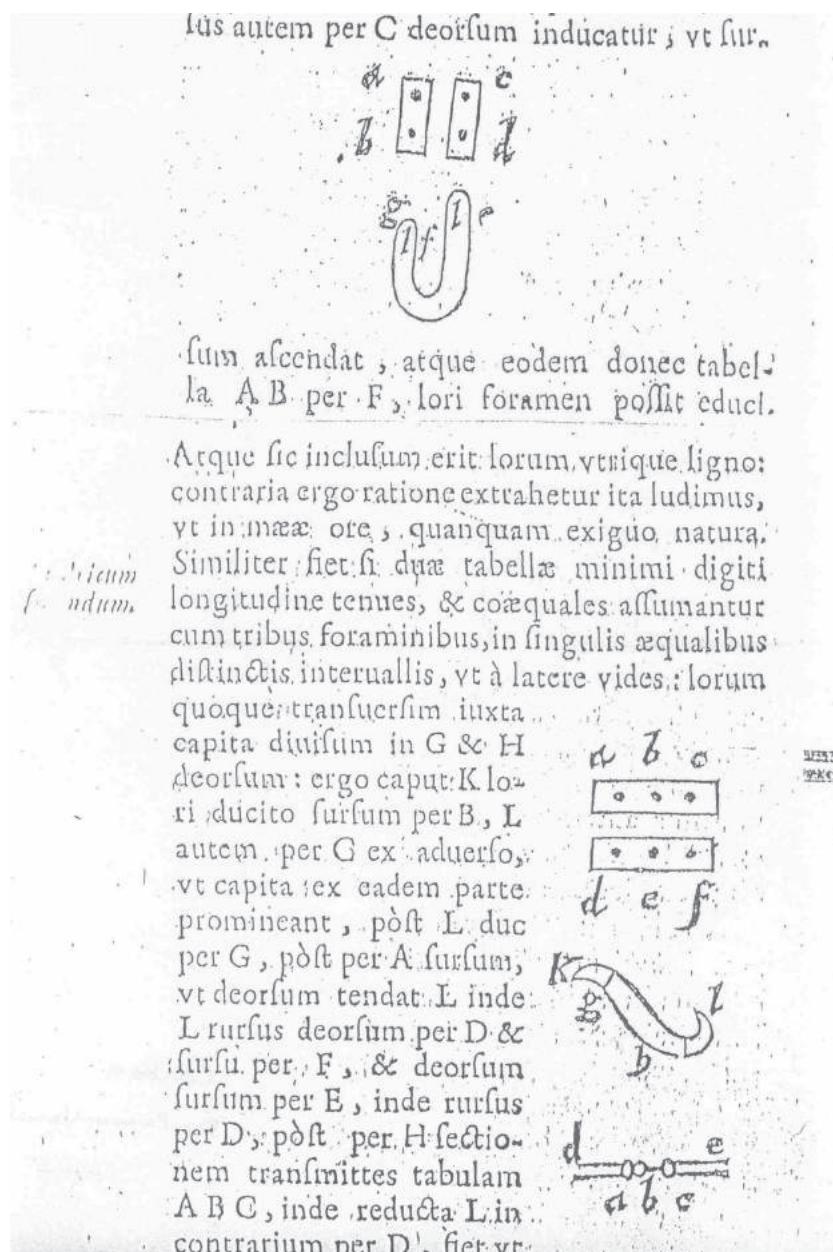
Figure 24: Cardan, [3], vol. III, pp. 245-246.

More recently, I have discovered a marginal drawing in Luca Pacioli's *De Viribus Quantitatis* MS of c1500 [Pacioli, ff. 206r - 206v, Part 2, Capitulo C] (Figure 25) *De cavare una stecca de un filo per 3 fori* (To remove a stick from a string through 3 holes). On f. 206r is the marginal drawing clearly showing the string through three holes in one board.
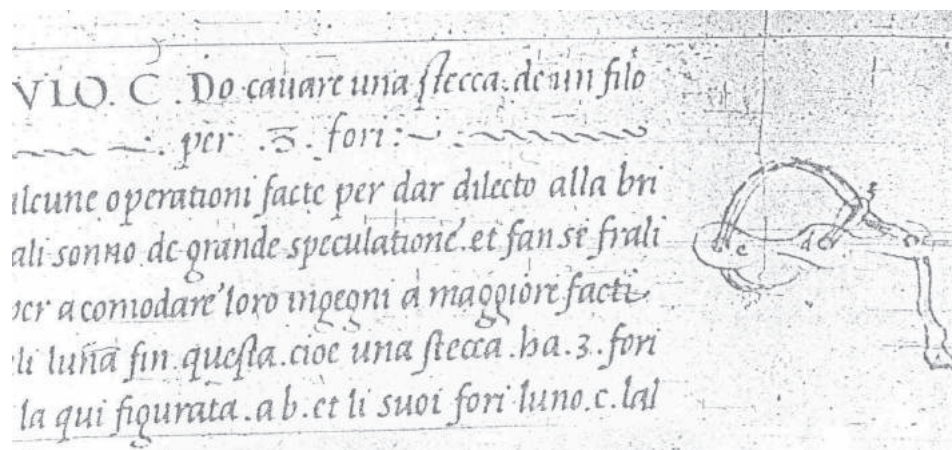
Figure 25: Pacioli 206r.

Both two hole and three hole versions occur in Prévost in 1584 ([9], pp. 133-136) (Figures 26 and 27).

To Enclose Two Tiny Pieces of Wood with Two Straps,⁹⁰ So That One May Not Take Them Out Without Breaking the Wood or the Straps

Have constructed two small, straight, and long pieces of wood, as you see drawn hereafter, each one having two round holes near the ends, which are marked with the four letters A, B, C, and D. Then take a slim leather strap of a
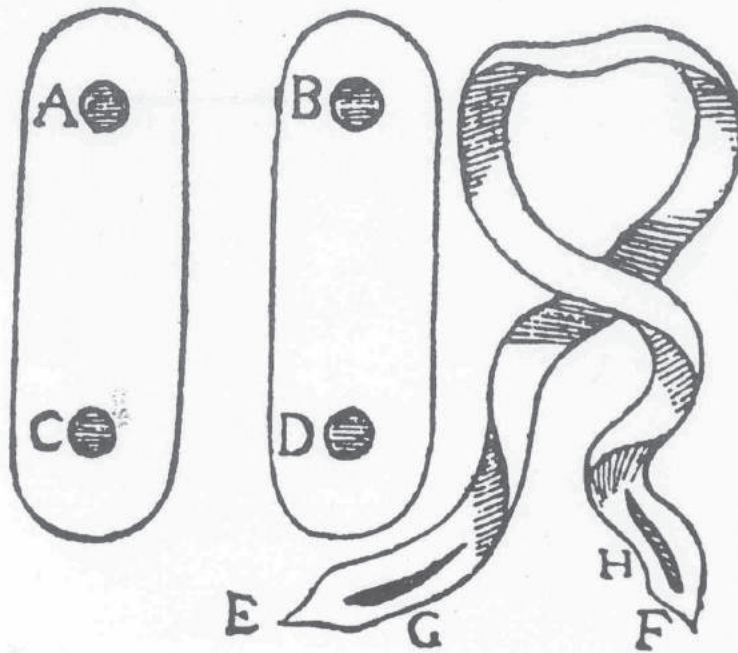
Figure 26: Prévost 133.

G through·hole A, and end H through hole B,
so that the two ends shall be on one side. Then
pass end G through slit K and through hole C.
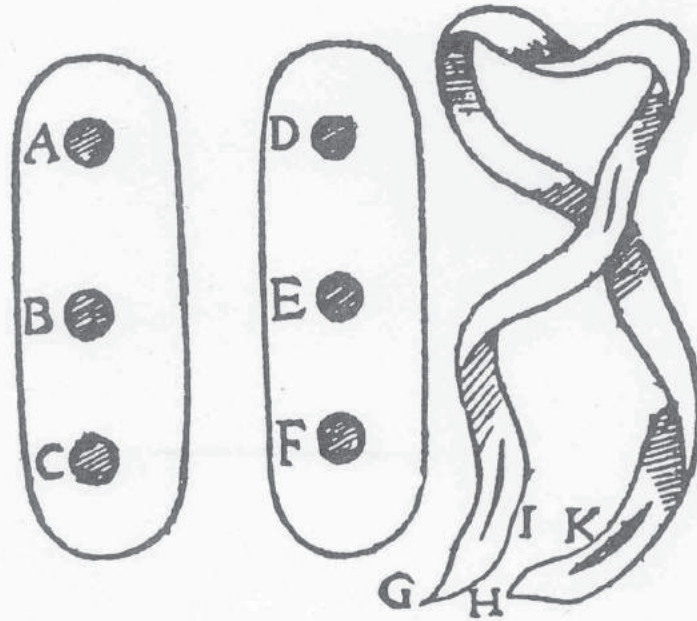
After this, to enclose the other piece, in a simi-
lar way pass end G through D and F, then
through E and again through D, where the end
is, holding the other piece, which you shall pass
through slit I, and you shall pull through the
strap from hole D. And the end shall be thor-
oughly caught in this piece as is the other end

[135]

Figure 27: Prévost 135.

Schwenter gives a two hole version ([12], Part 10, exercise 29, pp. 410-411) (Figure 28).
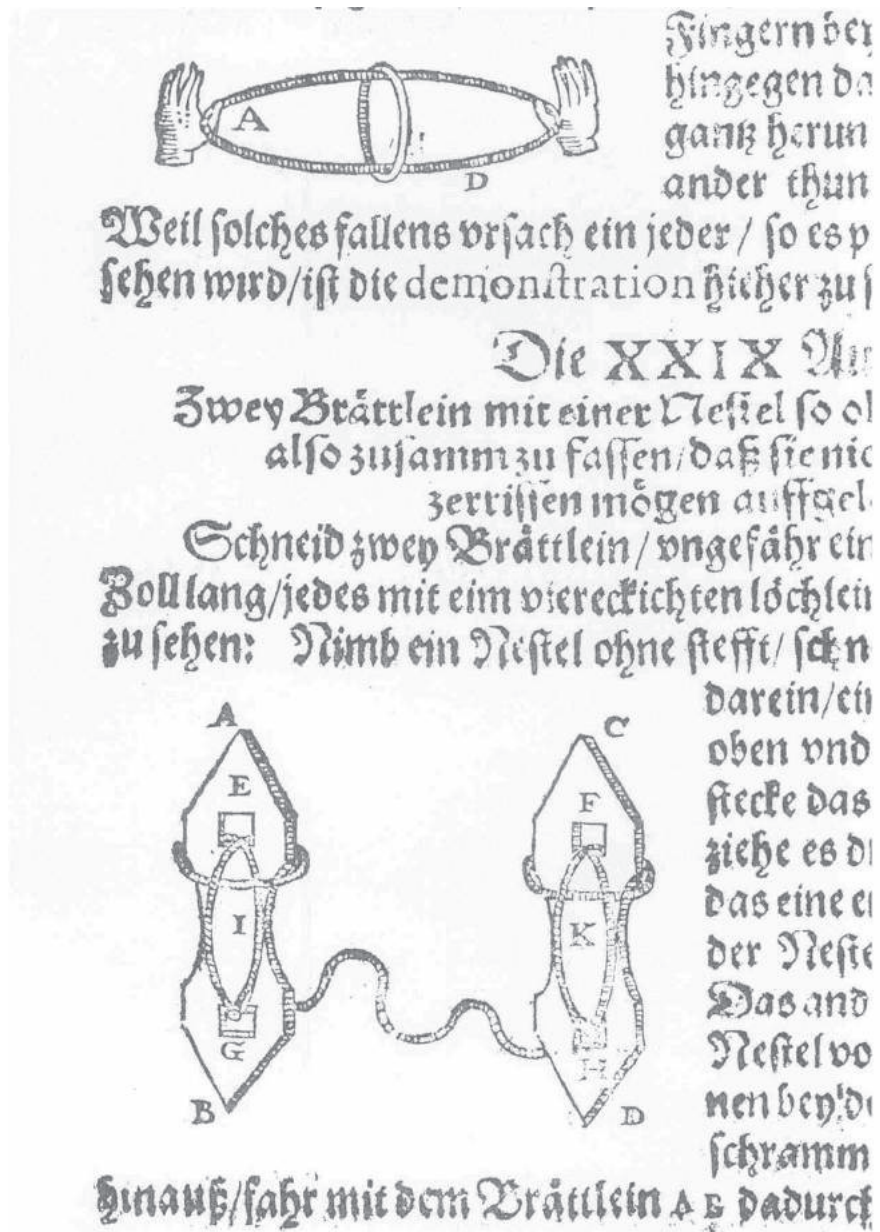
Figure 28: Schwenter 410.

([15], prob. 44, p. 35-36) is a two hole version taken from Schwenter (Figure 29).
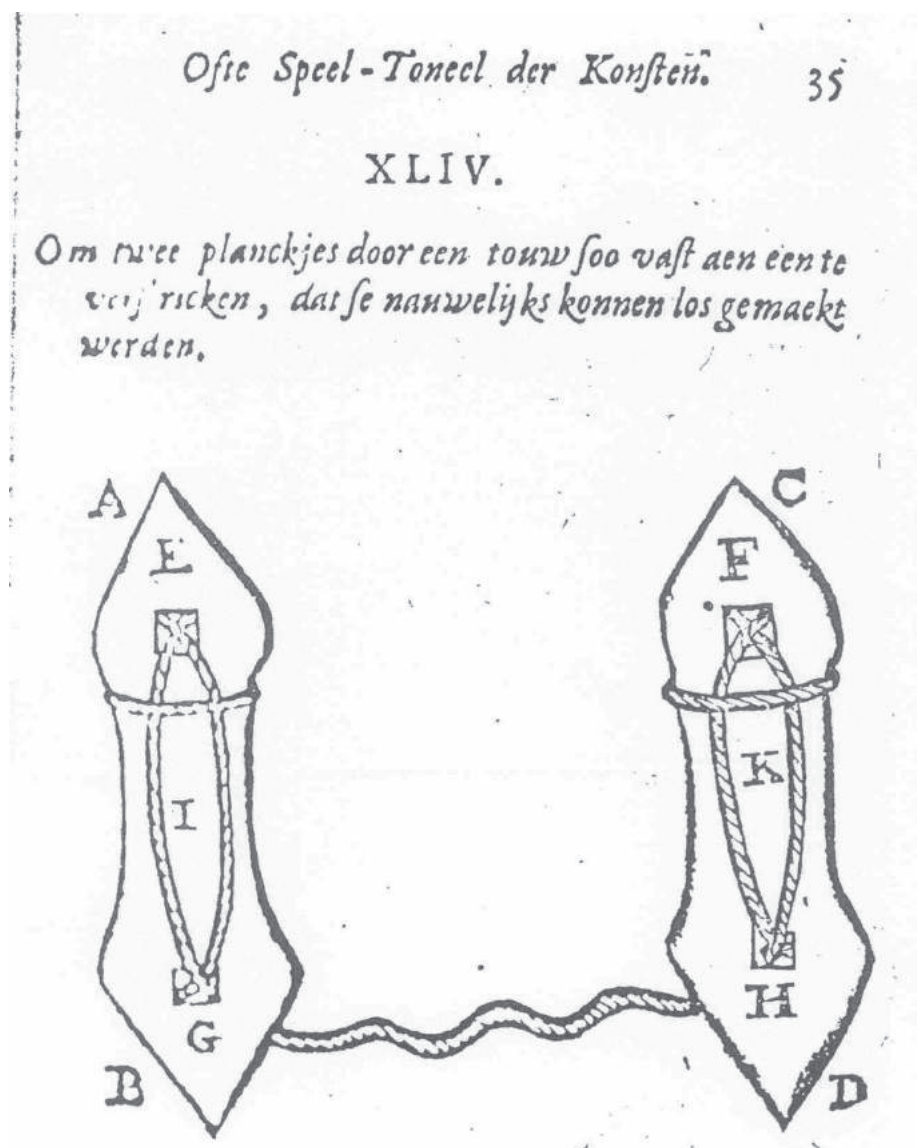
Figure 29: Witgeest 35.

5 Solomon's Seal or African Beads Puzzle

Again let us start with the version in the Ozanam [6], vol. IV, prob. 40, pp. 439-440 & fig. 47, plate 14 (16) (Figure 30), which calls it *Le Sigillum Salomonis*, or *Sceau de Salomon*. But this doesn't seem to relate to the shape called "Solomon's Seal" which we saw earlier.

Figure 30: Ozanam Prob 40.

I was pleased to recently find this in Schwenter ([12], Part 10, exercise 27, pp. 408-410).

Figure 31: Schwenter 409.

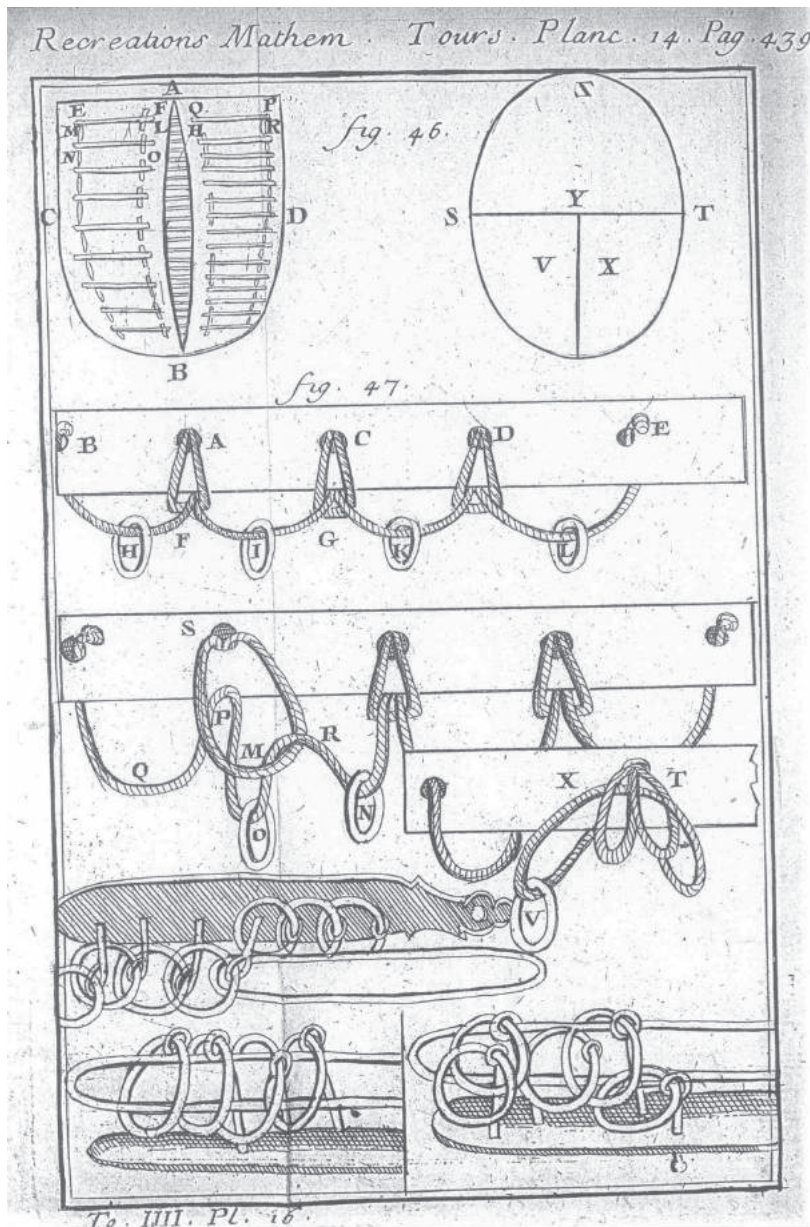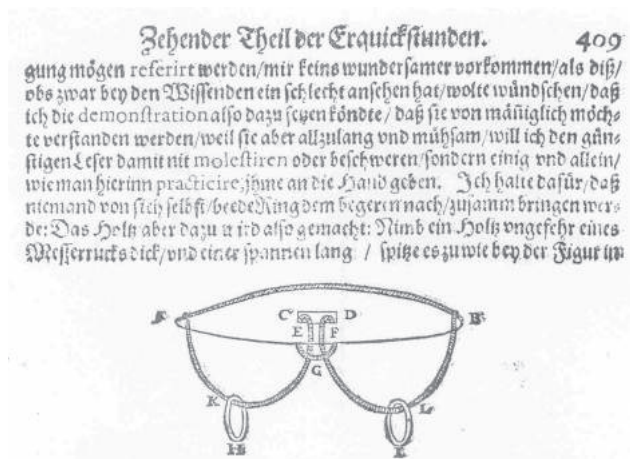Witgeest ([12], prob. 43, pp. 33-34) (Figure 32) is clearly taken from Schwenter.
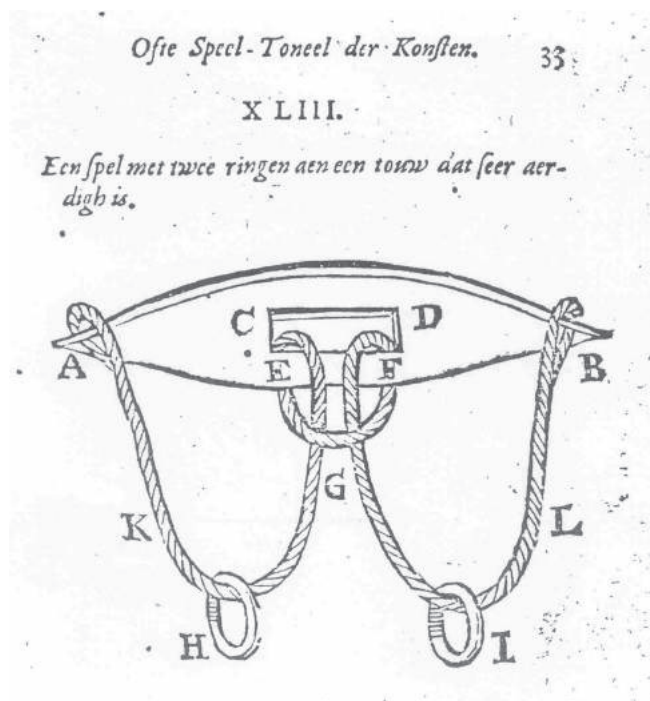

Figure 32: Witgeest 33.

While researching this work, I read some of the text adjacent to the Pacioli cited above. I had been putting off working on this because there is a lot of it and it is not always clear since there are rarely any pictures. Though I haven't yet made a detailed translation, it seems clear that [Pacioli, ff. 206v-207r, Part 2, Capitolo CI]: *De un altro filo pur in 3 fori in la stecca con unambra per sacca far le andare' tutte in una* is describing this same puzzle. The chapter titles vary between the actual problem and the Table of Contents and the latter shows that "unambra" should be "una ambra". Sacca means pocket or bay or inlet and it seems clear he means a loop which has that sort of shape. Ambra is amber, but seems to mean an amber bead here. So the chapter title can be translated as: "On another string also in three holes in the stick with one bead per loop, make all of them go onto one". Sadly there is no picture.

[I later made an inquiry on NOBNET about why this puzzle has recently been called an African puzzle and this turned up the following.

R. P. Lelong. "Casse-tête guerzé". *Notes Africaines* 22 (Apr 1944) 1. Not Yet Seen – cited and described by Béart. Says M. Gienger found the variant with an extra ring encircling both loops in the forest of the Ivory Coast in 1940, named *kpala kpala powa* [body of a toucan] or *kpa kpa powa* [body of a parrot].

Paul Niewenglowski. Bulletin de l'IFAN [Institut Français d'Afrique Noire] 14:1 (Jan 1952). Not Yet Seen – cited and described by Béart. Describes his invention of an interesting, rather simpler, variant as a result of seeing a standard version from Bêart.

Charles Béart. *Jeux et jouets de l'ouest africain.* Tome I. Mémoires de l'Institut Français d'Afrique Noire, No. 42. IFAN, Dakar, Senegal, 1955. pp. 413-418 discusses and carefully illustrates several versions. The standard version, but with several beads on one loop, is called *pèn* and is common in the forests of Guinea and Ivory Coast. Describes variants of Gienger/Lelong and Niewenglowski.

Fred Grunfield. *Games of the World.* Ballantine, NY, 1975. On p. 267, he calls this "African String Game", but gives no reference. Probably based on Béart.

Pieter van Delft & Jack Botermans. *Creative Puzzles of the World.* Abrams, New York, 1978. African ball puzzles. "It was once used in magic rites by tribes living in the jungles of the Ivory Coast. The puzzle is still used for amusement in this part of Africa, not only by the people who inhabit the remote outlying areas but also by city dwellers. . . . The puzzles were not restricted to this part of Africa. Variations may be found in Guinea, and some . . . were made in China." No reference given, but I suspect it must come from Béart, although this is not listed in their bibliography. My thanks to Mark Peters for the reference to van Delft and Botermans.]

References

[1] L. Beltrami. *1512-1532: materiale di studio*, Milano, Tip. U. Allegretti, 1911.

[2] B. Berenson. "North Italian Painters of the Renaissance", London: Forgotten Books pp. 251-2 (Original work published 1907), 2013.

[3] G. Cardano. *Opera omnia*, 1663.

[4] L. Daiken. *Children's games throughout the year*, London: Batsford, 1949.

[5] V. Huber. "The Yawning Mouth", Gibecière, 1, 13-46, 2005.

[6] J. Ozanam. *Recreations Mathematiques et Physiques*,... (2 vols.), Jombert, Paris, 1694. [1]

[7] P. Johnson. *Romano-British Mosaics*, Shire, 2nd edition, 2008.

[8] H.P. L'Orange, P. J. Nordhagen. *Mosaics: From Antiquity to the Early Middle Ages*, London:Methuen, 1966.

[9] J. Prévost, *Clever & Pleasant Inventions, Part One*, 1584.

[10] M. Rule. *Floor Mosaics in Roman Britain*, Macmillan, 1St Edition, 1974.

[11] T. Sammartini. *The Decorative Floors of Venice*, Merrell Publishers Ltd, 2000.

[12] D. Schwenter. *Deliciae Physico-mathematicae oder Mathematische und Philosophische Erquickstunden*, 1686.

[13] J. Shearman. *The Early Italian Pictures in the Collection of Her Majesty The Queen*, CUP, 1983

[14] J. J. Wecker. *De secretis libri*, 1582.

[15] S. Witgeest. *Het Natuurlijk Toverboek*, 1686.

[16] D. Wolfe and T. Rogers. *Puzzlers' Tribute, a Feast for the Mind*, A.K. Peters, 2002.

[17] http://www.uriland.it/matematica/DeViribus/Presentazione.html

---

[1]Numerous editions then appeared in Paris and Amsterdam, some in one volume; About 1723, the work was revised into 4 volumes, sometimes described as 3 volumes and a supplement, published by Claude Jombert, Paris, 1723. "The editor is said to be one Grandin.". In 1778, Jean Étienne Montucla revised this, under the pseudonym M. de C. G. F. [i.e. M. de Chanla, géomètre forézien], published by Claude Antoine Jombert, fils aîné, Paris, 1778, 4 volumes. The author's correct initials appear in the 1790 reissue] This is a considerable reworking of the earlier versions. In particular, the interesting material on conjuring and mechanical puzzles in Vol. IV has been omitted. The bibliography of Ozanam's book is complicated. I have prepared a detailed 7 pp. version covering the 19 (or 20) French and 10 English editions, from 1694 to 1854, as well as 15 related versions - this is part of my *The Bibliography of Some Recreational Mathematics Books.*

Mathematics and Arts

ALMADA NEGREIROS
AND THE REGULAR NONAGON

Pedro J. Freitas

Department of Mathematics and CEAFEL, University of Lisbon

pjfreitas@fc.ul.pt

Abstract: *Almada Negreiros was a Portuguese artist of the 20th century, one of the most relevant figures of Portuguese modernism. In the course his studies in visual art, he postulated the existence of a* canon, *a set of constants and geometric constructions underlying all art. Among these constructions were the n-th parts of the circle.*

In this paper we consider the regular nonagon. It is known that there is no exact construction for this polygon using just straightedge and compass. Nevertheless, there are some very beautiful approximate constructions, which we describe in the first part of this paper. In the second part, we describe two constructions with a very small error. The last one is due to Almada Negreiros, one that he found in his description of the canon.

Key-words: Regular polygons, nonagon, Almada Negreiros.

1 Almada Negreiros

José de Almada Negreiros (São Tomé e Príncipe, 1893 – Lisbon, 1970) was a key figure of 20th century Portuguese culture, in both visual arts and literature. He was a quite diverse artist, having produced paintings, murals, drawings, prose, poetry, theatre and critical essays. He is maybe less known for his studies of Portuguese renaissance art. These studies started as a compositional analysis of two famous Portuguese paintings, the *Ecce Homo* and the *Panels of St Vincent*, both on display at the *Museu de Arte Antiga* in Lisbon. Eventually, he thought of extending them to other Renaissance works, at first, and to all art works in general, at a later stage. He postulated the existence of an artistic canon, which could be understood as a set of proportions and constants, underlying all art. He enumerated the canonical elements as follows, in [5]:

> The simultaneous division of the circle in equal and proportional parts is the simultaneous origin of the constants of the relation nine/ten, degree, mean and extreme ratio and casting out nines.

Since the 1930s, Almada produced a very large number of geometric drawings in an attempt to produce these constants and relate them, partially inspired in [3]. In particular, he has many drawings in which he suggests processes for the division of the circle in n parts. He denotes by $\frac{\odot}{n}$ the n-th part of the circle and by $\frac{\bar{\odot}}{n}$ the chord of this arc, which corresponds to the side of the inscribed n-gon.

In this paper we take a look at the construction for the nonagon that is suggested by this drawing, which is part of a series called *The Language of the Square.*
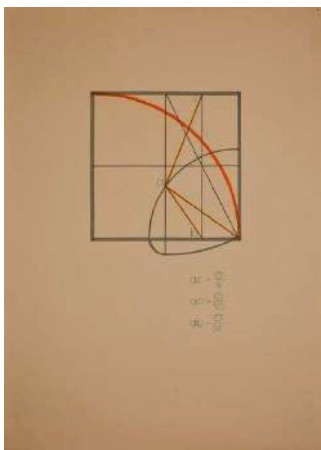
The construction also produces the sides of the decagon and the 14-sided polygon, as we can see, but we will direct our attention especially to the nonagon. Before that, we present other possible artistic constructions for this polygon.

2   Regular polygons

As many of us have learned in high school, there are geometric constructions for some regular polygons, using only straightedge and compass. These constructions usually start off with a circle, which is then divided in n parts, and the points obtained will be the vertices of the polygon. However, this is not possible for not all polygons. There is a well known result of Galois theory that describes completely which polygons are constructible — meaning, which ones admit a straightedge and compass construction. This is Gauss-Wantzel's theorem.

Gauss-Wantzel's Theorem. *The division of the circle in n equal parts with straightedge and compass is possible if and only if*

$$n = 2^k p_1 \ldots p_t$$

where $p_1, \ldots, p_t$ are distinct Fermat primes.

A *Fermat prime* is a prime of the form $2^{2^m} + 1$. Presently, the only known Fermat primes are 3, 5, 17, 257 and 65537. See for instance chapter 19 of [4] for a detailed account.

As we said, this result implies that the nonagon is not constructible with straightedge and compass since $9 = 3 \times 3$, the Fermat prime 3 appears twice in the factorisation of 9.

Therefore we can only hope to have approximate constructions for the nonagon. We will consider two types of construction: ones that provide all the vertices (even if the sides are not exactly equal), and then others that determine good approximations for the side of the nonagon, which then can be translated to the circle using the compass.
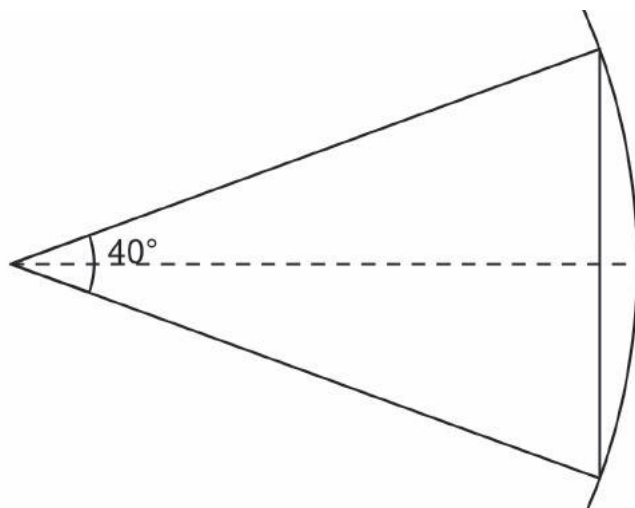
It is possible, nonetheless, to have an exact value for the side of the nonagon. If we take a circle of radius 1, the of the regular nonagon would be

$$2 \sin \frac{40^{\text{o}}}{2} = 2 \sin 20^{\text{o}}.$$

The value up to nine decimal digits is 0,684040287. All approximations of this value that we present in this paper are given by geogebra.

3 The nonagon, all vertices at once

In this section we analyze two constructions for the nonagon. The following one is sometimes associated with Dürer.
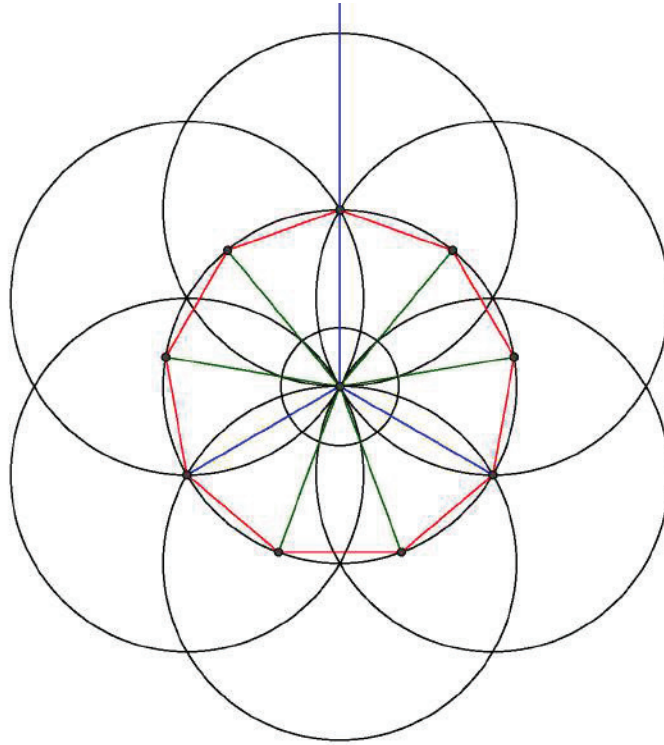
The construction protocol for this figure is rather straightforward. You start with a circle and draw a ray from its center (here we chose a vertical ray, marked blue, going up). Mark a point on this ray, at a distance from the center equal to three times the radius. Now take the distance between these two points as radius (three times the original radius) and draw two circles, one centred at each of the points. We then have two intersection points, which we use as new centres for circles (keeping the same radius). Again, new intersection points appear, and we draw more circles. When we are done, we have a central circle and six other ones around it.

Now we need to draw two types of rays stemming from the center: some are determined by some intersections of the large circles (the blue ones), while others are directed by points of intersection of the large circles with the small one (the green ones). There are three rays of the first type, and they make angles of 120º between them. The others would have to trisect these angles, which is something that cannot be done with straightedge and compass.

This construction leads to two different lengths for the sides of the nonagon. If we set the radius of the original circle as 1/3, then the larger circles have radius 1 and the lengths of the sides obtained are as follows.

side	length	error
between two green rays	0,697338	+2%
between a blue ray and a green ray	0,677378	−1%

Here is another construction, known as the flower of life.
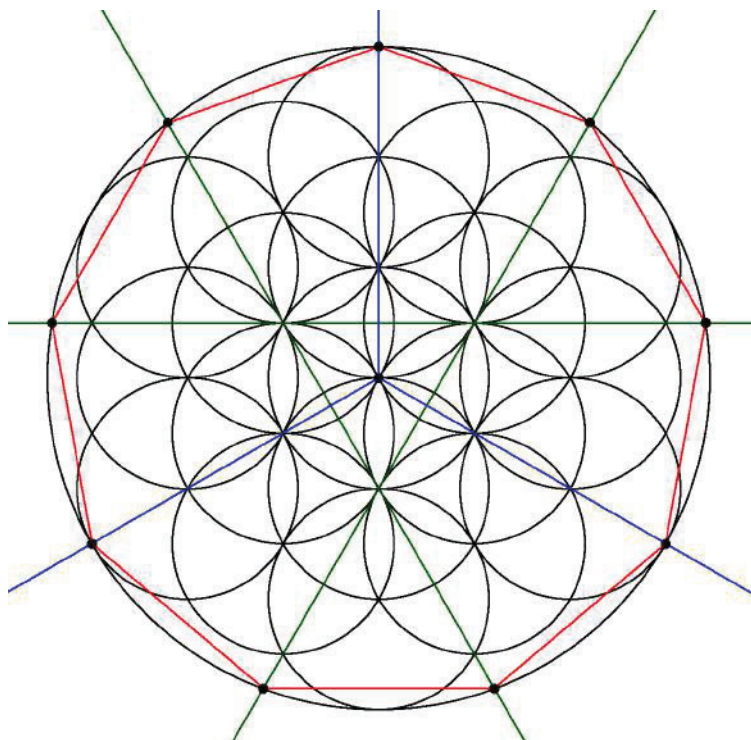
Again, the protocol is long, but rather straightforward. One starts with a circle and a ray from its its center (again we choose one going up). Then, keeping the radius, draw circles centred on the intersection points — over and over again, 19 of them on the whole.

In the end, you draw a big circle, with radius three times the original one. Some points of the nonagon are intersections of this circle with some smaller ones, which are tangent to it, marked by blue rays. Finally, add a regular triangle inside the original circle and draw lines based on its sides. This provides the remaining vertices, marked by green lines.

Is is a bit surprising that the points determined by this construction are exactly the same as the ones determined by the previous one — so the lengths of the sides and the errors are equal.

Here is a proof of this fact. Consider the following figure.
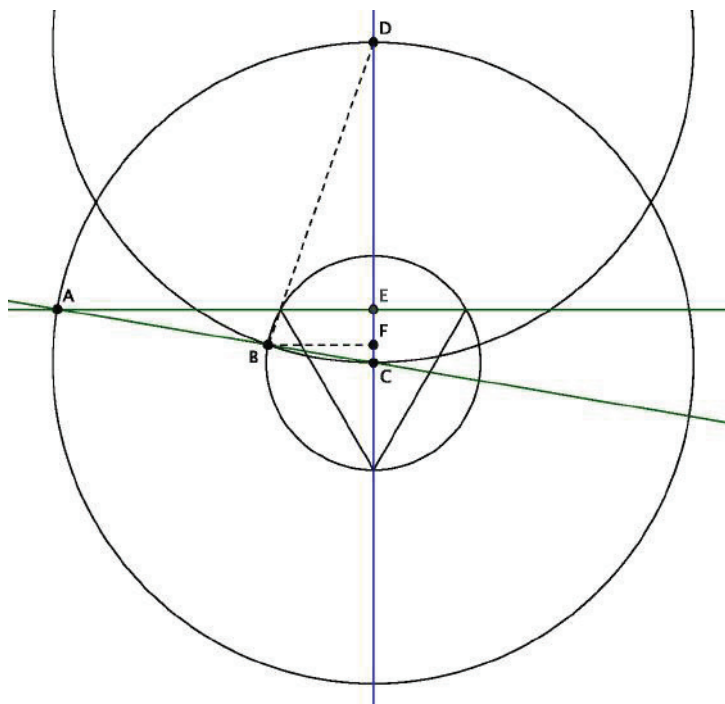
It is easy to see that, in both constructions, the blue lines are in the same position, at angles 120º from each other. For the green lines, we need to prove that A belongs to both lines shown — the one through the centre defines the vertex from point B in the first method, and the horizontal one is used in the second method.

Consider the dilation with center C and ratio 3. It clearly maps the smaller circle to the big one. We now need to prove that it maps point F to point E. Take BC to have unit length, we then have $CE = 1/2$. Thus we need to prove that $CF = 1/6$.

For this we use the Pythagorean theorem twice, applied to triangles BCF and BFD:
$$CF^2 + FB^2 = 1 \text{ and } FB^2 + FD^2 = 9.$$
Eliminating FB^2 from the equations, we get $1 - CF^2 = 9 - FD^2$. Replacing $FD = 3 - CF$ and working out the computations, we conclude that $CF = 1/6$ as we wished.

4   The nonagon — determining the side

In this last section we present two constructions that determine the side of the nonagon with great precision. By this we mean: if you use any of these methods to draw a nonagon in a sheet of paper of size A4 or US letter, you will not be able to notice the error.

The first method is very simple and elegant, it needs only four circles and a straight line.
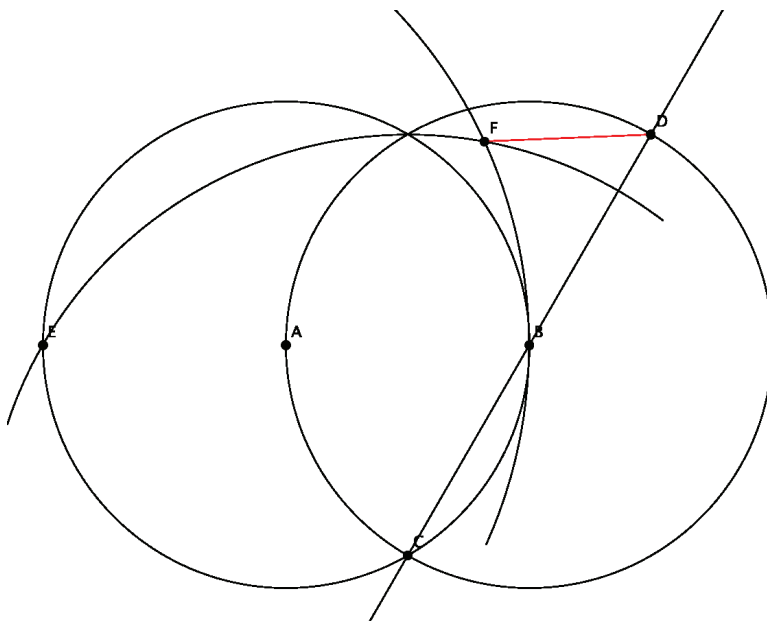
Here is the construction protocol.

- Draw a circle with center A.

- Take a point B on the circle and draw another circle with the same radius, centred at B. Let C be one of the intersection points of the two circles.

- Draw the line CB. Let D be the intersection point of this line with the second circle.

- Open the compass with radius AD and draw a circle with center C. Let E be one of the intersection points of this circle with the original circle, as in the figure.

- Draw a circle with center E and radius EB. Let F be the intersection point of this circle with the previous circle (centred at C).

- The segment FD is approximately the side of the regular nonagon, inscribed in the first circle.

There are two coincidences in this drawing. First, the length AD, used as radius for the third circle, is equal to the distance form C to the other intersection point of the circles centred at A and B. Second, the segment EB is a diameter. These are interesting facts (which the reader can verify easily), but they are not central to the construction.

If we take the first circle to have radius 1, the side obtained has length 0.68412, an error of 0,01%. In a circle of radius 10 cm, for instance, this error would be in the order of the hundredth of milimiter.

Finally, we look at Almada's drawing. The protocol in the drawing is rather simple, but Almada has another one, which we consider even simpler, for the same segment. We sketch it here.
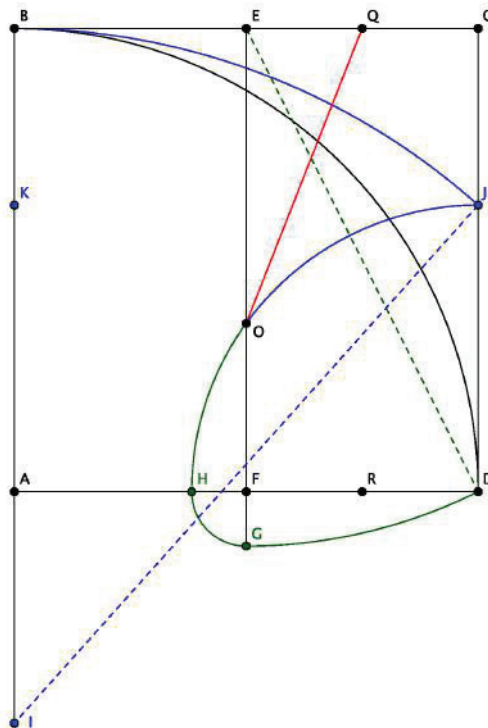
Points E and F are midpoints of the sides, point Q is the midpoint of EC and point R is the midpoint of FD. Point I is on the line AB, such that $IA = AF$ (half the side of the square). All these points appear in Almada's drawing, its positions can be determined just by taking measurements, since the square has side 20 cm.

The aim of the construction is to determine point O, so that OQ is the side of the nonagon, inscribed in the circle with center C and radius AB. The protocol sketched in the drawing is the following one.

- With center E draw arc DG.

- With center F draw arc GH.

- With center D draw arc HO.

This protocol is marked in green. The other one, marked in blue, which appears in another drawing, is the following.

- With center I draw arc BJ.

- With center D draw arc JO.

As we can see, this one is even shorter and avoids an arc of very small radius which might be hard to draw (arc GH).

Nevertheless, one should prove that the point O obtained by both methods is the same. For this we need to see that $DJ = HD$. We consider the square to have unit side.

In the first construction, using the Pythagorean theorem for the triangle DEF, we have

$$ED^2 = \left(\frac{1}{2}\right)^2 + 1,$$

so $EG = ED = \frac{\sqrt{5}}{2}$. Therefore $HF = FG = \frac{\sqrt{5}}{2} - 1$ and

$$HD = HF + FD = \left(\frac{\sqrt{5}}{2} - 1\right) + \frac{1}{2} = \frac{\sqrt{5} - 1}{2}.$$

For the second construction we take point K symmetric to J with respect to line EF and use the Pythagorean theorem for the triangle IJK:

$$\left(\frac{3}{2}\right)^2 = 1 + IK^2$$

which yields $IK = \frac{\sqrt{5}}{2}$. Therefore

$$DJ = \frac{\sqrt{5}}{2} - \frac{1}{2} = \frac{\sqrt{5} - 1}{2}.$$

This proves $DJ = HD$, as we wished. And actually, it proves a bit more. Almada had noted in the drawing that OD should be the side of the decagon, and with this computation we can actually verify this claim. The side of the decagon is $2\sin(360º/20) = 2\sin 18º$, and indeed $\sin 18º = (\sqrt{5} - 1)/4$, confirming Almada's claim.[1]

Almada also writes that OR is the side of the 14-sided regular polygon. Like the nonagon, this polygon is not constructible by straightedge and compass, according to Gauss-Wantzel's theorem. The error in this length is 1%.

Finally, we now present two constructions based on this drawing. The second one is the full construction, the first one is a partial one, which yields a reasonable approximation. The reason we show it is just because it is a nice and simple figure, consisting just of two midpoint constructions (well, three, if we count the one necessary to draw the perpendicular diameters, which is not shown).

---

[1]Consult WolframAlpha, for instance, for the exact value of the sine.
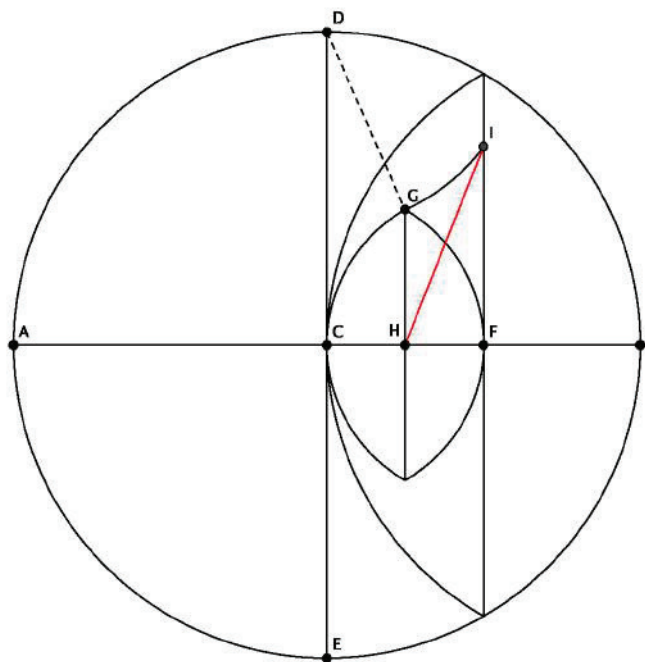
The protocol is quite clear from the figure.

- Draw a circle with center C and two perpendicular diameters, AB and DE.

- Draw a circle with center B and radius BC (as in the original circle). This construction leads to point F (midpoint of BC).

- Draw two arcs of circle with radius CF, marking the midpoint of CF, H and point G.

- Draw an arc with center D and radius DG, leading to point I.

- The segment HI is approximately the side of the regular nonagon.

For a circle with unit radius, this construction leads to a segment of length 0,681486, which represents an error of 0,4%.

We stumbled upon this construction as we tried to find a reasonable protocol for Almada's original drawing. It turned out that point G in the previous construction was close enough to the defining line to justify a shortcut. . .

We now present the full construction, inspired by Almada's drawing.
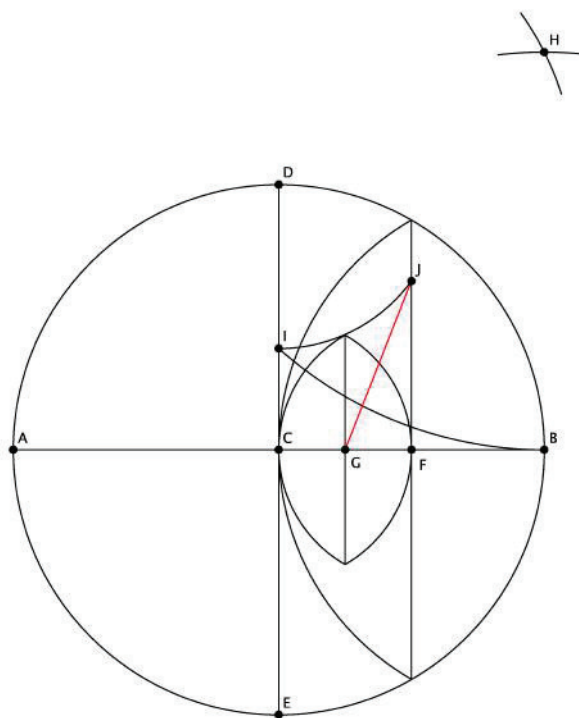
The construction protocol is as follows.

- Draw a circle with center C and two perpendicular diameters, AB and DE.

- Use the usual construction to determine the midpoint of CB, F.

- Use the same construction to determine the midpoint of CF, G.

- Draw a circle with center B and radius AF, and another one with center D and radius DF. Let H be the point of intersection of these two circles, as in the figure.

- Draw a circle with center H and the same radius HB and mark point I.

- Draw a circle with center D and radius DI and mark point J.

- The segment GJ is approximately the side of the regular nonagon.

Now we come to the main (and amazing) conclusion: the side of the nonagon obtained this way, for a circle of radius 1, is 0,6840493281, with an error of 0.001%. This corresponds to one thousandth of a millimetre in a circle of radius 10cm, or 1 mm in 100 m!

We believe that Almada reached these conclusions by endlessly drawing geometric constructions, trying to find relations between the elements of the canon. He never presented any calculations. Therefore, we believe he was unaware of the accuracy of his drawings, which makes this approximation, truly, an amazing find.

5   Acknowledgements

The ongoing study of Almada's estate is revealing quite a large number of geometry-based artworks and notebooks, which have only started to be analysed. So far, only the papers [1] and [2] have been published on this subject, but much more work remains to be done.

References

[1] Simão Palmeirim Costa and Pedro J. Freitas. "A Linguagem do Quadrado/The Language of the Square", *Cine Qua Non* (8), 2014.

[2] Pedro J. Freitas and Simão Palmeirim Costa. "Almada Negreiros and the Geometric Canon", *Journal of Mathematics and the Arts*, 2015 (DOI: 10.1080/17513472.2015.1012699).

[3] Jay Hambidge. *The Elements of Dynamic Symmetry*, Dover Publications, 1967.

[4] Ian Stewart. *Galois Theory – Third Edition*, Chapman Hall/Crc Mathematics, 2003.

[5] António Valdemar. "Assim Fala Geometria", a series of interviews to Almada Negreiros, *Diário de Notícias*, 1960.

Mathematics and Arts

Patterns, mathematics and culture: The search for symmetry in Azorean sidewalks and traditional crafts

Ricardo Cunha Teixeira
Department of Mathematics, University of the Azores
rteixeira@uac.pt

Abstract: *In the hustle and bustle of daily life, how often do we stop to pay attention to the tiny details around us, some of them right beneath our feet? Such is the case of interesting decorative patterns that can be found in squares and sidewalks beautified by the traditional Portuguese pavement. Its most common colors are the black and the white of the basalt and the limestone used; the result is a large variety and richness in patterns. No doubt, it is worth devoting some of our time enjoying the lovely Portuguese pavement, a true worldwide attraction. The interesting patterns found on the Azorean handicrafts are as fascinating and substantial from the cultural point of view. Patterns existing in the sidewalks and crafts can be studied from the mathematical point of view, thus allowing a thorough and rigorous cataloguing of such heritage. The mathematical classification is based on the concept of symmetry, a unifying principle of geometry. Symmetry is a unique tool for helping us relate things that at first glance may appear to have no common ground at all. By interlacing different fields of endeavor, the mathematical approach to sidewalks and crafts is particularly interesting, and an excellent source of inspiration for the development of highly motivated recreational activities. This text is an invitation to visit the nine islands of the Azores and to identify a wide range of patterns, namely rosettes and friezes, by getting to know different arts & crafts and sidewalks.*

Key words: Portuguese pavement, traditional Azorean arts & crafts, symmetry groups, rosettes, friezes.

1 Introduction

For being associated to the need of counting, calculating and organizing space and shapes, Mathematics is generally known as the science of quantity and space. Yet, this is a simplified and incomplete definition, particularly if one takes into account its extraordinary evolution over the last centuries and the

many branches that have meanwhile been created. Nowadays, "Mathematics: the science of patterns" is the definition that is widely accepted by the academic community. The mathematician's work is, then, to find, study and classify all types of patterns. This sometimes-strenuous job helps to understand better the reality around us.

The nine islands of the Azores Archipelago are divided in three geographical groups, located in the Atlantic Ocean: the Eastern Group, comprising Santa Maria and S. Miguel, the Central Group, including Terceira, Graciosa, S. Jorge, Pico and Faial, and the Western Group, composed by Corvo and Flores. In the Azores, there are many sidewalks and squares beautified by the traditional Portuguese pavement. Due to a great diversity of materials and techniques employed, the quality of the local arts & crafts proves astonishing to whomever visits the islands.

By embracing the challenge to discover and classify the patterns that can be found in the Azorean Portuguese Pavement and in the Traditional Azorean Crafts, those interested in the subject can achieve a better understanding of the mathematician's concern with the organization of information by "shelves", according to certain previously defined criteria. They can also enjoy the feeling of harmony of proportions and the search for an order, inherent to the concept of symmetry. Therefore, this is an excellent topic for promoting interesting recreational mathematical activities, thus connecting Mathematics with Culture.

2   What is symmetry and how Mathematics measure it

To simplify the classification of patterns, we will treat the examples of decorative and ornamental art (belonging to the 3-dimensional world) as if they were sets of points of the plane (2-dimensional), in other words, *figures* of the plane.

In the chosen examples, there should be a basic motif repeating itself. We will not be interested in the configuration of the motif (it could be a star, a flower, a petal, an abstract drawing or anything else), but in how the repetition occurs. To simplify the mathematical analysis, we should disregard minor flaws or irregularities and consider only two colors: the color of the figure and the background color.

Let us remember some important concepts. A *symmetry* of a figure is an *isometry* of the plane (a way of moving the points of the plane by keeping the distances among them) that maps the figure back onto itself. For example, when we rotate a square 90 degrees around its center, we still get a square in exactly the same position as we started (Fig. 1). We say the square has a rotational symmetry (in this case, of 90 degrees).
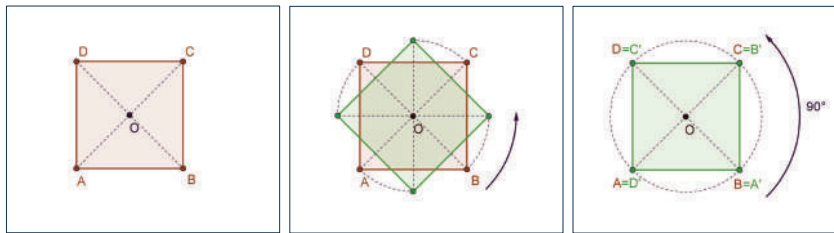
Figure 1: Rotational symmetry.

There are four types of symmetry:

- *reflection symmetry* or *mirror symmetry* (associated to a line, called the *axis of symmetry*);

- *rotational symmetry* (associated to a point, called the *rotation center*, and to a given amplitude);

- *translational symmetry* (associated to a vector, with a given direction and magnitude);

- *glide reflection symmetry* (resulting from the composition of a reflection with a translation of a vector parallel to the line that defines the reflection).

We can easily find these four types of symmetry in the Portuguese pavement. Some examples are given in Fig. 2.

Mirror symmetry (Terceira)   **Rotational symmetry (Corvo)**

Translational symmetry (Pico)

Glide reflection symmetry (S. Miguel)

Figure 2: Four types of symmetry.

Next, it is shown in detail how a basic pavement can comprise the four types of symmetry (Fig. 3). Let's start with the best known type: the mirror symmetry. In (a), if we place a mirror perpendicularly to the plane of the figure, so that the edge of the mirror is set on t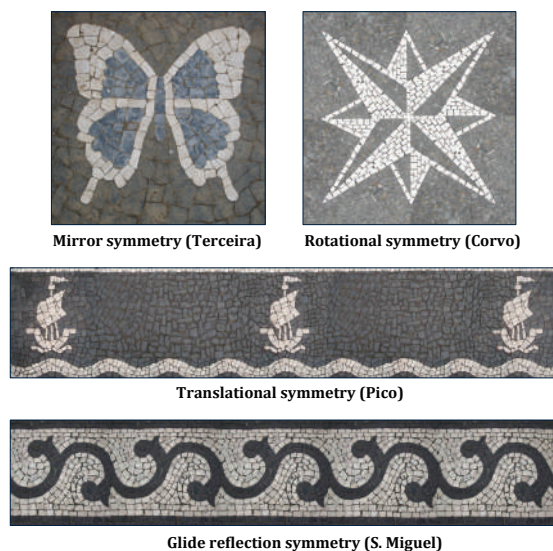he vertical red line, then we will see that each side of the image is indeed the reflection of the other (the same conclusion can be achieved by folding the paper that represents the plane along the red line). This line is called an axis of symmetry. Other axes of symmetry can be easily found as long as one bears in mind that the same pattern is repeated indefinitely to the right and to the left, beyond the photograph.
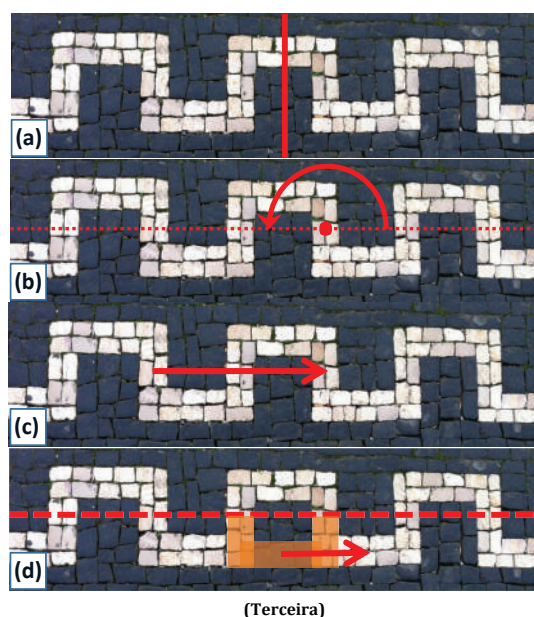

(Terceira)

Figure 3: A survey.

There are also other types of symmetry apparently less perceptible. In (b) one can explore the concept of rotational symmetry. First, we have to choose an appropriate point: the rotation center. The idea is to rotate the figure around the fixed point according to an angle with a given amplitude. Generally, it's counterclockwise, or the positive direction, as it's called. If by rotating the figure according to an amplitude lower than 360 degrees it matches the initial position, then we say that it has rotational symmetry: the initial figure and the one resulting from the rotation occupy the exact same position; when the symmetry was applied to the plane the design moved onto itself. The reader can use a pencil and tracing paper to duplicate the outline of the pavement in (b). Then, overlap the image drawn to the original image, place the tip of a pen on the tracing paper, on the point marked in (b), and rotate the paper 180 degrees around that point (two right angles). You will reach the conclusion that the figure obtained overlaps entirely the original figure. This is a 180-degree ro-

tational symmetry, also known as *half-turn*. If we bear in mind that the pattern repeats itself indefinitely and if we choose other appropriate rotation centers, we will find more half-turns.

There are two more types of symmetry to take into account. The translational symmetry is the symmetry that a figure has if it can be made to fit exactly onto the original when it is translated a given distance towards a given direction (according to a given vector). To illustrate that concept, use the sketch done on the tracing paper, overlap it to the figure in (c) and drag the tracing paper according to the direction and magnitude of the vector represented in (c). At the end of the process, you will reach the conclusion that there is a perfect overlapping of the two outlines.

Let's have a look at the last type of symmetry: the glide reflection symmetry. In (d), there is a dashed horizontal line. A closer look shows that this line is not an axis of symmetry of the figure. Nevertheless, if we apply the reflection in the given line, followed by a translational movement according to a vector parallel to the line with half the magnitude of the vector represented in (c), we quickly understand that the resulting figure overlaps the initial one[1].

For more details on isometries and symmetries, see [3, 5, 6, 8].

Now, let's see how to classify a figure based on its symmetries. The set of all symmetries of a figure forms a group under composition: the *symmetry group* of the figure. The classification of symmetry groups is summarized in Fig. 4 ([2]).
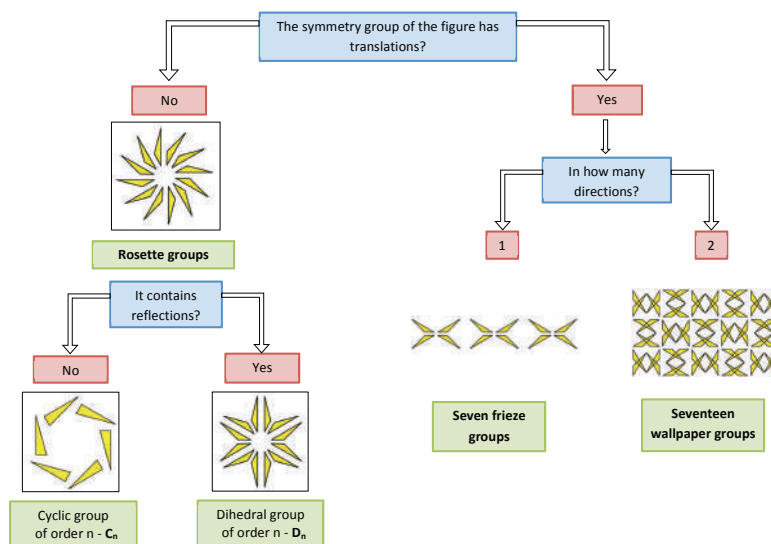

Figure 4: Symmetry groups.

---

[1]The exploration of these symmetries can be seen in: `https://youtu.be/aIgl9T658bk`.

Rosettes are figures with rotational symmetries and, in some cases, mirror symmetries. It can be proved [3] that only two situations can occur: their symmetry group is a *cyclic group* C_n (figures with n rotational symmetries) or a *dihedral group* D_n (figures with n rotational symmetries and n mirror symmetries). The rotational symmetries have all the same center and are associated to amplitudes of $360/n$ degrees and their multiple numbers. The axes of symmetry, when existing, all go through the rotation center.

Actually, you only have to identify the basic motif that is repeated around the rotation center and count the number of repetitions (n). Then all you have to do is check for rotational symmetries alone (C) or whether there are also mirror symmetries (D), as shown in Fig. 5.
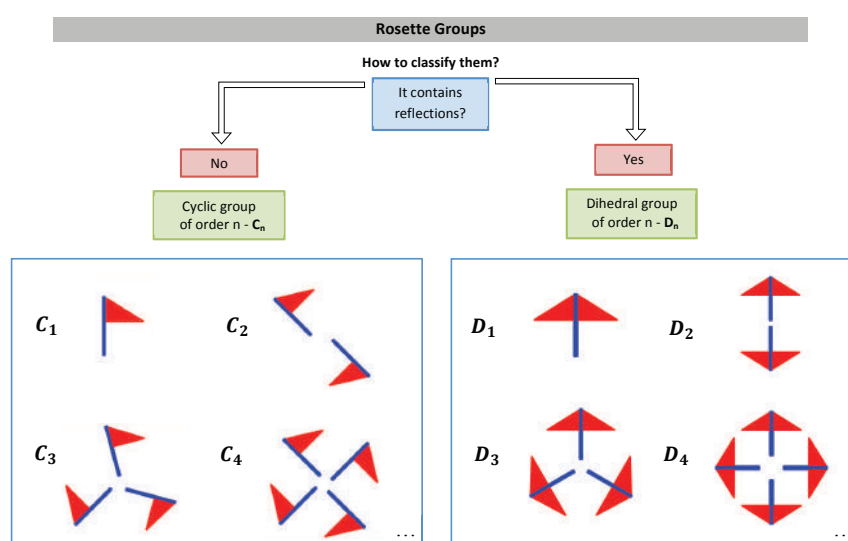

Figure 5: Rosettes.

A figure with symmetry group C_1 is called *asymmetric* (because it lacks symmetry), since the only way it can match the initial position is by the trivial rotation of $360/1 = 360$ degrees (or 0 degrees, if you prefer). A figure with symmetry group D_1, besides the trivial rotation, presents a mirror symmetry. For the symmetry group C_2, we have a rotational symmetry of $360/2 = 180$ degrees and another of $180 + 180 = 360$ degrees (that is, the trivial rotation). For group D_2 we also have to include two mirror symmetries (with perpendicular axes of symmetry). Furthermore, group C_3 comprises rotations of $360/3 = 120$ degrees, $120 + 120 = 240$ degrees and $120 + 120 + 120 = 360$ degrees. For group D_3, we have to add three mirror symmetries. And so forth and so on.

The *friezes* are figures that have translational symmetries in one single direction. Mathematics capacity to systematize the information has again prevalence here for it can be proved that there are only seven different ways of repeating a given basic motif along a strip by using the four types of symmetry [3].

In friezes, a rotational symmetry (if it exists) must be a 180 degrees one, also known as a half-turn. The reason for that is quite simple. As the motif is repeated across a plane following only one direction, using a rotation with an amplitude different from 180 degrees would subsequently displace the motif to a direction different from the one desired, i.e., outside the strip.

Another issue to take into account when classifying a frieze is its position. To avoid confusions, it's preferable to study it in a "horizontal position", that is, we should consider that the pattern repeats itself along a strip "parallel to the ground". Thus we can say without doubt that there is a *horizontal reflection* (when the axis of symmetry has the same direction of the strip) or a *vertical reflection* (when the axis of symmetry is perpendicular to the strip).

What are the differences between the seven frieze groups?[2] Fig. 6 presents a flowchart for the classification of the symmetry group of a frieze. We use the notation of Fejes Tóth [7]. The seven symmetry groups are represented by the letter F. When there is a half-turn a 2 is placed in the subscript position; otherwise a 1 is placed in that position. In the superscript position, it is placed a 1 (when there is a horizontal reflection), a 2 (when there is a vertical reflection) or a 3 (when there is a glide reflection). The absence of an exponent means that there are no mirror symmetries, as well as no glide reflection symmetries.
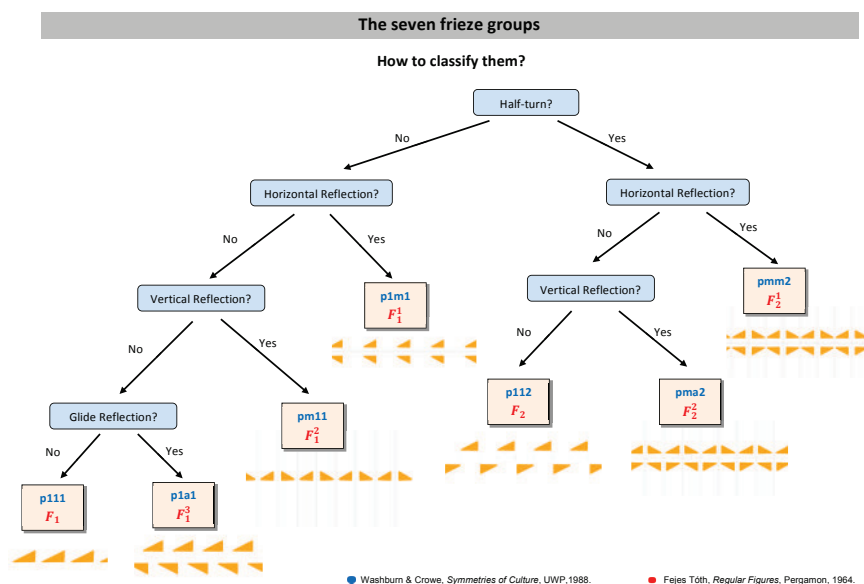

Figure 6: Flowchart for the classification of the friezes.

There are other notations. For instance, the four-symbol notation given in Fig. 6

---

[2]The exploration of the seven frieze groups using one domino can be seen in: `http://youtu.be/GKTNCNrOMhw`.

is the classical one introduced by Russian Crystallographers. It can be determined for a given frieze as follows: the first symbol, p, signals the existence of a translation; the second symbol is m if there is a vertical reflection and 1 otherwise; the third symbol is m if there is a horizontal reflection, a if there is a glide reflection and 1 otherwise; the fourth symbol is 2 if there is a half-turn and 1 otherwise [9].

If a figure is neither a rosette nor a frieze, then it must have translational symmetries in more than one direction (Fig. 4). The multiple translational directions force the pattern to cover the entire infinite plane. It can be proved that there are only 17 different ways of producing patterns under these criteria, which correspond to the 17 *wallpapers groups* [3].

Given the abundance of rosettes and friezes on Azorean sidewalks and Azorean traditional crafts, we propose a brief journey through the nine islands of the Azores to explore their symmetries.

3   Artistic crochet from Faial and Pico

In Faial and Pico, *needlework* or *artistic crochet* gives life to different decorative elements: flowers (the most characteristic element of the Azorean needlework, with interconnected rosettes representing the passion fruit flower, blackberries and grapes); geometrical (based on Greek motifs); and representations of everyday life in fine and delicate items either for domestic use or for decoration, as tablecloths, centerpieces, bed linen, doilies and other linen or cotton objects. This traditionally womanly craft had its highpoint in mid-twentieth century and is tightly connected to the emigration phenomena to the USA[3]. For more information, see [4].

Figure 7: Ana Baptista.

In the beginning of the 1950s, there were about five hundred full time needle

---

[3]The Regional Centre for Handicrafts's website: `http://www.artesanato.azores.gov.pt`.

lace workers. Nowadays that number has really decreased. What was once a fundamental means of subsistence for many families and the predominant activity for women, currently is a landmark of Azorean cultural heritage duly certified and deserving a place of honor in national and international exhibitions. The work of Ana Baptista (Fig. 7), of all the needle lace workers still active and still keeping this art alive, is worth mentioning for the perfection with which it is executed.

Ana Melo Baptista was born in the village of Flamengos, on the island of Faial, and currently lives in the city of Horta. She first began lace confection under the guidance of her older sister. Not yet out of school, she was already producing needle lace gloves. Stitches as the button hole and the eyelet were some of the most difficult challenges to overcome in the early years of apprenticeship of this young artisan, who spent all her available time, including evenings, learning. After some time, she was already her own boss and even employed several other workers, which serves to show how very interesting her professional profile is.

Let us have a look at some works made by this artisan (Fig. 8). Ana Baptista brings forth two pieces worth studying for their originality: a sunflower (a) and a rectangular doily with a different array of blackberries in the outer strip (b). The blackberries design has been complimented by many and orders for replicating the design on bed sheets and doilies for tea trays have been placed. The artisan has another interesting design: a doily with roses and beech leaves (c) which was chosen as the cover image of [4], a book on traditional lace.

Next, we will analyze the symmetries of some traditional lace pieces produced by Ana Baptista, whose promptitude and responsiveness we appreciate immensely.

We could analyze the sunflower, in (a), as a whole or we could focus on the flower in the center and on each of the following circular strips. We immediately find rotational symmetries: if we rotate the doily around its center according to a given amplitude, the figure obtained totally overlaps the initial figure. The amplitude to be used depends on the number of the motif's repetitions. For example, on the center flower there are 12 petals (12 repetitions), for which the rotation angle should have an amplitude of 360/12=30 degrees (or any of its multiples) as to obtain a symmetry of that flower. If we analyze the following circular strips of the doily, we find a strip with 24 "sunrays" (now the minimum amplitude is of $360/24 = 15$ degrees) and the outer strip has 18 sunflower petals (the minimum amplitude is of $360/18 = 20$ degrees).

In the situations analyzed, we can also find mirror symmetries (the number of axes of symmetry is equal to the number of repetitions of each motif). Such is the case of the flower in the center (D_{12}) and of the outer strip showing the sunflowers petals (D_{18}), but no longer the case of the "sunrays" strip (C_{24}). The use of a mirror allows the immediate conclusion that this strip does not have reflection symmetries. Thus, we are left with just rotational symmetries for the "sunrays" strip, which convey a feeling of movement around a point, as we get from looking at a weather vane or the sails of a windmill.

If we analyze the edge of the doily present in (b) and overlap the motif that
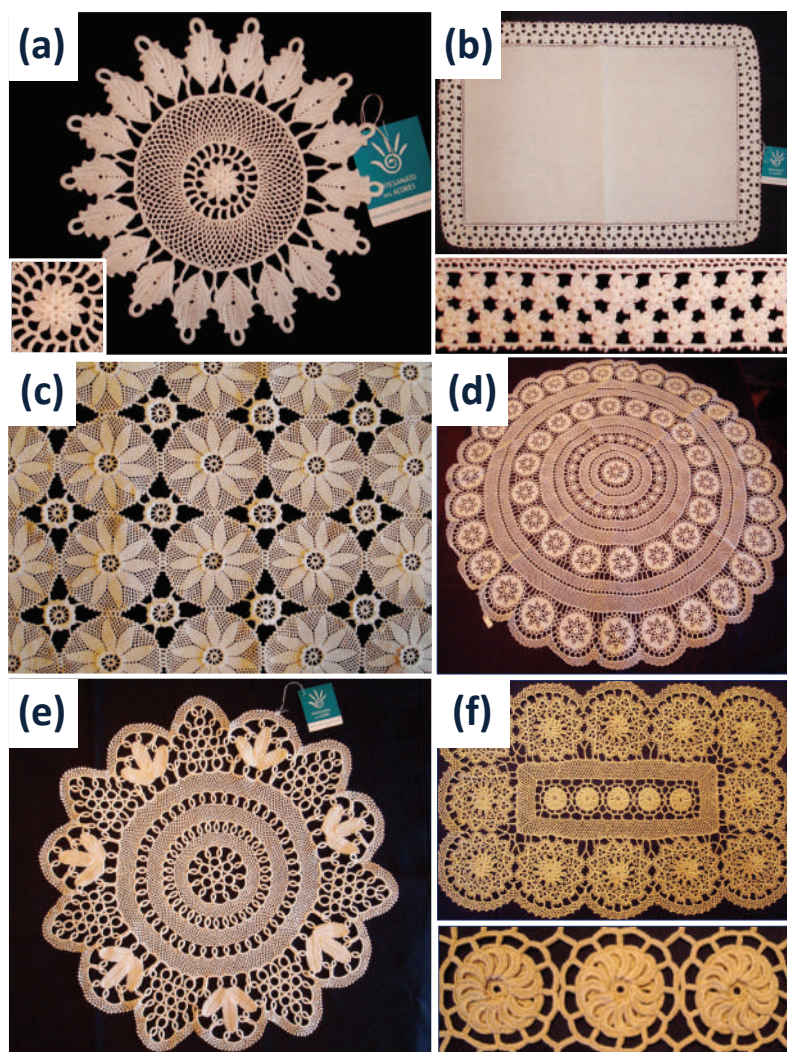
Figure 8: Works of Ana Baptista.

is repeated indefinitely from left to right, we get a frieze, that is, a figure with translational symmetries in only one direction. The strip in (b) has other types of symmetry as well: half-turn symmetries ($360/2 = 180$ degrees rotations; if we imagine the strip upside down, its configuration is not altered); vertical reflections (the axes of symmetry are perpendicular to the frieze); and glide reflections (following the same direction as the frieze, these symmetries produce an effect similar to our footprints when walking barefoot on the sand). Therefore, the symmetry group of the strip in (b) is F_2^2.

In (c) we can regard the flowers individually as rosettes. We can also consider strips of flowers, for example, horizontally or vertically and then get a frieze. If

we analyze the doily as a whole, we get a 2-dimensional pattern (a wallpaper), for the flowers pave the whole plane.

Traditional lace, often called *Artistic crochet*, is an unending source of symmetries. Some more examples are put forth in (d), (e) and (f) and we dare the reader to find their symmetries!

4   Wheat Straw Embroidery from Faial

As if it were gold thread, wheat or rye straw is used by the embroiderers of Faial to decorate white or black tulle, creating unique evening gowns, bridesmaid dresses, scarves or doilies. The main decorative element is the wheat spike, although other vegetal or even figural elements are part of the motifs chosen by the embroiderers of the island[4]. For more information, see [1].

The uniqueness and splendor of the Azorean embroidery totally justifies its promotion on the regional, national and international levels. It is a genuinely Azorean product with quality and origin certification since 1998. On the island of Faial, the major embroideries are made of wheat straw on tulle.

We sat down and talked with Isaura Rodrigues, a well-known artisan famous for her wheat-straw-on-tulle expertise (Fig. 9).

Figure 9: Isaura Rodrigues.

An interesting fact pops up when talking about her life experience: this artisan is not just an expert on embroidery using wheat thread and tulle. Isaura rewinds the film of her life: "In 1998 I had to go live temporarily in a prefabricated unit. I missed the roomy house where I lived. I tried to overcome this less positive moment of my life by exploring various forms of artisanship. I started

---

[4]The Regional Centre for Handicrafts's website: `http://www.artesanato.azores.gov.pt`.

by doing some stitch embroidery. Then I moved to building boxes with ribbons and flowers for napkins, for decorating a child's room or as sewing boxes. There was even an exhibition with those boxes, which had some demand. In addition, I became interested in working with clay. I attended a course and with the kiln my husband offered me, I made Nativity scenes, decorative plates and chimes. However, the opportunity to attend a course on wheat straw embroidery on tulle in the Capelo Crafts School arose. Whenever I had some free time, I carried on perfecting the technique at home. Later, I was encouraged by the Regional Centre for Handicrafts to request my certification."

We shall now analyze the symmetries found in some wheat straw on tulle works created by Isaura Rodrigues (Fig. 10), whose promptitude and responsiveness we appreciate immensely.

Let's begin with the scarf of images (a) and (b). It's easy to identify a half-turn (a 180-degree rotational symmetry). This means that if we turn the scarf upside down its configuration is not altered. This kind of symmetry is very usual not just in arts & crafts but in sidewalks and balconies as well. This abundance has a very practical reason. For example, a centerpiece with a half-turn symmetry has exactly the same configuration if seen from one or the other side of the table, in front of us. The same goes for a sidewalk with this kind of symmetry. As for the scarf, the person wearing it has just to worry about which is the right side of the piece; otherwise, it can be rotated 180 degrees as much as you like for the configuration will not change.

This scarf is an example of a rosette. Due to not presenting reflection symmetries, the scarf of (a) and (b) has C_2 as its symmetry group.

Let's consider some rosettes with mirror symmetries. The skirt (c) has two perpendicular axes of symmetry: a horizontal axis and a vertical axis. In (d), we can take a look to the rosette in more detail. If we place a mirror perpendicularly to the skirt, so that the edge of the mirror is set on the horizontal line (or the vertical line) passing through the center, then we will see that each side of the skirt is indeed the reflection of the other. This means that if we fold the skirt along one of the axes of symmetry, the two halves should overlap completely. It is also easy to see that the figure has a half-turn symmetry, thus D_2 is the symmetry group of this rosette.

Let's take a look to another skirt (e). The centre of the rosette is again of type D_2. However, if we consider the whole visible pattern in the image, it is possible to identify only one vertical axis of symmetry. In addition, if we turn the skirt upside down its configuration is changed (due to the branches of the left and the right). Therefore, the whole visible pattern of the skirt (e) is of type D_1. The same conclusion applies to the example (f).

The skirt of the image (g) is decorated with numerous flowers of 6 petals. Each of these flowers has D_6 as its symmetry group. In fact, it is possible to count 6 repetitions (6 petals), for which the rotation angle should have an amplitude of $360/6 = 60$ degrees (or any of its multiples) as to obtain a symmetry of that flower. We can also identify 6 axes of symmetry (all axes pass through the rota-

Figure 10: Works of Isaura Rodrigues.

tion center; 3 of them cut in half opposite petals and the remaining 3 separate consecutive petals).

Finally, in (h), we identify two motifs that are alternately repeated along a strip. The spacing between consecutive copies of these motifs is always the same. Therefore, this pattern is a frieze, which is characterized by the presence of translational symmetries in one direction. The configuration of the frieze is slightly different if we turn the skirt upside down, so we don't have half-turns. On the other hand, it is possible to identify vertical reflections, whereby F_1^2 is the group of symmetry of this frieze.

5   Angra do Heroísmo: the city of the seven friezes

The sidewalks and squares paved in Portuguese Pavement or Portuguese mosaics are one of the most characteristic aspects of the heritage of many Portuguese towns. We step on them every day but most of the time we do not pay attention to its historical, artistic and geometrical heritage.

Mosaic is as old as the most remote civilization in History. Nevertheless, it was the Roman civilization that took mosaic to the paving of the domus and the villae. In Portugal, its use as a decoration is a nineteenth-century derivation of the Roman way. It was in Lisbon that for the first time, in 1848, paving was used in urban spaces: the Mar Largo project, a composition shaped as waves, was built at the D. Pedro IV Square (Rossio nowadays). But six years before this project, by initiative of Lieutenant-General Eusébio Cândido Cordeiro Pinho Furtado, the narrow streets leading to the São Jorge Castle were paved with white (limestone) and dark (basalt) stones.

Promptly, Portuguese pavement spread to the rest of the continental country and later on to the archipelagos of the Azores and Madeira. It also went beyond the national borders for Portuguese pavior masters were asked to implement and teach their art abroad. This application can be seen in projects such as the S. Sebastião Square in Manaus (Brazil); the famous Calçadão of the Copacabana Beach in Rio; and in Macau, Cape Town (South Africa) and many other places.

In the Azores, artistic paving dates from mid-twentieth-century and replaced the basaltic paving of the sidewalks of the former and main streets of the towns. It also spread to squares and nowadays to private atriums and gardens, bearing different artistic patterns where the black basalt contrasts with the white limestone. For reasons of economy, the prevalence goes for the local black basalt stone and the white limestone (imported from Lisbon) is used on a smaller scale.

We should point out a major achievement for the valuation of the sidewalk as an Azorean heritage, including as a tourism attraction. In June 2014, Angra do Heroísmo (Terceira) reached the status of "City of the Seven Friezes" for having the seven possible types of friezes in its sidewalks, thus following in the footsteps of Lisbon. It is the first Azorean city to achieve this feat and, most likely, the second of Portugal, after Lisbon.

In Fig. 11, it is shown an example of each of the seven types of friezes. One should bear in mind that all friezes have a common property: the translational symmetries in one direction, which implies precisely the repetition of a motif along a strip. For example, in (a) it is possible to identify a basalt parallelogram repeated successively along the strip, with their consecutive copies showing equal spacing between themselves.

The first examples can be found on the sidewalks of Rua de São João (a), Rua da Conceição (b) and Avenida Tenente Coronel José Agostinho (c). Besides the
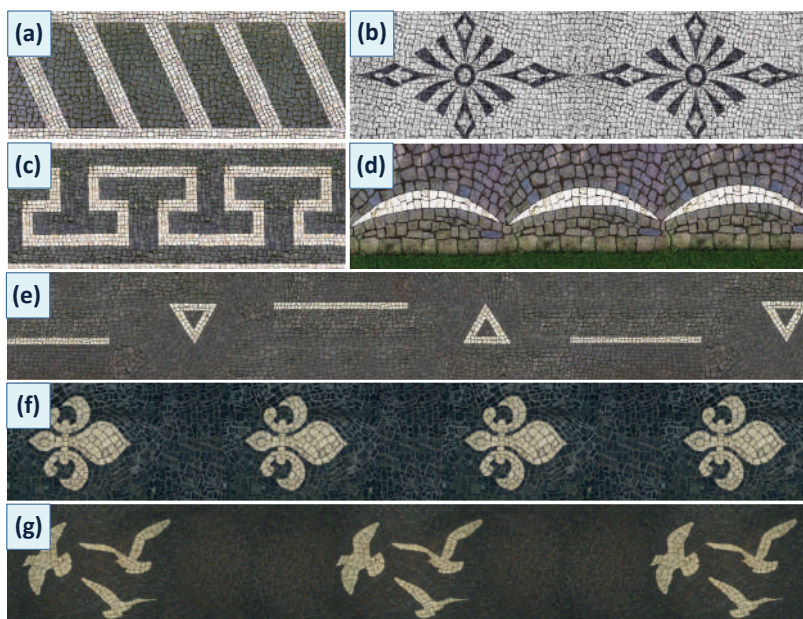
Figure 11: Seven types of friezes.

translational symmetries, these three types of friezes share another property: the half-turn symmetries. If the reader imagines each one of these upside down, the result is that the configuration does not change. The practical effect of this property explains why these three types of friezes can be found so abundantly in our sidewalks, our balconies, and several crafts: if you look to the frieze from one or the other side of the sidewalk its configuration is not altered. Then, what tells them apart? Besides the translational and half-turn symmetries, the first (a) does not possess other symmetries – F_2; the second (b) has horizontal and vertical reflection symmetries – F_2^1; finally, the third (c) shows vertical reflection symmetries and also glide reflection symmetries, which produce a kind of "zigzag" effect similar to our footprints when walking on the sand barefoot – F_2^2.

Let's look at a very peculiar example. Resembling the frieze at Avenida Tenente Coronel José Agostinho (c), the frieze at Rua da Queimada (e) also has glide reflection symmetries. Yet, the latter does not have half-turn symmetries – F_1^3. Should the reader focus on the position of the triangles and the line segments of the frieze at Rua da Queimada (e) and imagine it upside down, you will find out that the configuration you get is different from the original one. So, a new frieze is obtained, with a different disposition of the triangles and the line segments, but keeping the glide reflection symmetries. Interestingly, this new frieze can be spotted farther ahead, at Rua Madre de Deus. Thus, these are two different friezes with the same group of symmetry, since both have only translational and glide reflection symmetries. These are the only examples of this type of frieze in Angra do Heroísmo.

The next type of frieze is characterized by having only translational and vertical reflection symmetries – F_1^2. We can find examples of this type at the Duque da Terceira Garden (d), Rua de Cima de Santa Luzia and a short section of Rua Direita.

Before June 2014, there were only these five types of friezes in Angra do Heroísmo. Two types of friezes were still missing so the city of Angra, a World Heritage Site by UNESCO, could reach the status of the "City of the Seven Friezes". The quest ended precisely in June 2014 when the two friezes missing were inserted in the sidewalk near the Colégio Square: a frieze (f) with translational and horizontal reflection symmetries – F_1^1, and a frieze (g) with just translational symmetries – F_1. The motifs used for implementing these friezes are a creation of Architect Maria João Miranda, aided by Paulo Mendonça. The professional approach of the staff of the municipality of Angra do Heroísmo, lead by Professor Álamo Meneses, and the enthusiasm with which my proposal for achieving this status was welcomed cannot be forgotten. In a nutshell, "Angra, the City of the Seven Friezes" is an achievement that:

- brings added value to the cultural heritage of sidewalk paving;

- can be a tool for teachers for field trips (since the theme of symmetries is part of the present syllabus), connecting Mathematics to every day life;

- brings forth the opportunity to discover a growing tourism segment, the Mathematical Tourism ("Mathourism");

- is a fine example of the potential of Recreational Mathematics.

6   The endless search for symmetry

The Holy Spirit festivities take place on all the islands of the Azores and beyond that, in the diaspora. The Holy Spirit cult is one of the oldest and best-known practices of popular Catholic religiosity. The cult originated on mainland Portugal (probably with queen Santa Isabel) and were brought to the Azores by the early settlers. Gastronomically, the hosts of the festivities offer the traditional soup, the roast beef, *massa sovada* (a sweet bread) and *arroz doce* (a rice pudding). This rice pudding is usually decorated with cinnamon. And why not take the chance to replicate here the seven types of friezes? In Fig. 12, we show some examples of rice pudding's decoration that took place in a Holy Spirit festivity, in June 2014, for which there was no limit to mathematical imagination!

We wish to thank to Edna Soares and Pedro Soares, who hosted the Holy Spirit festivity, and also Goreti Rosa Carvalho, who cooked the Rice Pudding.

The half-turn friezes were selected as to pay homage to the three Azorean cities with more frieze types. There is the sidewalk of Rua Dr. Aristides da Mota, in Ponta Delgada (f); the sidewalk at Praça da República, in Horta (h); and a section of the sidewalk of Rua da Sé, in Angra do Heroísmo (j).
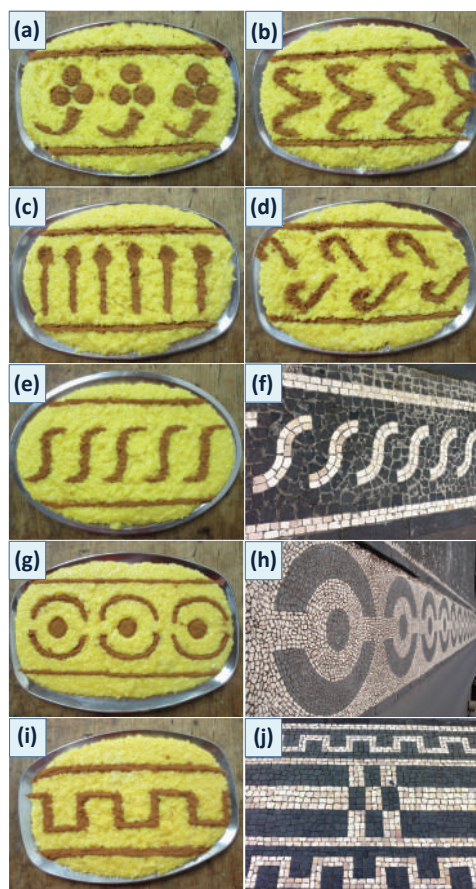
Figure 12: Symmetry groups: (a) – F_1; (b) – F_1^1; (c) – F_1^2; (d) – F_1^3; (e) and (f) – F_2; (g) and (h) – F_2^1; (i) and (j) – F_2^2.

Furthermore, the types of friezes selected are very usual in the aforementioned cities:

- In Ponta Delgada the friezes F_2 are very common: at Rua Dr. Guilherme Poças, at Av. Gaspar Frutuoso, at Largo da Matriz, at Portas da Cidade, at Rua 6 de Junho, at Largo de Camões and at Rua dos Clérigos;

- In Horta there are many friezes with symmetry group F_2^1: at Alameda Barão de Roches, at Largo Duque d'Ávila e Bolama, at Rua Walter Bensaúde, at Rua Conselheiro Medeiros and at Rua José Azevedo;

- In Angra do Heroísmo one can find many friezes F_2^2: at Rua Beato João Batista Machado, at Canada dos Melancólicos, at Rua Dr. Henrique Braz, at Rua Dr. Luís Ribeiro, at Rua Padre Manuel Joaquim Máximo and at Avenida Tenente Coronel José Agostinho.

We end with a brief visual catalog comprising a few more examples of the symmetries that can be found in the Azorean sidewalks.

Some of these examples shall be present in a deck of cards bearing the sidewalks of the Azores to be launched soon by the Ludus Association, with co-authorship of Alda Carvalho, Carlos Pereira dos Santos, Jorge Nuno Silva and Ricardo Cunha Teixeira.
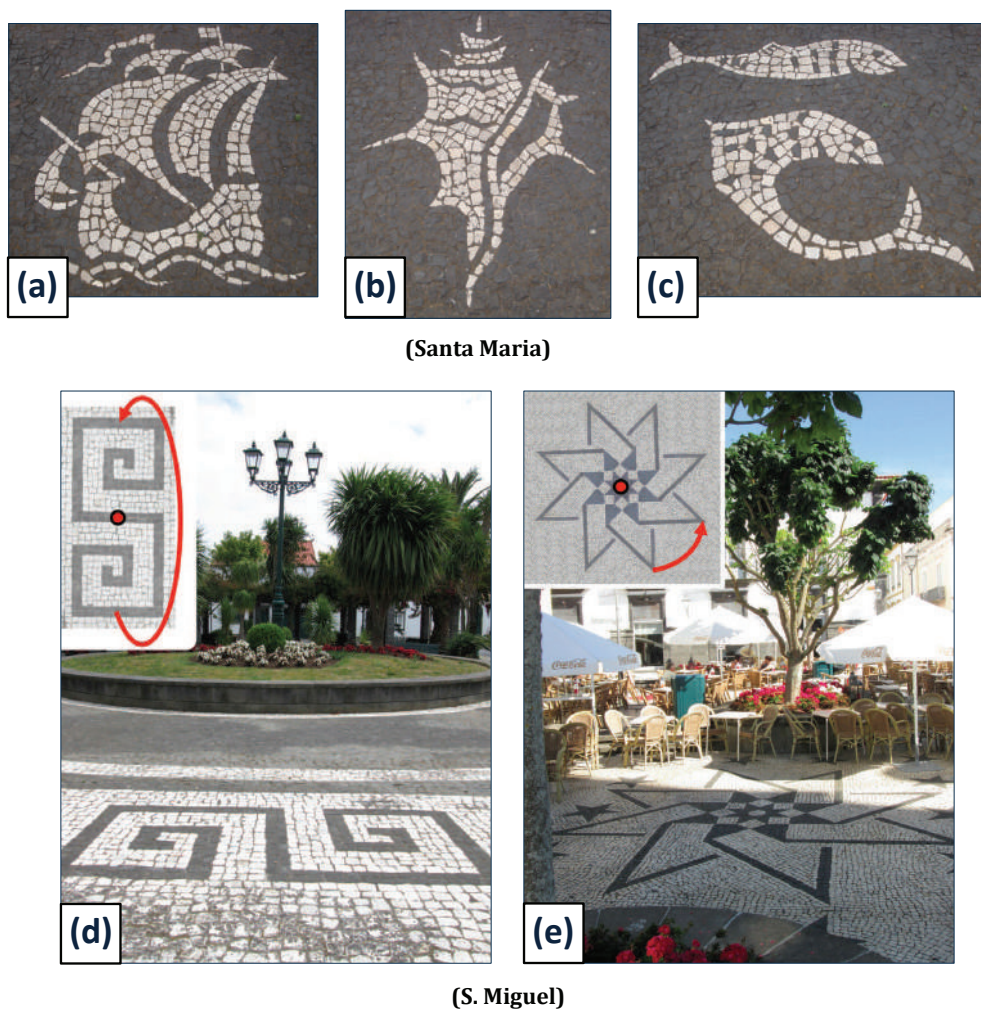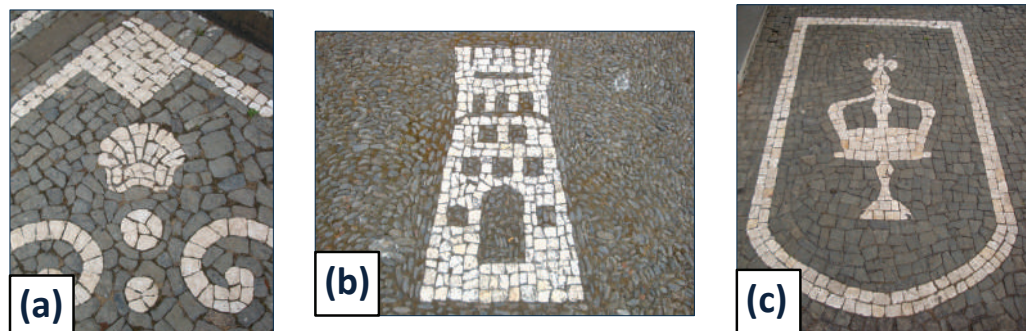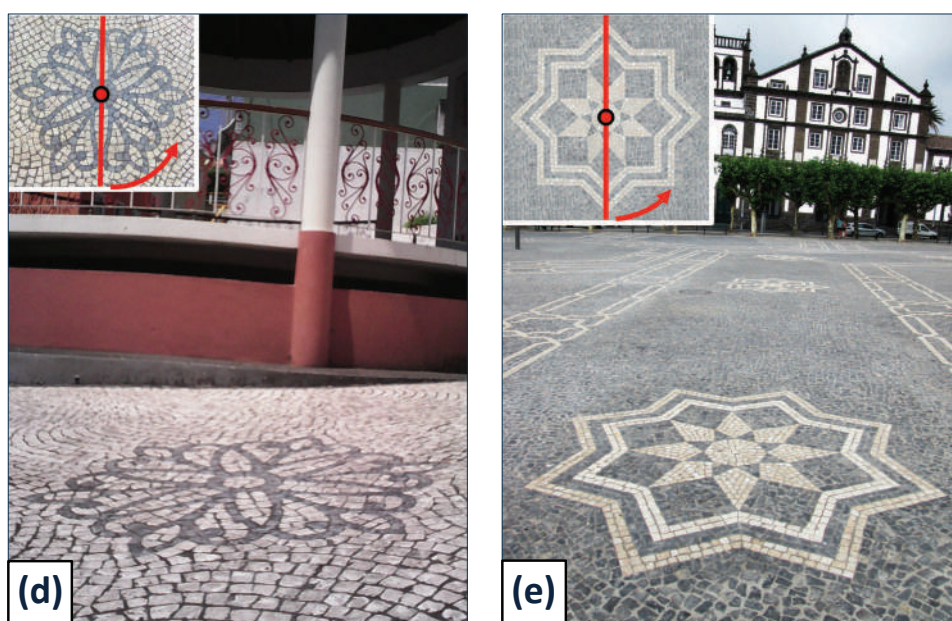

(Santa Maria)

(S. Miguel)

Figure 13: Symmetry groups: (a), (b) and (c) – C_1; (d) – C_2; (e) – C_8.

(Terceira)

(S. Jorge)   **(S. Miguel)**

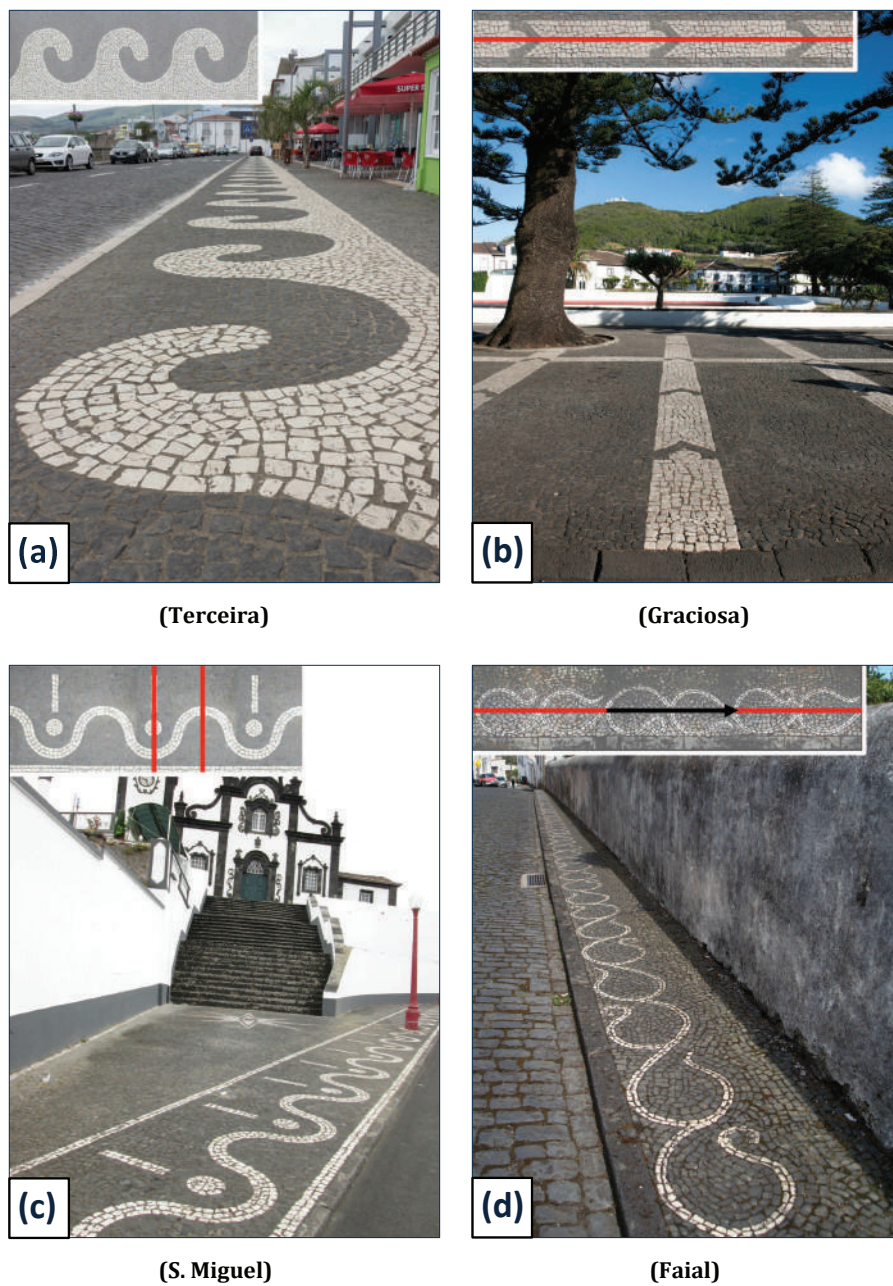Figure 14: Symmetry groups: (a), (b) and (c) – D_1; (d) – D_6; (e) – D_8.

Figure 15: Symmetry groups: (a) – F_1; (b) – F_1^1; (c) – F_1^2; (d) – F_1^3.

(Faial)

(S. Miguel)
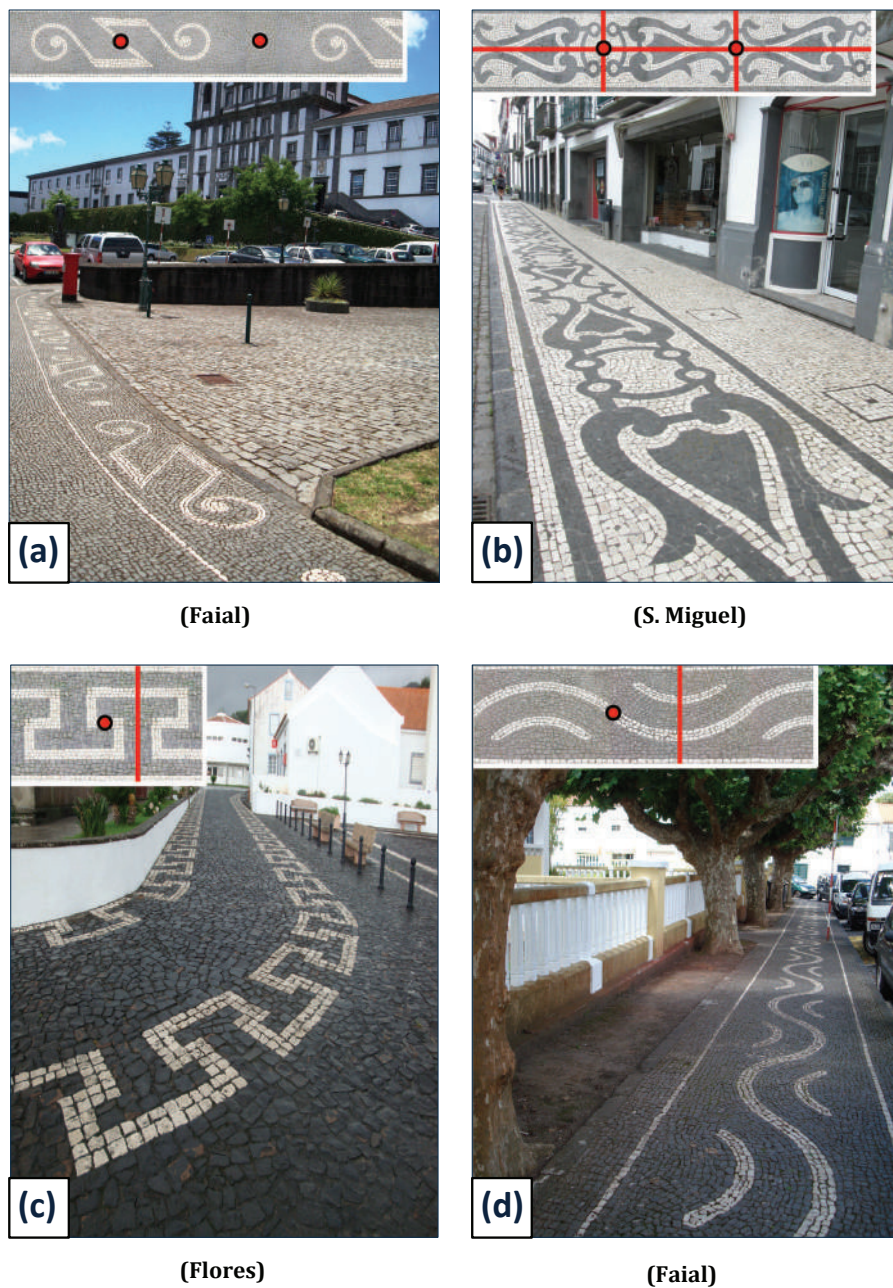
(Flores)

(Faial)

Figure 16: Symmetry groups: (a) – F_2; (b) – F_2^1; (c) – F_2^2; (d) – F_2^2.

References

[1] A. Andrade. *Bordados dos Açores*, Açores: Centro Regional de Apoio ao Artesanato, 2008.

[2] S. G. Costa, V. Moniz, R. C. Teixeira. "Symmetry Groups: Identifying Patterns in the Azorean Heritage", *Proceedings of Recreational Mathematics Colloquium III*, 93-104, Ponta Delgada, Portugal, 2013.

[3] G. Martin. *Transformation Geometry: An Introduction to Symmetry*, New York: Springer-Verlag, 1982.

[4] T. Perdigão, M. N. Amorim, A. Correia. *Rendas dos Açores – Ilhas do Pico e Faial*, Açores: Centro Regional de Apoio ao Artesanato, 2004.

[5] P. Stevens. *Handbook of Regular Patterns – An Introduction to Symmetry in Two Dimensions*, Cambridge: The MIT Press, 1999.

[6] R. C. Teixeira. "As simetrias das calçadas dos Açores", *In* A. P. Garrão, M. R. Dias, R. C. Teixeira (Coord.), *Investigar em Educação Matemática: diálogos e conjunções numa perspetiva interdisciplinar*, Ponta Delgada: Letras Lavadas, 2015.

[7] F. Tóth. *Regular Figures*, London: Pergamon, 1964.

[8] E. Veloso. *Simetria e Transformações Geométricas*, Lisboa: Associação de Professores de Matemática, 2012.

[9] D. Washburn, D. Crowe. *Symmetries of Culture – Theory and Practice of Plane Pattern Analysis*, Seattle: University of Washington Press, 1988.