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email: rmm@ludus-opuscula.org
URL: http://rmm.ludus-opuscula.org

Informations
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The magazine has the following sections (not mandatory in all issues):

Articles
Games and Puzzles
Problems
MathMagic
Mathematics and Arts
Math and Fun with Algorithms
Reviews
News

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Contents

Hat Puzzles

Tanya Khovanova
MIT's Research Science Institute
tanya@math.mit.edu

Abstract: *This paper serves as the announcement of my program – a joke version of the Langlands Program. In connection with this program, I discuss an old hat puzzle, introduce a new hat puzzle, and offer a puzzle for the reader.*

Key-words: hat problems, mathematical jokes.

1 Tanya's Program

This is one of my favorite jokes:

Three logicians walk into a bar. The waitress asks, "Do you all want beer?"
The first logician answers, "I do not know."
The second logician answers, "I do not know."
The third logician answers, "Yes."

This joke reminds me of hat puzzles. In the joke each logician knows whether or not s/he wants a beer, but doesn't know what the others want to drink. In hat puzzles logicians know the colors of the hats on others' heads, but not the color of their own hats.

Here is a hat puzzle where the logicians provide the same answers as in the beer joke. Three logicians wearing hats walk into a bar. They know that the hats were placed on their heads from the set of hats in Figure 1. The total number of available red hats was three, and the total number of available blue hats was two.

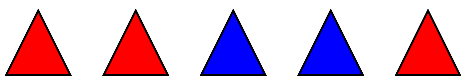

Figure 1: The set of available hats

Three logicians walk into a bar. The waitress asks, "Do you know the color of your own hat?"

The first logician answers, "I do not know."
The second logician answers, "I do not know."
The third logician answers, "Yes."

The question is: What is the color of the third logician's hat?

This process of converting jokes to puzzles reminds me of the Langlands Program, which tries to unite different parts of mathematics. I would like to unite jokes and puzzles. So here I announce my own program:

Tanya's Program. *Find a way to convert jokes into puzzles and puzzles into jokes.*

2 Old Hat Puzzle

Enough jokes. Let us move on to hat puzzles. The following is one of my favorite hat puzzles:

Some logicians are put to the test. They will have time to develop a strategy before the test. Here is the test:

The logicians stand in a line, one behind the other (as in Figure 2), so that the last person in line can see everyone else. Each wears a red or a blue hat and, as is usual in such puzzles, none of them knows the color of his or her own hat. Also, unlike in the puzzle above, there is no limit on the number of available hats of each color. The logicians can only see the colors of the hats on all of the people in front of them. Then, one at a time, according to the strategy they have agreed in advance, each logician guesses out loud the color of the hat on his or her own head. They are not permitted to give any other information: they can't even sneeze, fart or poke the person in front of them. They can only convey one bit of information: red or blue.

Their goal is to maximize the number of logicians who are guaranteed to guess correctly.

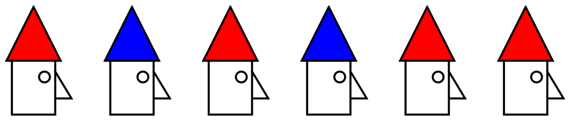

Figure 2: An old hat puzzle

I don't know the origins of this puzzle. It appeared in the 23rd All-Russian Mathematical Olympiad in 1997 as folklore. You can also find it in Peter Winkler's math puzzle book for gourmands [3] and in an article by Ezra Brown and James Tanton with a dozen hat puzzles [1].

2.1 Two Logicians

Let us start with two logicians in line as in Figure 3.

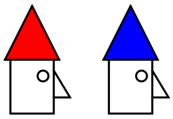

Figure 3: Two logicians

I teach a Math Competitions class at the Advanced Math and Sciences Academy Charter School. I gave my students this puzzle. In class we started with two logicians. It is impossible to guarantee that the last person can correctly guess the color of his or her hat, because no one sees it. The best we can hope to do is to guarantee that one person guesses the color correctly, namely the first logician in line. Most of my students solve this case very quickly. They realize that the last person in line should name the color of the hat in front of him/her, and the front logician should repeat this color.

2.2 Three Logicians

Three logicians (See Figure 4) give my students a moment's pause.

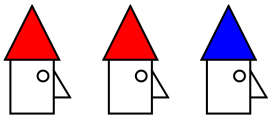

Figure 4: Three logicians

The students start by suggesting that the last person names either the color of the hat in front of him/her, or the color of the first person's hat. They soon realize that these strategies do not work. The last person needs to combine together all the colors s/he sees. Then they discover the solution to guarantee that all but the last logician guess correctly: the last person signals whether the two people in front have the same color or different color hats.

For example, the logicians can agree beforehand that "red" means the same color and "blue" means different colors. In the situation depicted in Figure 4, the last person sees different colors, so s/he says blue. The second to last logician now knows that the colors are different, and s/he sees blue, so she says "red." Now the first logician in line knows that his/her hat color is different from the middle person's color. Since the middle person said "red," the first person says "blue."

2.3 Many Logicians

The case with four logicians takes some more thinking. The students usually solve the four-logician case together with the any-number-of-logicians case (See Figure 5). Surprisingly, not more than one logician needs to be mistaken.

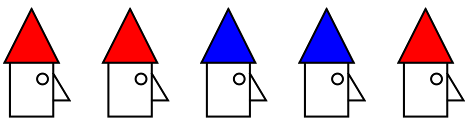

Figure 5: Many logicians

By this time the students understand that the last logician must start. S/he is the only one who can't have any information about his/her hat's color. So the last logician is doomed and uses this as an opportunity to pass information to everyone else. It is also clear that the logicians should state their colors from back to front. This way all logicians—except for the last one—are in the same situation. By the time the logicians have had to guess their own color, they know all the other colors: they see all the colors in front of them and have heard all the colors of the logicians behind them.

The solution: The last logician has to announce the parity of the number of red hats. For example, the last person says "Red" if the number of red hats s/he sees is even. After that the other logicians sum up the number of times they see or hear the word "Red," and say "Red" if this number is even and "Blue" if it is not.

I make my students re-enact this test. If more than one student is wrong, I chop off all of their heads (See Figure 6). Just kidding.

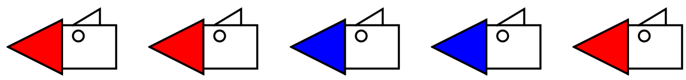

Figure 6: Heads chopped off

2.4 Many Colors

It is natural to generalize this problem to many colors. The thee-color version was suggested by Konstantin Knop for the 23rd All-Russian Mathematical Olympiad in 1997. You can also find the 100-color version in the above-mentioned paper with a dozen hat puzzles [1].

We have the same logicians standing in a line and now they might have a hat of any color—not just red or blue (See Figure 7). The important thing is that the set of hat colors is finite and needs to be known in advance.

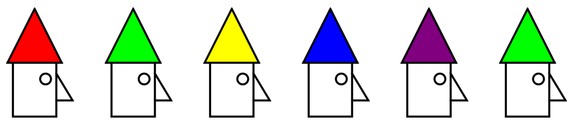

Figure 7: Many colors

It might sound surprising, but again the logicians can guarantee that not more than one person, namely the last person, is mistaken. The strategy is similar. Logicians replace colors by numbers from 0 to $N-1$, where N is the number of possible colors. The last person sums the colors modulo N and announces his/her own color so that the total sum is zero. As before, the other logicians state their colors from back to front. Each logician sums up all the numbers, that is colors, that s/he sees and hears. Then s/he calculates his/her own color by assuming that the total sum modulo N is zero.

3 New Hat Puzzle

This new variation is very recent and very beautiful. It was invented by Konstantin Knop and Alexander Shapovalov and appeared (with different wording) in March of 2013 at the Tournament of the Towns:

The same logicians stand in line, one behind the other, so that the last person in line sees everyone else. They were previously shown all the hats from which the hats on their heads have been chosen. Every available hat is a different color and there is one more hat than the number of logicians (see Figure 8). As before, in any order they agreed earlier, each logician guesses out loud the color of the hat on his/her own head, but is not permitted to signal anything else. No one can repeat a color that has already been announced. The logicians have time to design the strategy before the test and they need to maximize the number of people who are guaranteed to guess correctly. What should that strategy be?

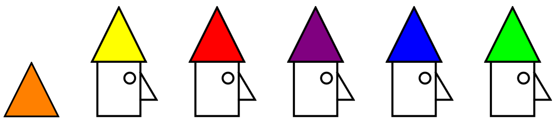

Figure 8: Distinct colors

It is natural to try to reuse the solution we know for the previous puzzle. Let us say that the number of hats is N and the number of logicians is $N-1$. Suppose the last person sums the colors modulo N and announces his/her own color so that the total sum is zero. The problem is that s/he might announce the color of one of the logicians in front of him/her.

Suppose the last logician says "Red." When it's time for the logician who is actually wearing a red hat to speak, s/he will realize that his/her hat is red, and oops, s/he can't repeat this color. If s/he says a different color randomly, s/he will throw off the strategy and mislead everyone in front.

Is there a way to rescue this strategy so that only a small number of logicians might be mistaken? It appears that there is. Suppose the last logician says "Red," and the logician in the red hat says "Blue." The blue color wasn't announced yet so the logician in the blue hat is most probably way in front of the red-hatted logician. That means that all the logicians between the red-hatted and the blue-hatted logicians will know that something wrong happened. They will not be misled. They will realize that the logician who said "Blue" said so because s/he were not allowed to say his/her own color. So that color must be red. That means all the logicians in between can still figure out their colors.

Here is the strategy where not more than three people will name a wrong color:

The last person announces the color so that his/her color and the sum of the colors s/he sees modulo N is zero. If there is a logician with this hat color, s/he says the hat color of the first person. This way the three people who might be mistaken are the last one, the logician with the hat color announced by the last person and the first person in line. Everyone else is guaranteed to be correct.

This is quite a good solution, but there is a completely different solution that guarantees that not more than **one** person is mistaken. I invite you to try and solve it yourself. If you can't do it, you can find the solution in my paper devoted to this puzzle [2].

4 A Puzzle for the Reader

Convert the above puzzles into jokes!

References

[1] Ezra Brown and James Tanton. "A Dozen Hat Problems", *Math Horizons*, 16 (8), 22–25, 2009.

[2] Tanya Khovanova. "A Line of Sages", to appear in *The Mathematical Intelligencer*, 3, 2014.

[3] Peter Winkler. *Mathematical Puzzles: A Connoisseur's Collection*, A.K. Peters, Natick, MA, 2004.

Camões, Pimenta and the Improbable Sonnet

Carlota Simões[*], *Nuno Coelho*[†]
Coimbra University
carlota@mat.uc.pt, ncoelho@dei.uc.pt

Abstract: *This article tells the story of a Renaissance sonnet gaining new life, new form and new meaning during the 20th Century.*

During the Eighties of last century, the poet Alberto Pimenta took the sonnet Transforma-se o amador na cousa amada *(The lover becomes the thing he loves) from Luís de Camões (16th Century), reorganized the letters of each verse of the poem and came up with a new sonnet,* Ousa a forma cantor! Mas se da namorada *(Dare the form, songster! But if the girlfriend). Who is the author of the second poem? We should say Pimenta, but, ironically, this author did not manage to organize a new verse from the last one of the original poem until he put aside the letters L and C, the initials of the author of the original sonnet. It seems that, in some mysterious and magical way, Luís de Camões came to reclaim the authorship of the second poem as well.*

Recently, the designer Nuno Coelho challenged his Design and Multimedia students at the University of Coimbra with a new project: to produce a multimedia transformation of the sonnet of Camões into Pimenta's, and a new breath was given both to poems and their authors.

Key-words: Luís de Camões, Alberto Pimenta, mathematics, literature, multimedia, OULIPO.

1 Introduction

To make a long story short, we may say that the poet Luís de Camões (1524 -1579) is for the Portuguese as important as Shakespeare is for the British. He is

[*]Department of Mathematics of the Faculty of Sciences and Technology and Center for Computational Physics, Coimbra University, Portugal.
[†]Department of Informatics Engineering of the Faculty of Sciences and Technology and Centre of 20th Century Interdisciplinary Studies, Coimbra University, Portugal.

most famous for his epic masterpiece, published for the first time in 1572, *Os Lusíadas* [1] (*The Lusiads* [2, 3]), but the subject of this article is his sonnet TRANSFORMA-SE O AMADOR NA COUSA AMADA [4].

During the Eighties of the 20th Century, the poet Alberto Pimenta (born in 1937) proposed himself a project: to create a new sonnet starting from TRANSFORMA-SE O AMADOR NA COUSA AMADA of Camões, using exactly the same letters, verse by verse, just by permutation of the characters in each verse of the original poem. What is the total number of permutations of this type? It seems a rather simple question to ask, just use combinatorial. But what is the probability of getting a new poem, with sense and literary value? And is there any solution at all? The fact is that Pimenta succeeded and the result is the poem OUSA A FORMA CANTOR! MAS SE DA NAMORADA. We may wonder how many more sonnets could we produce with this rule, but probably not much would make sense in any existing language.

Recently, during the second decade of the 21th Century, the designer Nuno Coelho challenged his Design and Multimedia students at the University of Coimbra with a new project: to produce a multimedia transformation of the sonnet of Camões into Pimenta's. There are some very interesting proposals, which we will show later in this text.

2 Combinatorial and poetry: Raymond Queneau and OULIPO

> *The work you are holding in your hands presents, itself alone, a quantity of text far greater than everything man has written since the invention of writing.*
>
> *François Le Lionnais, (postface to Cent Mille Milliards de Poèmes)*

The book by Raymond Queneau *Cent Mille Milliards de Poèmes* [6] (A Hundred Thousand Billion Poems or One Hundred Million Million Poems) was published in 1961. The book is a set of ten sonnets, each printed in a different card, each card cut in pieces such that each verse appears on a separated strip, which may be turned individually. It is possible for the reader to construct a sonnet using the first verse of any of the 10 sonnets, the same for the second verse, as well as for each of the 14 verses. In this way, the book contains not ten sonnets, but $10^{14} = 100\,000\,000\,000\,000$ = one hundred million million = one hundred thousand billion (or in French *cent mille milliards*) different sonnets [1].

This book was just one of many works produced by the group OULIPO - *Ouvroir de littérature potentielle* (workshop of potential literature), a gathering of writers and mathematicians who seek to create works with techniques that included self-imposed restrictions or, in OULIPO own words, "literature in unlimited quantities, potentially producible until the end of time, in huge quantities, infinite for all practical purposes". Indeed, if someone decided to read the book

[1]For an interactive version of this book, both in French and in English, see `http://www.bevrowe.info/Queneau/QueneauRandom_v4.html`

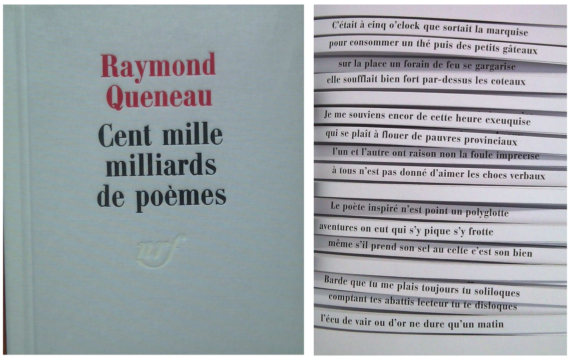

Figure 1: Raymond Queneau's *Cent Mille Milliards de Poèmes* and one of its 10^{14} sonnets.

taking one minute with each poem, he would spend 10^{14} minutes, which is about 190 million years, the amount of time from the beginning of Jurassic until today.

3 The Camões sonnet TRANSFORMA-SE O AMADOR NA COUSA AMADA

Many details concerning the life of Luís de Camões (1524-1579) remain unknown. His date and place of birth are not officially known, but the study of one of his sonnets, with the help of astronomical ephemerides for the second decade of the 16th century, allowed Mário Saa to conclude that Camões was born on January 23, 1524 [8].

Camões is most famous for his epic masterpiece, published for the first time in 1572, *Os Lusíadas*[2] [1] (*The Lusiads* [2, 3]), but this text deals with his sonnet TRANSFORMA-SE O AMADOR NA COUSA AMADA [4].

Figure 2: Luís de Camões (1524-1579)

[2]The second edition, also from 1572, is available here:
`http://purl.pt/1/4/cam-3-p_PDF/cam-3-p_PDF_24-C-R0150/cam-3-p_0000_capa-capa_t24-C-R0150.pdf`

Transforma-se o amador na cousa amada *Por virtude do muito imaginar;* *Não tenho logo mais que desejar,* *Pois em mim tenho a parte desejada.* *Se nela está minha alma transformada,* *Que mais deseja o corpo de alcançar?* *Em si somente pode descansar,* *Pois consigo tal alma está liada.* *Mas esta linda e pura semideia,* *Que, como o acidente em seu sujeito,* *Assim co'a alma minha se conforma,* *Está no pensamento como ideia;* *[E] o vivo e puro amor de que sou feito,* *Como matéria simples busca a forma.* Luís de Camões, 16th Century	*The lover becomes the thing he loves* *by virtue of much imagining;* *since what I long for is already in me,* *the act of longing should be enough.* *If my soul becomes the beloved,* *what more can my body long for?* *Only in itself will it find peace,* *since my body and soul are linked.* *But this pure, fair demigoddess,* *who with my soul is in accord* *like an accident with its subject,* *exists in my mind as a mere idea;* *the pure and living love I'm made of* *seeks, like simple matter, form.* Luís de Camões, 16th Century Translation: Richard Zenith [5]

We may now wonder how many different "sonnets" we obtain by permutation of the letters in each verse. For that, take the first verse, "Transforma-se o amador na cousa amada", which contains 31 letters. If all letters were different, there would be 31! different permutations. Since there are repeated characters, the total number of permutations will be smaller than that, but still rather large. Of the 31 letters, "A" appears 9 times, "O": 4 times, "M", "R", "S": 3 times, "D", "N": 2 times and "T", "F", "C", "E", "U" appear only once. This gives a total of $\frac{31!}{9!4!3!3!3!2!2!} = 1\,092\,782\,550\,735\,491\,616\,000\,000$ permutations, roughly 10^{24}, a much larger number than the complete amount of sonnets in Queneau's book, and we are still with the first verse. Tedious similar calculations lead to the following table.

Transforma-se o amador na cousa amada	$\frac{31!}{9!4!3!3!3!2!2!}$	1092782550735491616000000
Por virtude do muito imaginar;	$\frac{25!}{4!3!3!2!2!2!2!2!}$	561024668812608000000
Não tenho logo mais que desejar,	$\frac{26!}{4!4!3!2!2!}$	2917328277825561600000
Pois em mim tenho a parte desejada.	$\frac{28!}{5!4!3!2!2!2!2!2!2!}$	27568752225451557120000
Se nela está minha alma transformada,	$\frac{31!}{8!3!3!3!3!2!2!2!}$	1967008591323884908800000
Que mais deseja o corpo de alcançar?	$\frac{29!}{5!4!3!3!2!2!2!}$	1065991752717460208640000
Em si somente pode descansar,	$\frac{24!}{5!4!2!2!2!2!2!}$	673229602575129600 0
Pois consigo tal alma está liada.	$\frac{27!}{6!3!3!3!3!2!}$	583465655565112320000 0
Mas esta linda e pura semideia,	$\frac{25!}{5!4!3!3!2!2!}$	374016445875072000000
Que, como o acidente em seu sujeito,	$\frac{28!}{6!4!3!2!2!2!2!2!}$	918958407515051904000 00
Assim co'a alma minha se conforma,	$\frac{27!}{6!4!3!3!2!2!2!}$	218799620836917120000 0
Está no pensamento como ideia;	$\frac{25!}{4!4!3!3!2!2!2!2!}$	46752055734384000000 0
[E] o vivo e puro amor de que sou feito,	$\frac{28!}{6!5!3!2!2!2!}$	7351667260120415232000 0
Como matéria simples busca a forma	$\frac{29!}{5!4!3!3!2!2!2!2!}$	532995876358730104320000 0

A new "sonnet" can now be obtained by choosing, to each verse, one permutation of the original one. This means that the whole number of new sonnets is the product of all entries in the last column, giving the unbelievable value of $5,3 \times 10^{312}$. But the question remains: among all these possible new poems, is there any real sonnet, written in Portuguese?

4 Alberto Pimenta and the project METÁSTASE I

And what is an OULIPO *author? It is "a rat who built himself the maze from which it proposes to come out."*

OULIPO *group*

During the Eighties of the 20th Century, the poet Alberto Pimenta (1937-) proposed himself a challenge: to create a new sonnet starting from TRANSFORMA-SE O AMADOR NA COUSA AMADA of Camões, using exactly the same letters, verse by verse, just by permutation of the characters in each verse of the original poem. The total number of permutations of this type is indeed very large. But what is the probability of getting a new poem, with sense and literary value? And is there any solution at all? The fact is that Pimenta succeeded and the result is the poem OUSA A FORMA CANTOR! MAS SE DA NAMORADA. The project METÁSTASE I [7] is the answer to this challenge. Who is the author of the second poem? We should say Pimenta, but, ironically, this author did not manage to organize a new verse from the last one of the original poem until he put aside the letters "L" and "C", the initials of the author of the original sonnet. It seems that, in some mysterious and magical way, Luís de Camões came to reclaim the authorship of the second poem as well.

Alberto Pimenta is not a member of the OULIPO group, but by doing this, he built himself a maze to come out of it, just like a real OULIPO member.

Figure 3: Alberto Pimenta during the performance METÁSTASE I in Funchal, Madeira Island, Colloquium "Imaginário do Espaço", April 1987.

METÁSTASE I

Transforma-se o amador na cousa amada	*ousa a forma cantor! mas se da namorada*
Por virtude do muito imaginar;	*nua d'imagem, tido rio por vir tu*
Não tenho logo mais que desejar,	*não tens, oh rola joga, sem que ide*
Pois em mim tenho a parte desejada.	*ia e mente passem, admito, d'harpejo,*
Se nela está minha alma transformada,	*nada, terno mar, falsamente ilhas. amas*
Que mais deseja o corpo de alcançar?	*desejando amor que cale cio, parcas*
Em si somente pode descansar,	*rimas e os desencantos pedem*
Pois consigo tal alma está liada.	*odi et amo, caos, sigla. ali plantas*
Mas esta linda e pura semideia,	*setas em ideia, ainda mel, puras,*
Que, como o acidente em seu sujeito,	*semente que caído sujeito como eu,*
Assim co'a alma minha se conforma,	*a lama minha informa. com acessos*
Está no pensamento como ideia;	*te penso e cato o mínimo. se nada*
[E] o vivo e puro amor de que sou feito,	*muda, vê que frio e pó e riso e voto ou*
Como matéria simples busca a forma.	*ímpio amor mat'o ser e busca famas.*
Luís de Camões, 16th Century	L. C.
	(Alberto Pimenta, 1987)

The remaining challenge would be now to translate Pimenta's poem ending up with a permutation of the English version of the Camões sonnet.

5 Nuno Coelho and the multimedia version of METÁSTASE I

Recently, during the academic year 2011/2012, the designer Nuno Coelho challenged his undergraduate Design and Multimedia students at the Coimbra University with a new project: to produce a multimedia transformation of the sonnet of Camões into Pimenta's. There are some very interesting proposals.

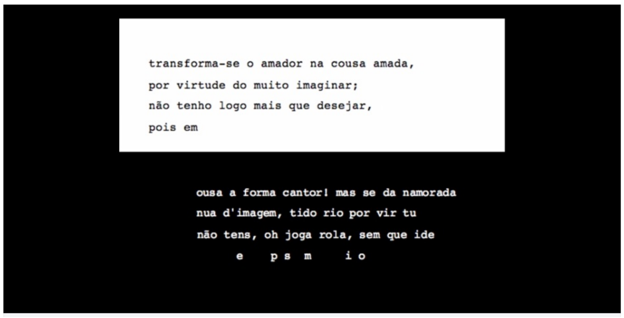

Figure 4: Joana Rodrigues's black and white screen.

Using a black and white screen, Joana Rodrigues types both poems simultaneously, with the sound of a typewriter, making clear the use of each character in both poems[3].

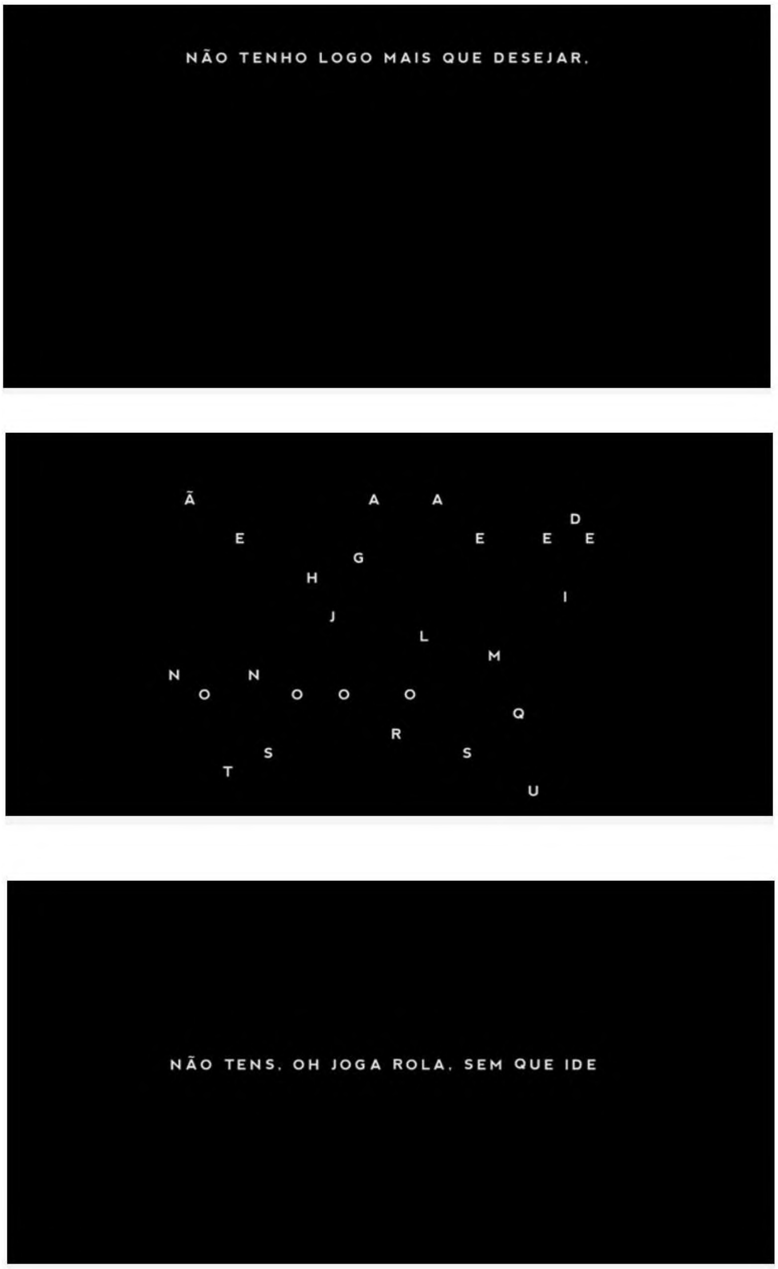

Figure 5: Bruno Santos's reorganization.

Bruno Santos picks each verse of the first poem and reorganizes it to the corresponding verse of the second poem, inspired on Jean Luc Godard's *Pierrot le*

[3]Joana Rodrigues: `http://vimeo.com/32167095`

Fou[4] opening titles[5].

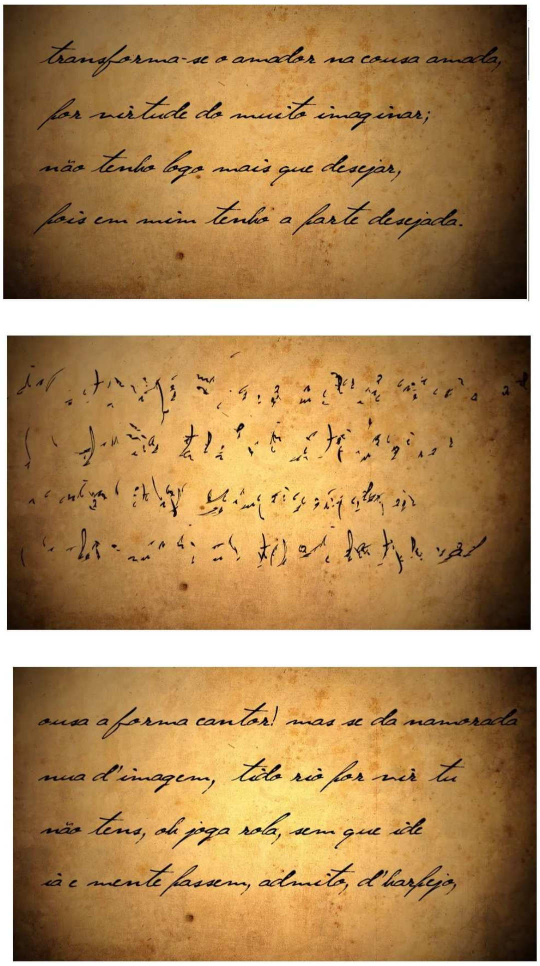

Figure 6: Mariana Seiça's proposal.

Mariana Seiça's proposal seems a palimpsest in which Pimenta's poem is revealed where we first saw the sonnet of Camões [6].

The interested reader may be interested as well in the works of Filipe Amaro [7], Ana Falé[8], João Oliveira [9] or Ernesto Cruz[10].

[4] *Pierrot le Fou* opening titles: `https://www.youtube.com/watch?v=-717PTVBCdw`
[5] Bruno Santos: `http://vimeo.com/35222462`
[6] Mariana Seiça: `http://vimeo.com/42437054`
[7] Filipe Amaro: `http://vimeo.com/32106514`
[8] Ana Falé: `http://vimeo.com/32207596`
[9] João Oliveira: `http://vimeo.com/32545705`
[10] Ernesto Cruz: `http://vimeo.com/32348937`

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Articles

Martin Gardner's Mathemagical Life

Tereza Bártlová
Charles University
bartlova@karlin.mff.cuni.cz

on the occasion of the 100th anniversary of birth

Abstract: *Martin Gardner has turned dozens of innocent youngsters into math professors and thousands of math professors into innocent youngsters.*

Persi Diaconis [32]

Key-words: Martin Gardner, 100th anniversary.

1 Introduction

There are a lot of people who devote a great deal of their lives to mathematics, but it is difficult to find those who have done more for the promotion of mathematics than Martin Gardner. Despite the fact that Gardner had not a formal mathematical education, his position in the world of mathematics is unique.

Being the author of the "Mathematical Games" column that ran for twenty-five years in Scientific American magazine, he opened the eyes of the general public to the beautiful and fascination of mathematics and inspired many to go on to make the subject their life's work. His column was the place where several important mathematical notions, such as Conway's Game of Life and Penrose tiles, first became widely known. It was also a place where the sheer fun of mathematical games and puzzles was celebrated and savored.

Allyn Jackson [26]

Martin Gardner had many lifelong passion and mathematics was one of them. In spite of, or perhaps because of, lacking proper mathematical education, Gardner's articles influenced generations of mathematicians. Thanks to his boundless enthusiasm and careful choice of topics, his articles got the general public interested in mathematics. Apart from mathematics he was an amateur magician, a well-known skeptic and also a leading figure in refuting pseudoscientific theories

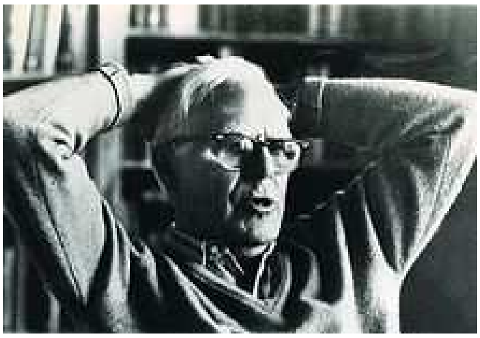

Figure 1: Martin Gardner.

ranging from modern diets to flying saucers. He showed great interest in religion, was a writer of fiction and poetry. He wrote more than 70 books concerning magic, philosophy, mathematics or commented on other authors' books.

2 Personal life and education

Martin Gardner was born October 21, 1914, in Tulsa, Oklahoma. His father was a geologist who owned a small oil company. His mother once worked as a kindergarten teacher, but after the delivery of her third child she decided to stay at home and became a housewife. From an early age Martin was fascinated by various puzzles, mathematical games, resolving paradoxes or mysterious stories. "My mother read the *Wizard of Oz* to me when I was a little boy, and I looked over her shoulder as she read it," he remembered from his childhood [2]. With his childhood friend John Bennett Shaw they collected different kinds of brain teasers. Later on, J. B. Shaw's extensive collection consisting of various brain teasers and Sherlock Holmes' mementos was awarded a prize[1].

Figure 2: Martin's brother Jim, Martin's father Dr. Gardner, and Martin.

[1]More detailed information about Gardner parents and his childhood is described in the article [2].

Martin Gardner loved physics in high school. He admired his physics teacher and his hopes were to become a physicist too. He applied for his dream job to Cal Tech[2]and found out that he was supposed to spend two years at university before he might be accepted. He wasn't discouraged and decided to study at the University of Chicago. During his studies Gardner got absolutely enthused by philosophy, especially philosophy of science. He chose to change his field of study and in 1936 obtained a bachelors degree in philosophy.

In his free time Gardner kept in touch with mathematics, unfortunately he couldn't take any maths courses at university because of his study plan. The main idea was to introduce the widest possible range of scientific disciplines to students in the first two years of their study. Optional subjects (maths in Gardner's case) could be chosen in the third year.

Gardner was always very keen on philosophy, but he already knew he would not make living as a philosopher [13]: "If you're a professional philosopher, there's no way to make any money except teach. It has no use anywhere." Gardner was sure he did not want to teach - partly this was due to his shyness but also because by then he knew really liked writing. He started occasionally to write some articles for various magazines but he didn't get paid. Then he worked shortly for the *Tulsa Tribune* as an assistant oil editor.

Before World War II he returned to Chicago to work in the public relation office of the *University of Chicago Office of Press Relation*, mainly writing science. During World War II he served for four years in the U. S. Navy. He spent about a year at Madison, Wisconsin, which bad a radio training school there [8]: "I handled public relations for the school, and edited a school newspaper," he described his work. The following three years he served as a yeoman on DE-134[3], a destroyed escort, in the Atlantic.

Figure 3: Martin Gardner in the Navy.

[2]Caltech or California Institute of Technology is private research university located in Pasadena, California, United States.

[3]USS Pope (DE-134) was an Edsall-class destroyer escort built for the U.S. Navy during World War II. She served in the Atlantic Ocean and provided destroyer escort protection against submarine and air attack for Navy wessels and convoys.

After World War II he went back to the University of Chicago, where he took some graduate courses and started to sell short stories to *Esquire Magazine.* His first story for magazine called "The Horse on the Escalator". It was a humorous and crazy story about a man who collected shaggy dog jokes[4] about horses. Shortly after his first story was published he wrote a second one named No-Sided Professor, in which Gardner explains the basis for topology with the help of the Möbius Strip. He made his living by writing for *Esquire Magazine* for about a year or two. Most stories are collected in the book called *The No-Sided Professor and Other Stories*[5].

In the early 1950s, Gardner moved to New York City and started to work for a children's magazine *Humpty Dumpty.* Not only did he write short stories and various columns, he was also in charge of inventing various paper-folding toys and cutouts for children. He was inspired by magazine called *John Martin's Book* where he found lots of interesting sources (activity features, where you cut things out of the page and fold them into different things, pictures that turn upsidedown, or you hold them up to the light and see through) [4]: "I grew up on this magazine." And these puzzles and folding paper toys had influence on his writing style for the magazine *Scientific American*, where he was employed from 1956 until 1981.

In 1979, Gardner moved with his wife Charlotte to Hendersonville in North Carolina. After her death in 2002, he decided to move to Norman, Oklahoma, where his son lived. In May 22, 2002 he died here.

Figure 4: Martin and with his wife.

[4] Shaggy dog joke is an extremely long-winded anecdote characterized by extensive narration of typically irrelevant incidents and terminated by an anticlimax or a pointless punchline.

[5] More detailed description we can find in the article [3].

3 Recreational mathematics and Scientific American

I dare say so, what Martin Gardner has done is of far greater originality than work that has won many people Nobel Prizes.

Douglas Hofstadter [22]

3.1 Recreational mathematics

Gardner always tried his best to give the general public insight into mathematics and mathematical research. He though that one of the reasons for unpopularity of mathematics lies in its isolation from the outside world [8]: "Well, that's a tough one, because almost all the really exciting research going on in mathematics is not the sort of thing that the public can understand. It takes considerable knowledge of mathematics to know what the breakthroughs are. And the really big breakthroughs that take place are just-it seems almost impossible to put it in terms that the general public can understand, whereas big breakthroughs in biology and so on are popularized, I think, fairly easily." In this connection he appreciated Sherman Stein's[6] job of popular articles on the subject, and books that the layman can read and understand. Gardner wished more similar mathematicians had written some popular articles and had introduced mathematical research to the public.

The method that Gardner chose to hook the interest in mathematics, was based on the attractiveness of the so-called *recreational mathematics* [8]: "...if they don't make mathematics to a certain degree fun to those first coming to it..." He believed that if mathematics teachers would use some problems from recreational mathematics in their lessons, the interest of their students could be attracted: "...the students are so bored that they get turned off by the topic, especially if the teachers are dull teachers." Gardner defined the term *recreational mathematics* in the very broad sense: "include anything that has a spirit of play about it".

Gardner was not only a "popularizer" of mathematics. American writer and cognitive scientists Douglas Hofstadter[7] considers Gardner's approach and his ways of combining ideas are truly unique and truly creative [22]: "In each column Martin managed to point out some little known but profound issue, and to present it in such a clear (and often humorous) fashion..."

[6]Sherman Stein (*1953) is a professor emeritus of mathematics at the University of California at Davis. He is the author of books *How the Other Half Thinks: Adventures in Mathematical Reasoning* (McGraw-Hill, 2002), *Strength in Numbers: Discovering the Joy and Power of Mathematics in Everyday Life* (Wiley, 1999) and *Mathematics: The Man-Made Universe* (Dover Publications, 1998) [49].

[7]Douglas Richard Hofstadter (*1945) is an American professor of cognitive scientists of wide interests. In his early career he focused on the logic, mathematics, computer science and other cognitive sciences. Later he got interested in interdisciplinary themes. He produced program in computer modeling of mental processes (he called "artificial intelligence research"). He was appointed adjunct professor of history and philosophy of science, philosophy, comparative literature, and psychology. He is best known for his book *Gödel, Escher, Bach: an Eternal Golden Braid*, first published in 1979. It won both the Pulitzer Prize for general non-fiction, in 1980 [16].

He put special emphasis also on the applications. He held the view that it is necessary to combine mathematics with its applications [8]: "...If the math can be applied somehow thats useful in the childs experience and things can be introduced so theyre challenging and have a play aspect..." With rapidly evolving technology and computers he saw success of popularization of mathematics in conjunction recreational maths with computer programming.

Martin Gardner didn't consider himself a "true mathematician". At the same time he believed that in this lies his advantage [8]: "If I can't understand what I'm writing about, why, my readers can't either." Maybe this is the reason why more people have probably learned more from him. He devoted a lifetime to work with mathematics and we could say that he kept himself busy with recreational mathematics in the U.S. for most of the twentieth century. However, he became really famous for his column entitled Mathematical Games in magazine *Scientific American*.

3.2 Scientific American and other recreational literature with mathematical topics

Everything began in 1952 when Gardner sent off an article on history of logic machines[8] to *Scientific American*. Editors of the magazine were so pleased with the article and they showed interest in an other contribution. So, in December 1956, Gardner published his article on hexaflexagons. A hexaflexagon is a paper model with hexapolygon shape, folded from straight strip of paper. One of the ways to construct such hexaflexagon is shown in the image below. Hexaflexagons have the fascinating property of changing their faces when they are "flexed". When we pinch two adjacent triangles together and push the opposite corner of the hexaflexagon toward to center, the model would open out again, a "budding flower" according to Gardner, and show a completely new face[9].

The article had overwhelmingly positive response from readers. A lot of people were folding hexaflexagon models, drew some motives on them and sent them back to the magazine or some of them were used in advertising. The magazine publisher Gerard Piel did not hesitate a minute to ask for a monthly column but also wanted to know if Martin thought there was enough material to warrant a monthly column. Although Gardner did not own any books on recreational mathematics, he knew that there are a big field out there [8]. His first column called *A new kind of magic square with remarkable properties*[10] appeared in the January 1957 issue and it was called Mathematical Games[11].

Since then the column appeared regularly every month exactly for a quarter of a century. It is remarkable that all this time Gardner was taking care of the articles on his own, as well as inventing new topics, gathering correspondence from his readers and answering to it. Gardner never wanted any assistants. He claimed that he had learned to type fast when being a yeoman in the Navy and

[8]Logic Machines 186, 68-73, Mar 1952

[9]A detailed description of the construction of hexaflexagon and its other curious property we can find, for example in the book [18].

[10]A new kind of magic square with remarkable properties Jan 1975 (169,1,-)

[11]Note that the initial letters of words in the title of the article are also Gardner's initials.

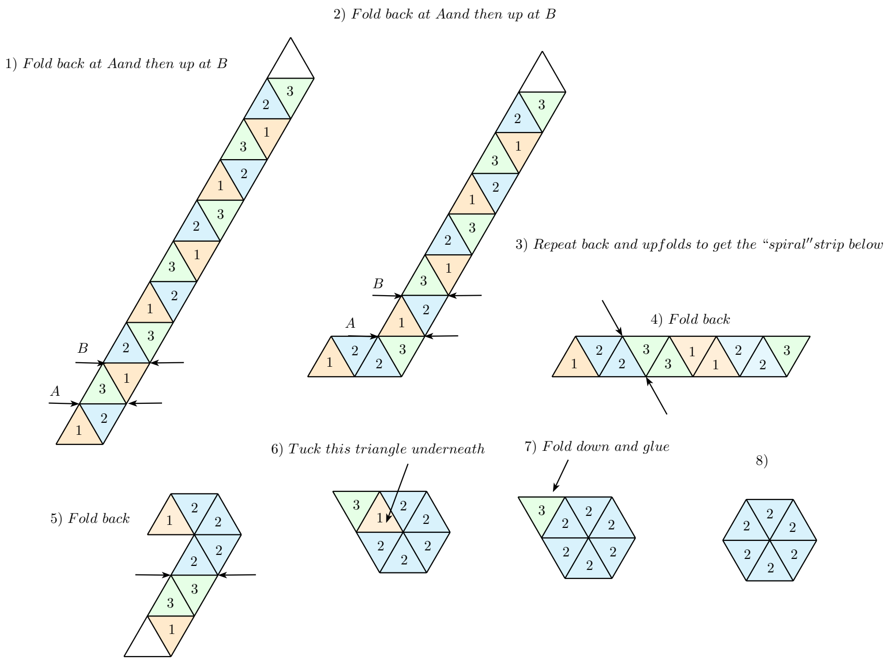

Figure 5: The construction of hexaflexagon

so that it was faster for him to type himself rather than to dictate anything. Only his wife Charlotte was allowed to help him. She proofread for him, checked the text for grammatical errors and spelling.

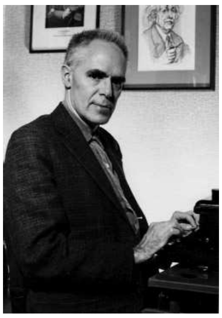

Figure 6: Martin Gardner during the writing

Gardner chose the topics of his column with great care [8]:"I try to pick a topic that is as different as possible from the last few topics; that's one of my criteria in choosing topics, so that I get a maximum variety from month to month," said Gardner. He kept gathering and piling possible future topics over the years. He drew inspiration from some books that came out in the recreational maths field, periodicals (he subscribed to about ten journals), and, of course, from a big correspondence with his readers who sent him ideas [8]: "Once the column became popular and the people interested in recreational math started reading it, why, they started writing to me. And then if I replied on my own stationery, why then they could write to me directly, and not have to go through Scientific American. So about half the correspondence I get comes through the magazine and about half I get directly." Among his column correspondents were several distinguished mathematicians and scientists as John Horton Conway[12], Persi Diaconis[13], Ron Graham[14], Douglas Hofstadter, Richard Guy[15], Donald Knuth[16], Sol Golomb[17] and Roger Penrose[18].

Gardner particularly enjoyed writing columns where philosophical things were interfering into mathematical issues and the other way round, for example a marvelous paradox called *Newcomb's paradox*. It is a thought experiment (a game) between two players, when one of them claims that he is able to predict the future [19]:

Two closed boxes, B1 and B2, are on a table. B1 contains $1 000. B2 contains either nothing or $1 million. You do not know which. You have an irrevocable choice between two actions:

1. Take what is in both boxes.

2. Take only what is in B2.

At some time before the test a superior Being has made a prediction about what

[12]John Horton Conway (*1937) is a British mathematician active in the theory of finite groups, knot theory, number theory, combinatorial game theory and coding theory. He has also contributed to many branches of recreational mathematics, notably the invention of the cellular automaton called the *Game of Life* [27].

[13]Persi Diaconis (*1945) is an American mathematician and former professional magician. He is the Mary V. Sunseri Professor of Statistics and Mathematics at Stanford University. He is particularly known for tackling mathematical problems involving randomness and randomization, such as coin flipping and shuffling playing cards [36].

[14]Ronald Graham (*1935) is an American mathematician credited by the *American Mathematical Society* as being "one of the principal architects of the rapid development worldwide of discrete mathematics in recent years". He has done important work in scheduling theory, computational geometry and Ramsey theory [41].

[15]Richard Guy (*1916) is a British mathematician, Professor Emeritus in the Department of Mathematics at the University of Calgary. He is best known for co-authorship of *Winning Ways for your Mathematical Plays* and authorship of *Unsolved Problems in Number Theory*, but he has also published over 100 papers and books covering combinatorial game theory, number theory and graph theory [1].

[16]Donald Knuth (*1938) is an American computer scientist, mathematician, and Professor Emeritus at Stanford University. He is the author of the multi-volume work *The Art of Computer Programming*. Knuth has been called the "father" of the analysis of algorithms [15].

[17]Solomon Golomb (*1932) is an American mathematician, engineer and a professor of electrical engineering at the University of Southern California. He is best known for his works on mathematical games. Most notably he invented *Cheskers* or *Polyominoes* and *Pentominoes*, which were the inspiration for the computer game *Tetris*. He has specialized in problems of combinatorial analysis, number theory, coding theory and communications [47].

[18]Roger Penrose (*1931) is an English mathematical physicist, recreational mathematician and philosopher. He is the Emeritus Rouse Ball Professor of Mathematics at the Mathematical Institute of the University of Oxford, as well as an Emeritus Fellow of Wadham College. He is best known for his scientific work in mathematical physics, in particular for his contributions to general relativity and cosmology [40].

you will decide. If the Being expects you to choose both boxes, he has left B2 empty. If he expects you to take only B2, he has put $1 million in it. If he expects you to randomize your choice by, say, flipping a coin, he has left B2 empty. In all cases B1 contains $1 000. What should you do?

Due to the popularity of Gardner's columns many of these articles have been collected in a book *The Scientific American Book of Mathematical Puzzles and Diversions*, published in 1959. Over the next forty years, he published another fourteen books[19].

In 1981, Gardner handed his coulmn over to Douglas Hofstadter. Gardner had so many other writing interests in these days that he felt he could no longer maintain the column. Hofstadter was looking up to Gardner. On one hand he feared being put in Martin Gardner's shoes, but on the other hand he understood if he hadn't taken the chance he would have regretted it later. He didn't want the readers to expect him to copy Gardner's style so he decided to rename the column. So Metamagical Themas (anagram of the earlier title) came to existence.

Hofstadter managed to publish his column regularly for almost three years. But in 1983 he was swamped with work to such an extent, it became clear that he would be unable to continue producing columns at a monthly pace. And so a Canadian mathematician Kee Dewdney[20] took over Hofstadter. The change of the author meant the change of title again. So this time Computer Recreations first saw the light.

In September 1987, a Scottish mathematician Ian Stewart[21] got the opportunity to contribute to this column. Stewart also gratefully accepted the chance of writing articles for *Scientific American*. Although he had never met Martin Gardner, Stewart admitted that was a regular and faithful reader of Gardner's column, since he was sixteen years old [48]: "Every column contained something new to attract my attention, and it was mathematical, and it was fun. There was plenty of room for new ideas and creative thinking. It is probably fair to say that Martin Gardners column was one of the reasons I ended up becoming a mathematician."

In December 1990, there was another change of the column's title. It was renamed the Mathematical Recreations and a few months later Ian Stewart became officially its author. Stewart had problem with choice of the topics. He thought that Martin Gardner had already used loads of interesting themes. He identified himself with Gardner's point of view [43]: "The way to explain math to nonspecialists is to understand it thoroughly yourself, to strip away needless technicalities, and to focus on the central story." And exactly this principle he tried to follow.

[19]Most Martin Gardner bibliography we can find on the Web side [31].

[20]Alexander Keewatin Dewdney (*1941) is a Canadian mathematician, computer scientist and author who has written a number of books on mathematics, computing, and bad science [6].

[21]Ian Nicholas Stewart (*1945) is a professor of mathematics at the University of Warwick, England, and a widely known popular-science and science-fiction writer [24].

The last one to finish the famous columns was an American mathematician Dennis Shasha[22]. He started to work on the column, which changed its name for the last time in early 2001. The Puzzling Adventures columns were published in print until May 2004, and since the following month the columns were accessible only on the website of the magazine. The very last article was published in June 2009, and then column came to an end.

From 1977 until 1986 Gardner also was contributive to the magazine *Asimov's Science Fiction*. His column was focused primarily on "puzzle tale".

Retirement did not stop Gardner from working. He only focused more on writing scientific literature and updating his older books such as *Origami, Eleuis and the Soma Cube*.

3.3 Gathering for Gardner

Despite the fact that Gardner was very popular among people, he was known for his shy personality. He refused to receive several awards just because he would have to take part in the public ceremony [17]: "I hate going to parties or giving speeches. I love monotony. Nothing pleases me more than to be alone in a room, reading a book or hitting typewriter keys." Once he told Colm Mulcahy [30]: "...I have never given any lecture in my life and most probably I wouldn't know how to do it."

However, in 1993, Atlanta puzzle collector Tom Rodgers persuaded Gardner to attend a special evening occasion devoted to Gardner's puzzle-solving efforts. The event met with roaring success and was repeated in 1996, again with Gardner's presence. No wonder that Rodgers and his friends decided to organize the gathering on regular basis. Since then it has been held every other year (even-numbered) in Atlanta, and the programme consists of any topic which is concerning Gardner and his writing career in any way. The event is named *Gathering for Gardner*, in short G4Gn, when n stands for the number of the event (the 2010 event thus was G4G9) [23]. Gardner attended the 1993 and 1996 events.

4 Pseudoscience

Even when a pseudoscientific theory is completely worthless there is a certain educational value in refuting it.

Martin Gardner [17]

Despite his introverted nature Gardner was considered to be one of the leading polemics against pseudoscientific and fringescientific theories, astounding

[22]Dennis Shasha is a professor of computer science at the Courant Institute of Mathematical Sciences, a division of New York University. He does research in biological computing (including experimental design), pattern recognition and querying in trees and graphs, pattern discovery in time series, cryptographic file systems, database tuning, and wireless [14].

[23]Detailed event program we can find, for example in the article [12].

discoveries, the paranormal and everything what became later known as *pseudoscience.*

In his articles, he tried to put all these misleading and confusing information appearing in the media straight. He was irritated by boundless human gullibility. He warned scientists that at the time they do not write any popular articles attacking pseudoscience and do not acquaint the general public with scientific discoveries, there is space for pseudoscientists to popularize their dubious discoveries and inventions and it may easily happen that the general public would consider pseudoscience a real science. He believed that if he explained everything rationally, he would be able to influence people's opinions and also mitigate the damage caused by pseudoscientists. "Bad science contributes to the steady dumbing down of our nation", declared Gardner [17].

For many years he tirelessly researched and studied different pseudoinventions and pseudofacts from a scientific view point, and wrote various articles concerning these topics. The first article which had the scent of distrustful spirit and reacted negatively to the results of pseudoscience was called The Hermit Scientist and was published in 1950 in the journal *Antioch Review.* This article wasn't definitely the last one and only two years later he published his first book dealing with these issues entitled *In the Name of Science.* It was a skeptical book by its nature - it explored myriads of dubious outlooks and projects including modern diet, fletcherism[24], creationism[25], Charles Fort[26], Rudolf Steiner[27], scientology[28], dianetics[29], UFOs, dowsing[30], extra-sensory perception[31] and psychokinesis[32]. Not only this book but many others, for example *Science: Good,*

[24]Fletcherism is a kind of special diets named after Horace Fletcher (1849–1919). The basic tenets of diet were these: one should eat only when genuinely hungry and never when anxious, depressed, or otherwise preoccupied; one may eat any food that appeals to the appetite; one should chew each mouthful of food 32 times or, ideally, until the food liquefies (see [23]).

[25]the belief that the universe and living organisms originate from specific acts of divine creation, as in the biblical account, rather than by natural processes such as evolution [28].

[26]Charles Fort (1874–1932) was an American writer and researcher into anomalous phenomena [10].

[27]Rudolf Joseph Lorenz Steiner (1861–1925) was an Austrian philosopher, social reformer, architect, and esotericist. Steiner gained initial recognition as a literary critic and cultural philosopher. At the beginning of the twentieth century, he founded a spiritual movement, anthroposophy (see [42]). Anthroposophy is a human oriented spiritual philosophy that reflects and speaks to the basic deep spiritual questions of humanity, to our basic artistic needs, to the need to relate to the world out of a scientific attitude of mind, and to the need to develop a relation to the world in complete freedom and based on completely individual judgments and decisions [7].

[28]Scientology a religious system based on the seeking of self-knowledge and spiritual fulfilment through graded courses of study and training. It was founded by American science fiction writer L. Ron Hubbard (191186) in 1955 (see [44]).

[29]Dianetics is a system developed by the founder of the Church of Scientology, L. Ron Hubbard, which aims to relieve psychosomatic disorder by cleansing the mind of harmful mental images. [45].

[30]Dowsing a technique for searching for underground water, minerals, ley lines, or anything invisible, by observing the motion of a pointer (traditionally a forked stick, now often paired bent wires) or the changes in direction of a pendulum, supposedly in response to unseen influences [38].

[31]Extrasensory perception (ESP) involves reception of information not gained through the recognized physical senses but sensed with the mind [34].

[32]Psychokinesis or telekinesis is an alleged psychic ability allowing a person to influence a physical system without physical interaction [39].

Bad and Bogus, (1981); *Order and Surprise*, (1983), *Gardner's Whys & Wherefores*, (1989), caused that number of fierce opponents and critics arose in the fields of fringe science and New Age philosophy, with many of them he kept up in touch (both publicly and privately) for decades.

Another reason for Gardner's uncompromising attitude towards pseudoscience it was its impact on the real science. It often happened that pseudoscientists used some serious scientific discovery, which they interpreted erroneously and applied it as the basis for their pseudoresearch. In the worst case, it became the very opposite - scientists did not recognize a pseudodiscovery was not based on the real facts and they took it seriously. In many cases, they made fools out of themselves.

Gardner wanted to prevent these situations, and so in 1976, he was a founding member of the *Committee for the Scientific Investigation of Claims of the Paranormal*, in short CSICOP[33]. It is to serve as a sort of neutral observer that examines various psychic phenomena from a scientific point of view. From 1983 until 2002 Gardner wrote a column called Notes of a Fringe Watcher (originally Notes of a Psi-Watcher) in magazine *Skeptical Inquirer* (originally *Zeletic*). All the articles were later collectively published in several books. Especially in his old age, Gardner was an excellent sceptic about paranormal phenomena. In August, 2010, Gardner´s contributions in the skeptical field earned him, in memoriam, an award rom the *Independent Investigations Group* on its 10th Gala Anniversary.

5 Religion

During his life Gardner found lifelong fascination for religion. As a youngster he was influenced by a Sunday school teacher and the Seventh-day Adventist Church. Young Gardner became convinced that the Second Coming of Jesus was close [52]:

"I grew up believing that the Bible was a revelation straight from God," he recounted. He had lived in this belief before he began studying at university and met some other points of view on Christianity and religion. University life and some ideas of authors whose books Gardner read slowly weakened his fundamental beliefs. Among these authors belonged e.g. Platon[34], Immanuel Kant[35],

[33]The mission of the Committee for Skeptical Inquiry is to promote scientific inquiry, critical investigation, and the use of reason in examining controversial and extraordinary claims. More detailed information we can find on the Web site www.csicop.org.

[34]Platon (427 BC – 347 BC) was a philosopher in Classical Greece. He is one of the most important founding figures in Western philosophy. Plato's sophistication as a writer is evident in his Socratic dialogues, his dialogues have been used to teach a range of subjects, including philosophy, logic, ethics, rhetoric, religion and mathematics [37].

[35]Immanuel Kant (1724–1804) was a German philosopher who is widely considered to be a central figure of modern philosophy. He argued that human concepts and categories structure our view of the world and its laws, and that reason is the source of morality [25].

Gilbert Keith Chesterton[36], William James[37], Charles Sanders Pierce[38], Rudolf Carnap[39] and Herbert George Wells[40]. Gardner tried to catch pearls of wisdom from every single one of them [52]: "From Chesterton I got a sense of mystery in the universe...", he explained. "From Wells I took his tremendous interest in and respect for science. That's why I do not accept the virgin birth of Christ or a blood atonement for the sin of Adam and Eve." Gardner was also inspired by the theology of a Spanish philosopher Miguel de Unamuno[41] According to Unamuno belief in God and the desire for immortality were as important as any scientific and rational view of the world. He claimed that one feels the need for faith in God and at the same time he yearns for recognition of his personality as an individual [33]: "the most tragic problem of philosophy is to combine the intellectual with the emotional needs, and also with free will."

With highest respect to all religious convictions Gardner described his own belief as *philosophical theism* [9]: "I am a philosophical theist. I believe in a personal god, and I believe in an afterlife, and I believe in prayer, but I dont believe in any established religion."

Gardner professed his faith in God as creator, but criticized and rejected everything which was beyond human understanding: God's revelation, prophecy, miracles, the authority of the Church. Despite all the criticism of the Church he believed in God and asserted that this belief cannot be confirmed or denied by science. At the same time, he was sceptical about claims that God has communicated with human beings through spoken or telepathic revelation or through miracles in the natural world [30]: "There is nothing supernatural, and nothing in human reason or visible in the world to compel people to believe in any gods. The mystery of existence is enchanting, but a belief in *The Old One* comes from faith without evidence. However, with faith and prayer people can find greater happiness than without."

Gardner often compared parapsychology with religion in his comments, and

[36] Gilbert Keith Chesterton (1874–1936) was an English writer,lay theologian, poet, dramatist, journalist, orator, literary and art critic, biographer, and Christian apologist [20].

[37] William James (1842–1910) was an American philosopher and psychologist who was also trained as a physician. James was one of the leading thinkers of the late nineteenth century and is believed by many to be one of the most influential philosophers the United States has ever produced, while others have labelled him the "Father of American psychology". He is considered to be one of the greatest figures associated with the philosophical school known as pragmatism [51].

[38] Charles Sanders Peirce (1839–1914) was an American philosopher, logician, mathematician, and scientist, sometimes known as "the father of pragmatism". He is appreciated largely for his contributions to logic, mathematics, philosophy, scientific methodology, and semiotics, and for his founding of pragmatism [11].

[39] Rudolf Carnap (1891–1970) was a German philosopher, mathematician and logician. He made significant contributions to philosophy of science, philosophy of language, the theory of probability, inductive logic and modal logic [35].

[40] Herbert George Wells (1866–1946) was an English writer, now best known for his work in the science fiction genre (see [21]).

[41] Miguel de Unamuno (1864–1936) is a Spanish writer, philosopher and one of the main leaders of the Group Generation 98. The central theme of his essays and poetry is faith. He touched topics such as finding personal spirituality, mental anguish, time, death, pain caused by confidentiality God and others. Second quote is from his the most famous book *Del sentimiento trágico de la vida*, published in 1913. Other information we can find, for example on the Web side [33].

claimed that he considered parapsychology and other researches on paranormal phenomena completely the same as "God temptation" and "looking for signs and wonders".

His attitude towards religion is best explained and described in his novel with autobilgraphical features *The Flight of Peter Fromm*[42] of 1973. This novel is not purely autobiographical, because Gardner does not identify himself with Peter, the main character. Nevertheless, the main character goes through the same changes of his own faith as Gardner which makes the book partly autobiographical.

6 Magic

Apart from brain teasers and puzzles Gardner expressed his interest in magic from his early age. Magic was hobby of his father who showed him some magic tricks (see [2]): "I learned my first tricks from him, in particular one with knife and little pieces of paper on it..." Gardner never made a living by magic. He only got paid once for doing magic at the occasion of presenting Gilbert´s magic set at the Marshall Field department store[43].

It is no surprise that Gardner prefered tricks with a touch of maths, particularly those that are breaking topological laws [2]: "The most important thing is to startle people, and have them wonder how it's done." Close-up magic is very different from the stage illusion that David Copperfield does. In close-up magic or micromagic hold true that "hands must be always quicker than eye".

It is not surprising that Gardner prefered tricks with a mathematical flavor and especially those that they are violating topological laws [2]: "In recent years magicians have gotten interested in rubben band tricks that are all topologically based... I did a book for Dover Publications on mathematical tricks that has a chapter on topological tricks."

Gardner wrote two voluminous books for magicians: *The Encyclopedia of Impromptu Magic* and *Martin Gardner Presents*. Both books have about five hundred pages where original tricks with cards, matches, dices, coins or mental magic tricks can be found.

[42] Character of novel is better explained in the article [8] and [3].

[43] Alfred Carlton Gilbert (1884–1961) was an American scientist, inventor, illusionist, athlete and scholar. Gilbert turned to magic by his studies at Yale. He began performing illusions on street corners and in shop windows. At these performances, Gilbert sold tricks and magic kits to his audience. He developed a local following and the Mysto Manufacturing Company became interested in publishing his magic toys. In 1990, Gilbert collaborated with the New Haven-based company to produce fis first toy, the Mysto Magic Set [46].

7 Literature

Gardner was considered a leading expert on Lewise Carroll[44]. They both shared love for mathematics, puzzles, formal logic and conjuring. Carroll was delighted to do simple magic tricks for his little audience and often took children to magic performances or wrote books for children. Among his best-known books belong *Alice's Adventures in Wonderland* and *Through the Looking Glass.* Although it might seem that it is just a fairy tale, in fact, both Alice books are full of logical and mathematical tricks and wordplays. Gardner admitted that he appreciated the depth of Carroll's stories when he was a grown-up [5]: "I did not discover the richness of this kind of humor in the Alice books until I was in my twenties, but since then I have felt a close kinship with Carroll."

In 1960, Martin Gardner published his annotated version of both Alice stories. Garnder revealed and explained all the mathematical riddles, wordplays, and literary references hidden in the Alice books. Later Gardner published a follow-up book with new annotations called *More Annotated Alice* and in 1999, the last edition combining the notes from earlier editions and new pieces of knowledge *The Annotated Alice: The Definitive Edition* was released.

Over the years Gardner's annotated Alice book has become a best seller [8]: "I was lucky there in that I really didn't have anything new to say much in The Annotated Alice, because I just looked over the literature and pulled together everything in the form of footnotes in the book. But it was a lucky idea because thats been the best seller of all my books." In following years more editions of the book appeared and it was translated into many languages.

Figure 7: Martin Gardner with Alice in Central Park

[44]Charles Lutwidge Dodgson (1832–1898), better known by his pen name, Lewis Carroll, was an English writer, mathematician, logician, Anglican deacon and photographer. His most famous writing are *Alice's Adventures in Wonderland* and its sequel *Through the Looking Glass and What Alice Found There*), as well as the poems "The Hunting of the Snark", all examples of the genre of literary nonsense [29].

From childhood Gardner fell in love with the books by Lyman Frank Baum about *The Wizard of Oz.* He wrote several forewords in additional issues of Baum's books and in 1998 Gardner published his own book called *Visitors From Oz.* Although Gardner's Visitors from Oz is an imitation of Baum's Wizzard of Oz book, again, he added some mathematics into it. Gardner made use of Klein's bottle, which appears throughout the story, as a magical feature for transition between parallel worlds[45].

Apart from Alice books Gardner published annotated edition of the books by Gilbert Keith Chesterton *The Innocence Of Father Brown* and *The Man Who Was Thursday.* He also commented on famous poems e.g. *The Rime of the Ancient Mariner, Casey at the Bat*, *The Night Before Christmas* and *The Hunting of the Snark* too.

Over the years he was expressing his concerns with many present-day problems. In 1993 he described his philosophical opinions and attitudes in his book *The Whys of a Philosophical Scrivener* [5]: "It is my favorite because it is a detailed account of everything I believe... Well, the book is controversial because almost everybody who believes in a personal god is into an established religion." Later Gardner harshly panned his own book in a review written under the pseudonym George Groth for *New York Review of Book* [52]: "I heard that people read the review and didn't buy the book on my recommendation."

After Martin Gardner's death his autobiographical book under title *Undiluted Hocus-Pocus* was released. The book was meant as a present for his fans. There are no dramatic revelations to be found, it only summarizes the story of his life, ideas and beliefs.

8 Closing

Martin Gardner was a man of wide interests. He was passionate about all types of paradox and revealing secrets. His columns and writings are unique considering constant novelty of human thoughts. He managed to get freed from expected patterns of thoughts, broke seemingy-solid laws, and discovered unexpected connections and revelations.

He died at the age of 95 and there is no doubt that he attracted many people of all ages to recreational mathematics during his lifetime. His infectious enthusiasm and brilliant choice of topics are unrivalled. There were plenty of those who tried to emulate him, but nobody has succeeded. American mathematics Douglas Hofstadter paid Martin Gardner a compliment saying [13]: "... is one of the great intellects produced in this country in the 20th century".

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A Very Mathematical Card Trick

Carlos Santos, Jorge Nuno Silva, Pedro Duarte
CELC, University of Lisbon, University of Lisbon
cmfsantos@fc.ul.pt, jnsilva@cal.berkeley.edu, pduarte@ptmat.fc.ul.pt

Abstract: *Andy Liu proposed a very interesting card trick whose explanation is based in the well-known Hamming codes. In this work, a performance for the same trick based in* NIM *mental calculation, is presented. A similar idea is also useful to analyze the game* TOE TACK TRICK *proposed by Colm Mulcahy.*

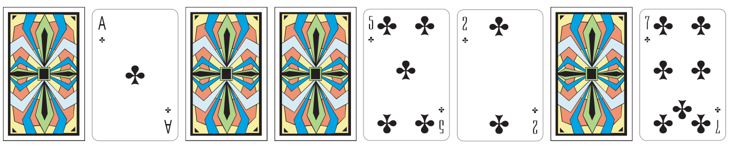
Key-words: mathematical card tricks, Hamming codes, NIM.

1 Introduction

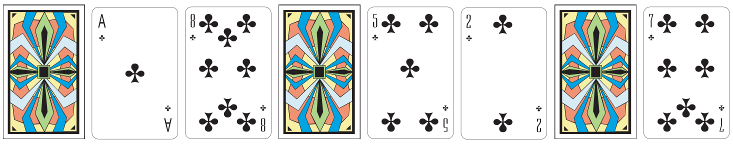
We start with the principal purpose of this paper: a magical card trick. It uses a small set of cards: ace through eight of clubs. The audience chooses one of them, giving the information about the choice to a magician's helper. Then, one volunteer of the audience shuffles the eight card deck, places the cards in a row, arbitrarily deciding which should be turned up. The helper does not do anything and the magician is not in the room.

Following, the helper turns **exactly** one card. After all this, the magician, who does not know what happened, enters the room and, looking at the cards, determines the card chosen by the audience.

Consider the following example. The audience chooses the deuce and leaves the following setup:



Following, the helper turns the third position leaving the following row:



To finish the trick, the magician enters in the room and shouts "deuce of clubs!".

2 Hamming Codes and the Liu's Card Trick

In [3], it is shown how the trick is conceived using Hamming codes. A Hamming code is a linear error-correcting code to detect single-bit errors. Exemplifying, consider a 8-bit word $a_0a_1a_2a_3a_4a_5a_6a_7$. The codification of the message includes 4 more digits (test-bits t_i). The t_i occupy the positions 1, 2, 4 and 8 (powers of 2). The values t_i are test-bits chosen by solving the following 4 equations:

$$\begin{aligned}
t_0 + a_0 + a_1 + a_3 + a_4 + a_6 &\equiv 0 \pmod 2 \quad (1)\\
t_1 + a_0 + a_2 + a_3 + a_5 + a_6 &\equiv 0 \pmod 2 \quad (2)\\
t_2 + a_1 + a_2 + a_3 + a_7 &\equiv 0 \pmod 2 \quad (3)\\
t_3 + a_4 + a_5 + a_6 + a_7 &\equiv 0 \pmod 2 \quad (4)
\end{aligned}$$

It is usual to organize the following chart related to the encoding process:

Bit Position	1	2	3	4	5	6	7	8	9	10	11	12
Encoded Bits	t_0	t_1	a_0	t_2	a_1	a_2	a_3	t_3	a_4	a_5	a_6	a_7
Equation 1	×		×		×		×		×		×	
Equation 2		×	×			×	×			×	×	
Equation 3				×	×	×	×					×
Equation 4								×	×	×	×	×

To understand the concept behind the codification, we note that the 4 equations can fail by 15 distinct ways (15 of the 16 subsets of a set with 4 elements). For instance, the simultaneous failure of the 1st and the 4th is one of these possible failures. The codification is done in such way that every single-bit error is related to exactly one of the possible failures. From code theory, it is known that the rules for the equations can be listed like this:

Eq 1: *skip* 0, *check* 1, *skip* 1, *check* 1, *skip* 1, ... → *positions* $1, 3, 5, 7, 9, 11, 13, 15, \ldots$

Eq 2: *skip* 1, *check* 2, *skip* 2, *check* 2, *skip* 2, ... → *positions* $2, 3, 6, 7, 10, 11, 14, 15, \ldots$

Eq 3: *skip* 3, *check* 4, *skip* 4, *check* 4, *skip* 4, ... → *positions* $4, 5, 6, 7, 12, 13, 14, 15, \ldots$

Eq 4: *skip* 7, *check* 8, *skip* 8, *check* 8, *skip* 8, ... → *positions* 8–15, 24–31, 40–47, ...

(...)

Eq k: *skip* $2^k - 1$, *check* 2^k, *skip* 2^k, *check* 2^k, *skip* $2^k, \ldots$

There is a unique bit coverage. For example, the bit responsible for the failure of the equations 1 and 4 is the 9th bit (a_4). The receptor of the message just has to check the congruences (1), (2), (3), and (4) to determine the bits with error.

Consider the message 10010111. The equations (1), (2), (3), and (4) produce the Hamming code 101000110111. Imagine a single-bit error and the sent message 101001110111 (with an error in the 6th position). The receptor calculates the congruences (1), (2), (3), and (4) and sees that (1) and (4) hold while (2) and (3) fail. The error occurs in the 2nd and 3rd equations so, by table inspection, the error-bit is in 6th bit. The detection of the error position can be made by visual inspection of a table. There are 2^k subsets of a finite set with cardinality k, so, if the message length (k) is such that $2^j \leqslant k < 2^{j+1}$ then the encoded message needs $j+1$ test-bits.

There are several card tricks based in the Hamming codes (see the chapter "Hamming It Down" of [4]). Andy Liu's idea is *to prepare the magician's reception*. Card's backs act like 1s and card's faces act like 0s. In our example, the trick's victim leaves a configuration encoded by 10110010 and the helper wants to "construct an error" in the second position (to inform the magician about the chosen card, the deuce). First he should organize the following table:

Bit Position	1	2	3	4	5	6	7	8
Encoded Bits	1	0	1	1	0	0	1	0
Eq 1	×		×		×		×	
Eq 2		×	×			×	×	
Eq 3				×	×	×	×	
Eq 4								×

Then, checking the four congruences, the helper finds that only (1) fails. As the helper wants just (2) to fail, he has to adjust (1) and (2). This can be done flipping the third bit. With cards, the helper has to turn the third card. The magician arrives and makes the same congruence calculations and table inspection. In this communication scheme, t_3 acts like neutral element. If the magician, after inspection, sees that (2) and (4) fail, it is the same as if only (2) fails. If the magician, after inspection, sees that nothing fails, it is the same as if only (4) fails. This is not an easy process. We repeat,

THIS IS NOT NICE FOR THE MAGICIAN!

3 NIM Approach

The main goal of this paper is to show how NIM sum can help the assistant and the magician to execute the trick.

The classic game of NIM, first studied by C. Bouton [1], is played with piles of stones. On his turn, each player can remove any number of stones from any pile. Under Normal Play convention, the winner is the player who takes the last stone.

The *nim-sum* of two nonnegative integers is the exclusive or (XOR), written $\oplus$, of their binary representations. It can also be described as adding the numbers in binary without carrying.

If a position is a previous player win, we say it is a P-position. If a game is a next player win, we say it is a N-position. The set of P-positions is noted $\mathcal{P}$ and the set of N-positions is noted $\mathcal{N}$. Bouton proved that $\mathcal{P}$ is exactly the set of positions such that the nim-sum of the sizes of the piles is zero.

The structure $(\mathbb{N}_0, \oplus)$ is an infinite group. It is very easy to execute mental calculations with nim-sum. It is just needed to write the summands in binary notation, canceling repetitions in pairs and using standard addition for the remaining powers. Some examples:

$$5 \oplus 3 = (4+1) \oplus (2+1) = (4+\cancel{1}) + (2+\cancel{1}) = 6$$

$$\begin{aligned} 11 \oplus 22 \oplus 35 &= (8+2+1) \oplus (16+4+2) \oplus (32+2+1) = \\ &= (8+\cancel{2}+\cancel{1}) + (16+4+\cancel{2}) + (32+2+\cancel{1}) = 62 \end{aligned}$$

The structures $(\{0, \ldots, 2^k - 1\}, \oplus)$ are finite groups with the property $x \oplus x = 0$. Following, the table for the case $(\{0, \ldots, 15\}, \oplus)$.

$\oplus$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
7	7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
12	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
14	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
15	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

For the “easy” implementation of the card trick it is important to prove the following theorem:

Theorem 1. *Let* $\{a_0, a_1 \ldots, a_j\} \subseteq \{0, \ldots, 2^k - 1\}$*. For all* $N \in \{0, \ldots, 2^k - 1\}$*, one of the following holds:*

1. $\exists\, i \in \{0, \ldots, j\} : a_0 \oplus a_1 \oplus \cdots \oplus a_{i-1} \oplus a_{i+1} \oplus \cdots \oplus a_j = N$;
2. $\exists\, b \in \{0, \ldots, 2^k - 1\} \setminus \{a_0, a_1, \ldots, a_j\} : a_0 \oplus a_1 \oplus \cdots \oplus a_j \oplus b = N$.

Proof. The proof is a direct consequence of the property $x \oplus x = 0$. We begin to consider the equation

$$a_0 \oplus a_1 \oplus \cdots \oplus a_j \oplus x = N.$$

As the inverse of a number is itself, the solution of the equation is

$$x = N \oplus a_0 \oplus a_1 \oplus \cdots \oplus a_j.$$

If $N \oplus a_0 \oplus a_1 \oplus \cdots \oplus a_j \notin \{a_0, a_1, \ldots, a_j\}$ then *2* holds and

$$b = N \oplus a_0 \oplus a_1 \oplus \cdots \oplus a_j.$$

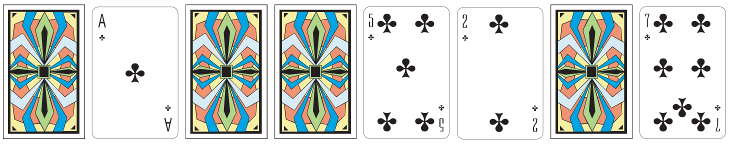
If $N \oplus a_0 \oplus a_1 \oplus \cdots \oplus a_j \in \{a_0, a_1, \ldots, a_j\}$ then *1* holds and

$$a_i = N \oplus a_0 \oplus a_1 \oplus \cdots \oplus a_j.$$

□

This theorem provides a very elegant communication between the helper and the magician. Consider again our first example.

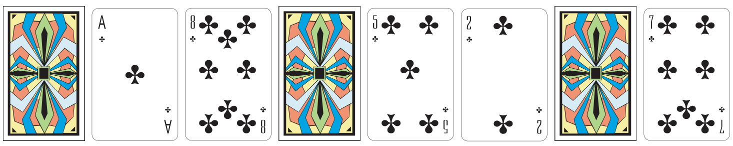
The audience chooses the deuce and leaves the following setup:



With the order $1, 2, \ldots, 6, 7, 0$ (0 corresponds to 8) and with the convention *On*→*Back* and *Off*→*Face*, the helper calculates

$$x = \underbrace{1 \oplus 3 \oplus 4 \oplus 7}_{a_i(Back)} \oplus \underbrace{2}_{N} = 3.$$

In this case, $x = 3$. Because the third card is backward, the situation corresponds to the first item of the Theorem 1. So, the helper turns the third card giving the following setup to the magician:



Now, the magician just calculates $N = 1 \oplus 4 \oplus 7 = 2$ and shouts "deuce of clubs!"

This is a MUCH EASIER execution of the trick.

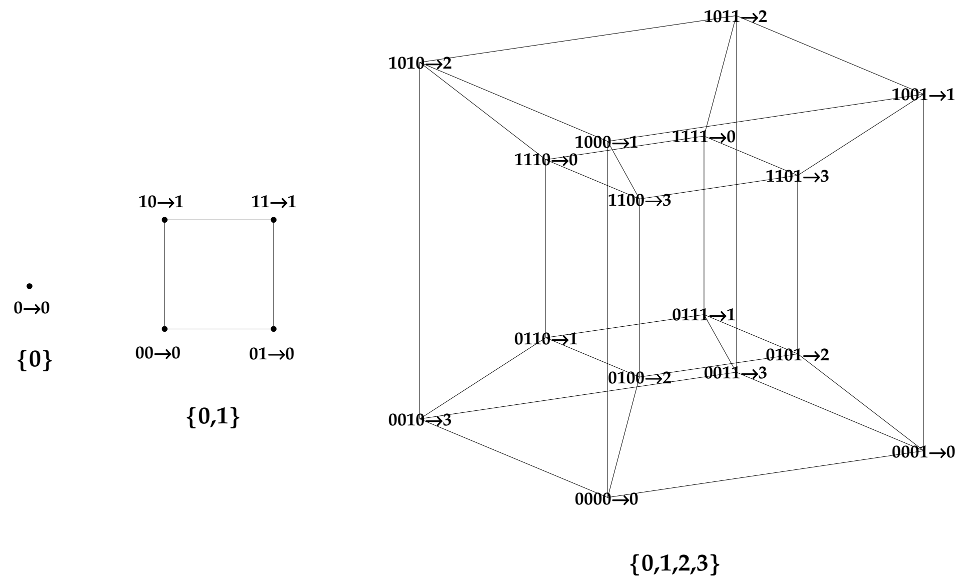
The trick has a nice geometric interpretation. In the first part of the card trick, the victim gives a configuration to the helper and the information about a chosen card $N \in \{0, \dots, 2^k - 1\}$. We can associate each configuration to a graph's vertex. The helper's move is to choose an adjacent vertex of the given configuration. If we can define a function $f : V(G) \to \{0, \dots, 2^k - 1\}$ over the set of vertices such that the helper can always choose a move giving $f(v) = N$, the trick is explained.

"Good" graphs are the hypercubes $I^k = \{0,1\}^{2^k}$ with 2^{2^k} vertices (the vertices are all the arrangements $\alpha_1\alpha_2 \dots \alpha_{2^k-1}\alpha_{2^k}$ $(\alpha_i \in \{0,1\})$. In those hypercubes, each vertex has degree 2^k. A function satisfying our goal is

$$f : I^k \to \{0, 1, \dots, 2^k - 1\} \text{ given by}$$

$$f(\alpha_1\alpha_2 \dots \alpha_{2^k-1}\alpha_{2^k}) = \alpha_1 \oplus 2\alpha_2 \oplus 3\alpha_3 \oplus \dots \oplus (k-1)\alpha_{2^k-1}$$

We can visualize the geometric idea:



If we perform the trick with just 4 cards, the action on the hypercube is easy to visualize. For instance, if the helper gets the configuration 1010 and wants to inform the chosen card 3, he must chose the vertex 0010 (he turns the first card).

4 Colm Mulcahy's TOE TACK TRICK

The knowledge about the structure $(\mathbb{N}_0, \oplus)$ gives a very practical way to deal with Hamming codes. Suppose the message 10010111 and the related scheme:

Bit Position	1	2	3	4	5	6	7	8	9	10	11	12
	t_0	t_1	a_0	t_2	a_1	a_2	a_3	t_3	a_4	a_5	a_6	a_7
Encoded Bits	?	?	1	?	0	0	1	?	0	1	1	1

The procedure starts NIM adding the *positions* of the $a_i = 1$ (N). In this example, $N = 3 \oplus 7 \oplus 10 \oplus 11 \oplus 12 = 9$. Because $(\{0, 1, \ldots, 15\}, \oplus)$ is a finite group, it is mandatory that, for all possible messages with the considered length, N is an element of $\{0, 1, \ldots, 15\}$. After, the t_i are chosen in such a way that the nim-sum of the positions of 1's is N. This is always possible because the t_i's positions are the powers of 2 (in this case, 1, 2, 4 and 8). In fact, the t_i are chosen in such way that $t_3t_2t_1t_0$ is the binary expansion of N. In the example, $t_3 = 1$, $t_2 = 0$, $t_1 = 0$, and $t_0 = 1$. The encoded message is 101000110111.

When the receptor receives the message, he calculates the nim-sum $\bigoplus p_i$ (nim-sum of the positions with digit 1). If the result is different than zero, an error occurred. Say that $\bigoplus p_i = x$. The question is: what is the position where the error occurred? That is, what is the value y such that $\bigoplus p_i \oplus y = 0$? It is possible to understand that x is the answer, revealing the position where the error occurred. If $x \notin \{p_i\}$ then the element in position x is 0 and must be replaced by 1. If $x \in \{p_i\}$ then the element in position x is 1 and must be replaced by 0. The receptor just has to compute $\bigoplus p_i$ to discover the position x. Suppose that the error transforms the encoded message 101000110111 into 101000110101. The receptor calculates $1 \oplus 3 \oplus 7 \oplus 8 \oplus 10 \oplus 12 = 11$ and immediately understands that the error was in the 11th position. The finite NIM groups provide an elegant explanation for the Hamming's idea.

At *Gathering for Gardner 9*, Colm Mulcahy showed his TOE TACK TRICK. The game starts with a an empty 3×3 grid as in the ordinary TIC-TAC-TOE, but in TOE TACK TRICK the grid is totally filled and the winner is the player who finishes with the smallest number of three-in-row. In this version, both players can use both symbols ("X" and "O"). However there is another important difference, in TIC-TAC-TOE players can place the symbols where they want, but, in TOE TACK TRICK, First can only play in the middle of each side and Second can only play in the corners.

2nd	1st	2nd
1st		1st
2nd	1st	2nd

During the first stage, First fills in **all** his four cells. **After**, during the second stage, Second fills in all his corners. During a third and last stage, the center is filled with “X” or “O”, depending of a coin toss.

A three-in-row counts for Second if the configuration ends in the corners (a side or a diagonal). A three-in-row counts for First if the configuration ends in the middle of the sides.

Suppose the final configuration:

X	O	O
X	O	X
O	O	X

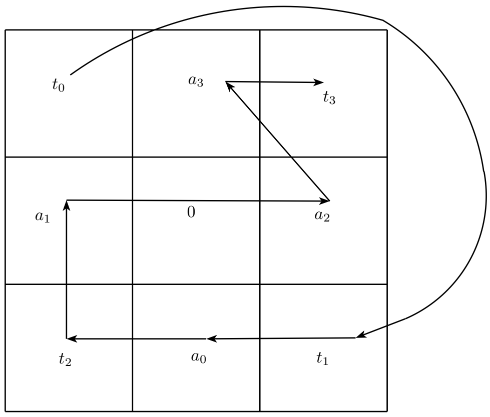
Each player has one three-in-row. The game results in a draw.

Mulcahy’s proposal is related, not to the game itself, but to a very interesting communication situation. Suppose that the second player, fearing that his opponent could be a cheater, prepares a communication scheme with a good friend (Sherlock). Second knows that, after playing a game, First always cheats, *switching exactly one symbol.* In the previous example, suppose that First changes the final grid to the following one:

O	O	O
X	O	X
O	O	X

Sherlock, who has witnessed nothing, enters in the scene and comments “You, First, are a cheater. You changed that mark in the top-left corner, winning the game $(1-2)$. Before the switch, you each had one three-in-row and the game was a draw.”.

How is this possible? Sherlock saw and heard nothing! Again, instead of Hamming codes, we can explain everything with recourse to nim-sum. Consider the following communication scheme:



During the first stage of the game, Second saw the First’s choices:

	0 (7th)	
1 (5th)		1 (6th)
	0 (3rd)	

He calculated $5 \oplus 6 = 3$ and, because, in binary, $3 = 11$, he made the choice $t_3t_2t_1t_0 = 0011$:

1	0	0
1		1
0	0	1

After the randomized central move and the "cheating switch", Sherlock entered in the room and observed the final configuration:

0	0	0
1	0	1
0	0	1

He calculated $2 \oplus 5 \oplus 6 = 1$ and discovered that the first position was changed. If the nim-sum was equal to 0, then the cheating switch should have been in the center.

For more relations between Error-Correcting Codes and NIM algebraic operations see [2]. This reference is concerned with various classes of *lexicographic* codes, that is, codes that are defined by a greedy algorithm: each successive codeword is selected as the first word not prohibitively near to earlier codewords (in the sense of Hamming distance, the number of positions at which the corresponding symbols are different). Among others, the authors proved a very interesting theorem: for a base $B = 2^{2^a}$, unrestricted lexicodes are closed under NIM addition and NIM multiplication.

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MathMagic

The Generalized Apex Magic Trick

Yossi Elran
Weizmann Institute of Science
Davidson Institute of Science Education
yossi.elran@weizmann.ac.il

Abstract: *Many card tricks are based on mathematic principles. One such trick is known as the "apex number triangle". In this trick, the basic tool the magician uses to make a prediction is the "modulo 9" vector product between a row of numbers chosen by a spectator and the corresponding row in Pascal's triangle. In this paper we will show how to explore the fractal structure of the modulo versions of Pascal's triangle to produce different "magic" procedures, how to "work" them under different initial conditions, and how to prove pictorially a theorem regarding these kind of problems.*

Key-words: mathematical card tricks, Martin Gardner, Pascal's triangle.

1 Introduction

There seems to be a deep connection between mathematics and magic. Perhaps this is because some results of mathematical reasoning, methodology and procedure are somewhat surprising and counter-intuitive. One example of such a mathematical "self-working" magic trick is the so-called "apex number triangle" that was introduced by Martin Gardner [1] in his book *Mathematical Carnival* and further studied and generalized by Colm Mulcahy [2] and others [3].

Five playing cards, face-valued 1-9, are chosen by a spectator and placed face-up in a row on the table. In this trick, the suit of the cards is not important, so for all purposes, instead of using playing cards, one could simply take five slips of paper and have the spectator write down a number between 1-9 on each slip. The magician predicts a certain number and writes down his prediction. Then, the spectator is asked to add the two numbers on each pair of adjacent cards in the row. If the sum is larger than 9, the two digits of the result are added again to get a one-digit number, in other words, addition mod 9. A card with the face value of the sum is chosen from the pack and placed in a row above and in-between the two cards. As a result of this, a 4-card row of the modulo 9 sums of the pairs of cards in the row beneath is created. This procedure is repeated to create a 3-card row, then a 2-card row and finally one card - the "apex" of the triangle of cards that has just been formed. This card, of course, turns out to be the "predicted" card.

The math "behind" the trick can be unveiled using algebra, assigning variables to the set of initial numbers and performing the procedure. Once done, it is easy to see that the magician "predicts" the "apex" card by multiplying the face value of the spectator's initial five cards by 1,4,6,4,1 respectively and adding the sum modulo 9. Interestingly, the coefficients 1,4,6,4,1 are the numbers in the fifth row of Pascal's triangle modulo 9. Figure 1 shows an example of the trick played out. The spectator chose the cards: 5,6,2,7,5.

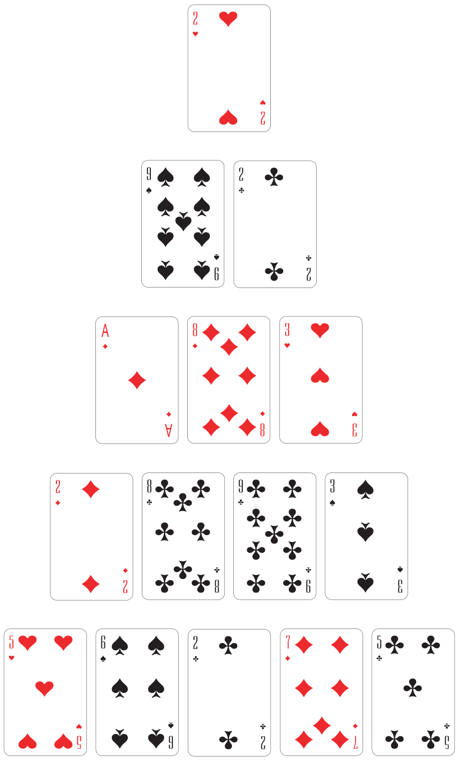

Figure 1: Example of the played out apex magic trick.

The magician predicts that the apex card will be

$$1 \times 5 + 4 \times 6 + 6 \times 2 + 4 \times 7 + 1 \times 5 = 2 \mod 9.$$

2 Generalizations

This trick can be generalized by using any initial number of cards and the corresponding row in Pascal's triangle modulo 9. This is due to the fact that the algebraic construction of the magic trick and Pascal's triangle is the same. Figure 2 shows the first 49 rows of the Pascal's triangle modulo 9. The fourth row, for example, is 1,3,3,1. These are the coefficients to be used for a 4-card trick. The magician predicts the apex card in this case by multiplying 1,3,3,1 by the numbers on the four cards that the spectator chooses, respectively. The tenth row is 1,0,0,3,0,0,3,0,0,1 and can be used for a 10-card trick. In this case, the magician predicts the apex card by multiplying these coefficients by ten numbers (or cards) that the spectator chooses. Since the coefficients have many zeroes, this is particularly easy to perform mentally. There is an infinite number of similar rows, where the magician only needs to remember four numbers in the bottom row of the triangle of cards. The next row is the twenty-eighth row: 1,0,0,0,0,0,0,0,0,3,0,0,0,0,0,0,0,0,3,0,0,0,0,0,0,0,0,1 and after that, rows 82, 244 and 730 etc. In general, any $3^n + 1$ row, where n is a positive whole number, will generate such a row. These rows can easily be spotted if we use a scheme to color the different numbers in the modulo 9 Pascal triangle, generating a fractal visual cue for similar-pattern rows.

Note that the fractal shows the general rules for the repetition of any row combination. The $3^n + 1$ rows, for example, are the 3-colored base of the top downwards-pointing Sierpinski style triangles in the fractal. In contrast, the 3^n rows are the 4-colored base of the top upwards-pointing Sierpinski style triangles in the fractal. This corresponds to rows where all the coefficients are non-zero.

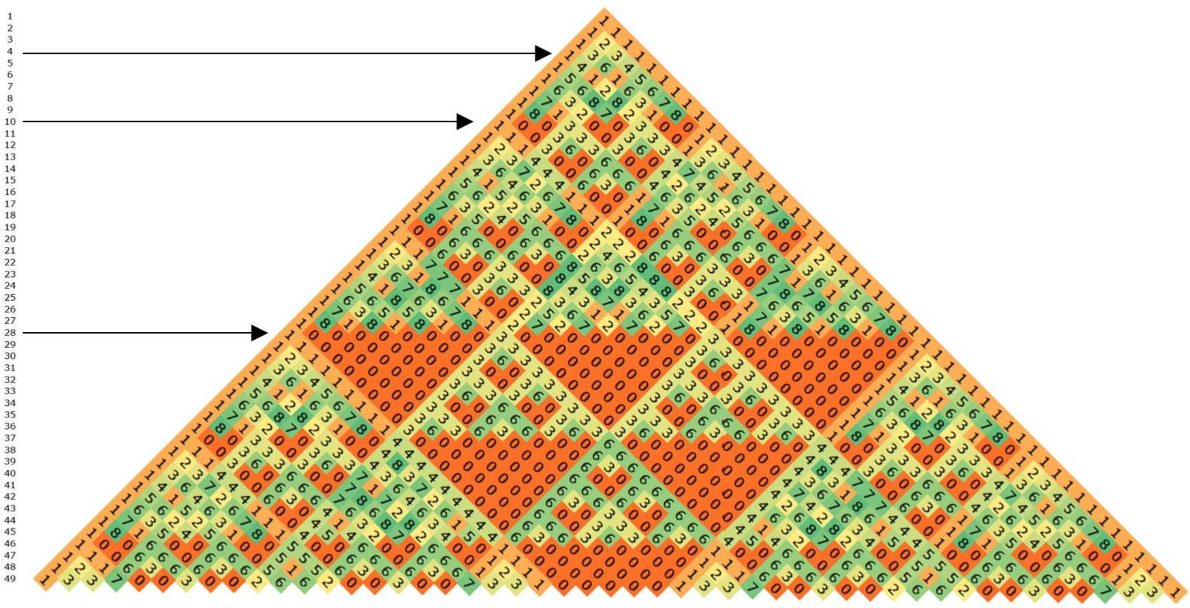

Figure 2: Color-coded mod 9 Pascal Triangle fractal.

The five-card trick can also be performed using different moduli. In a recent study, Behrends and Humble [4] consider the modulo 3 case. This is the best scenario for the magician since some rows in the modulo 3 Pascal triangle consist entirely of zeroes except the first and last numbers in the row that are 1. Behrends and Humble called these rows "Φ-simple". This means that all the magician needs to do is to add the first and last number in the row of cards that the spectator lays out and the result will be the apex number.

In particular, they rigorously prove that if a given row d is the smallest Φ-simple row, then a row $n > d$ is Φ-simple if and only if n is $(d-1)^s + 1$ where s is a positive whole number. In the modulo 3 case, the fourth row 1,0,0,1, is the smallest Φ-simple row, hence, rows 10, 28, 81 etc ... will also be Φ-simple. The fractal generation of the modulo 3 Pascal triangle can be used as a "pictorial proof" of Behrends and Humbles' theorem (Figure 3). It is obvious from the picture that the fractal grows by a factor of 3, hence, Behrend and Humbles theorem is proven, at least for this special case, and also proves that non-Φ-simple patterned rows repeat with the same factor.

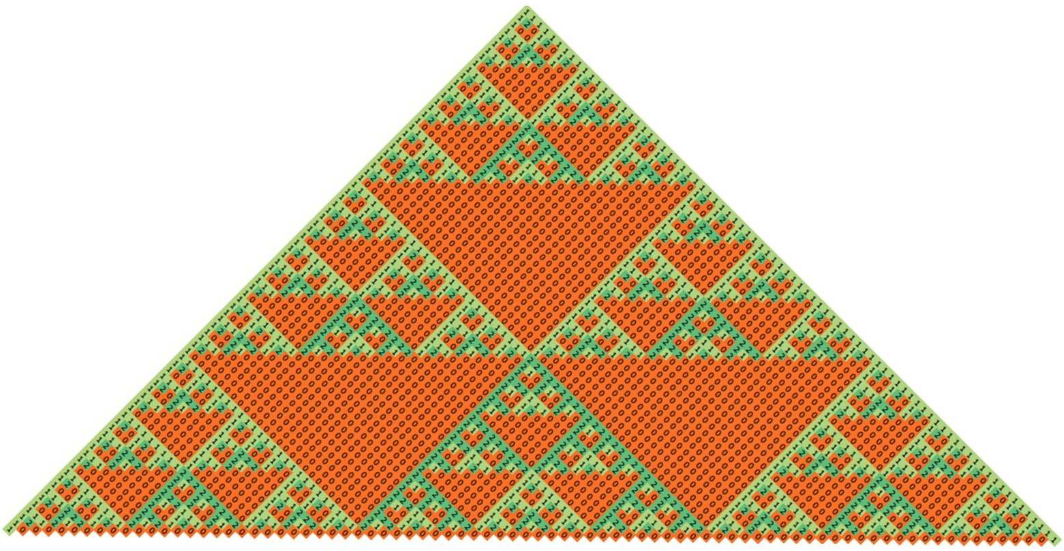

Figure 3: Color-coded mod 3 Pascal Triangle fractal.

We can explore this fractal structure to devise a "large audience" magic trick. Arrange a triangular formation of seats in an auditorium or large classroom so that there is one chair at the back of the room, two chairs in the row in front, three chairs in the next row and so on, until there are 10, 28 or 81 chairs in the last row (depending on the space available). Tell the people on the first row in the room, the row with the Φ-simple number of chairs, that they have to choose between three different gestures. They can raise one hand, raise both hands or sit with their arms folded. After each person chooses a position, you immediately predict and write down on a piece of paper one of the positions. You then ask each person in the second row to choose a gesture according to the combination of gestures of the two people in the row in front. If both have both hands raised or one of them has one hand raised and the other has folded his arms, then the person should raise both his hands. If both have one hand raised or one has both hands raised and the other has folded his arms, then the person should fold his arms. If both have arms folded or one has one hand raised and the other has both hands raised, then the person should raise one hand.

This procedure carries on until the person sitting in the last row has made his gesture, which of course, turns out to be the prediction. The analogy to the five card apex trick is immediate. All the magician does is to combine the gestures of the first and last people in the first row. Since the row is Φ-simple, this will be the apex gesture. The procedure itself is just addition modulo 3 where 0 = "both hands raised", 1="one hand raised" and 2="arms folded".

The nice thing about the fractal structure is that it allows one to perform this trick with any number of rows of chairs, albeit increasing the difficulty of the mental calculation. This is useful if one suspects that a clever audience might suggest that only the two end chairs predict the outcome. An easy choice would be to use rows of length: 7, 19, 55 etc. that correspond to the base of the downward pointing 2-triangle rows. The calculation is the same as before, except that twice the gesture of the middle chair in the first row is also added. Then, if the audience suspects as before that the end chairs cause the result, one can "prove" that this is not so.

Note that the Φ-simple rule for the modulo 3 case (3^n+1) is the same rule that we found for the case of modulo 9. Although none of the rows in the modulo 9 case are Φ-simple, the rule does in fact generate all the rows with the minimal number of non-zero elements. This is easily explained by the visual similarity of the Pascal triangle fractals (Figs. 2 and 3).

The modulo 10 case presents another "neat" magic trick. Write eleven numbers in a row. Add each pair of numbers, keeping only the units digit and create a triangle as before. Although the modulo 10 Pascal triangle is not Φ-simple, as can be observed by its corresponding fractal, Figure 4, the prediction is very easy. Multiply the value of the spectator's first and last numbers by 1, fourth and eighth numbers by 5 and sixth number by 2 - and then mentally sum the results, keeping only the units digit (this is akin to addition modulo 10). This is the apex number. Note that if the fourth and eighth number have the same parity, they can be ignored because they will cancel each other out, so summation of the first, last and twice the middle card is all that is needed and the prediction is made.

3 Summary

We have shown just a few magic tricks that can be performed using the different moduli versions of Pascal's triangle. Each trick can be adapted to different initial conditions by exploring the fractal structure of the triangle. Indeed, the fractal coloring of different moduli Pascal triangles is intriguing. Observation of the emerging structure of the fractals leads to pictorial proofs regarding the position of Φ-simple, "minimal non-zero element" and similar-structure rows. These immediately provide the magician with many more puzzles and tricks that can be generated. We intend to study further generalizations, including possible uses of different shapes in the fractal other than triangles, and developing tricks using the different diagonals of the moduli versions of Pascal's triangle. On a final note, we would like to encourage educators and others to use the ideas

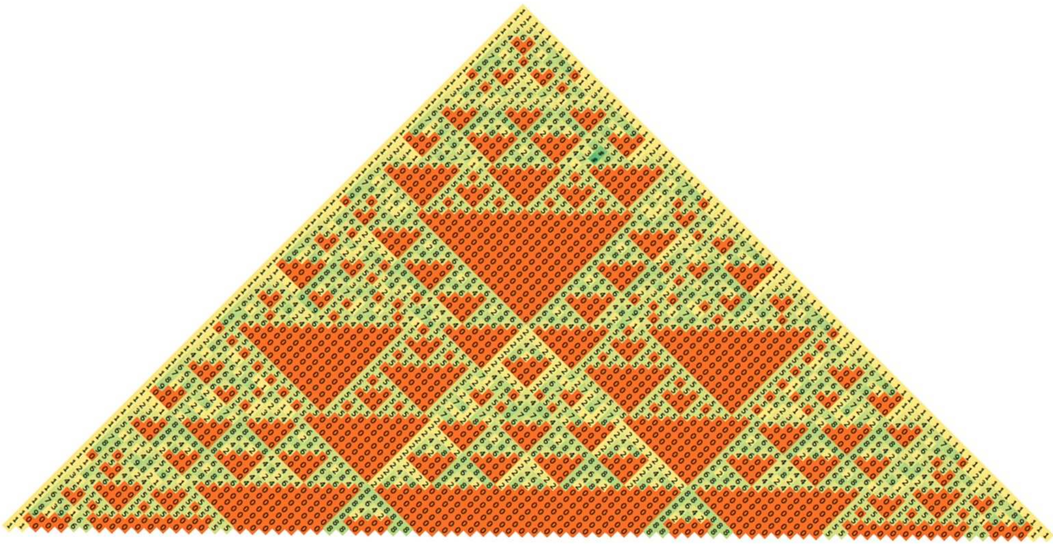

Figure 4: Color-coded mod 10 Pascal Triangle fractal.

developed in this paper for educational purposes. It is our firm belief, and indeed our own experience, that introducing mathematical topics through recreational mathematics is well received and inspiring for teachers and students and can help eradicate negative attitudes towards mathematics.

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