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Informations
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Articles
Games and Puzzles
Problems
MathMagic
Mathematics and Arts
Math and Fun with Algorithms
Reviews
News

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Contents

Games and puzzles

Mathematics of a Sudo-Kurve

Tanya Khovanova, Wayne Zhao
MIT, Bridgewater–Raritan High School
tanya@math.mit.edu, waynezhao2011@gmail.com

Abstract: *We investigate a type of a Sudoku variant called Sudo-Kurve, which allows bent rows and columns, and develop a new, yet equivalent, variant we call a Sudo-Cube. We examine the total number of distinct solution grids for this type with or without symmetry. We study other mathematical aspects of this puzzle along with the minimum number of clues needed and the number of ways to place individual symbols.*

Keywords: combinatorics, Sudoku.

1 Introduction

Sudoku, a massively popular puzzle, was likely invented in 1979 by Howard Garnes. It hopped over to Japan, which gave it its modern name of "Sudoku." After Wayne Gould got the first Sudoku published in Britain in 2004, the puzzle grew dramatically in popularity in 2005. As a result, Sudoku has been extensively studied [6].

Sudoku puzzles are special cases of Latin squares, which were studied by Euler two centuries before. In 2005, extensive casework determined the total number of Sudoku solution grids to be 6670903752021072936960 not accounting for symmetries [2]; enough Sudoku puzzles to use up the informational, energy, and storage capacities of all of human history. Accounting for symmetries there are 5472730538 solution grids. In 2013, the minimum number of clues needed to solve a given Sudoku was found. An extensive computer-assisted proof found the answer to be 17 [3].

"Regular" Sudoku has thus been well-studied. There are now several dozen variants of Sudoku, appearing in many books, world puzzle championships, and websites [7, 5, 8]. One of the websites at the forefront of this is the GMPuzzles blog [4]. One particularly interesting variant found on GMPuzzles and elsewhere is that of the Sudo-Kurve, first invented by Steve Schaefer and named by Adam R. Wood [4]. In Sudo-Kurve, each gray line connects the 9 cells that comprise a row or column. These rows and columns are twisted into

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each other, but the rule about each row or column containing one of each number from 1 to 9 still holds.

In this paper we examine various mathematical aspects of a particular type of Sudo-Kurve we call a Cube Sudo-Kurve. We begin by examining the rules for Cube Sudo-Kurves and go slowly over an example of a puzzle in Section 2. Then, we find an equivalence between the Cube Sudo-Kurve and a Sudoku on a $3\times3\times3$ cube, and use this equivalence to deduce some strategies for solving Cube Sudo-Kurves in Section 3. We use these strategies in Section 4 to determine that the total number of distinct solution grids without taking symmetry into account is 14515200. In Section 5 we give an alternative calculation of the same number by using some symmetries. Later, in Section 6, we compute the total number of distinct solution grids accounting for all possible symmetries to be 2. In Section 7 we consider Sudo-Cubes of other sizes. Then, in Section 8, we show that the minimum number of clues required to uniquely determine a solution of a Cube Sudo-Kurve is 8. Finally, we conclude with some observations on individual digit placement in Section 9.

2 Cube Sudo-Kurve

In the paper, we consider one of the most interesting Sudo-Kurves that contain only three 3-by-3 squares. We call it a *Cube Sudo-Kurve*.

The Cube Sudo-Kurve consists of three square blocks as in Figure 1. The gray bent lines indicate how rows and columns continue. For example, the first row of the top left block becomes the last column of the middle block and continues to the first row of the bottom right block.

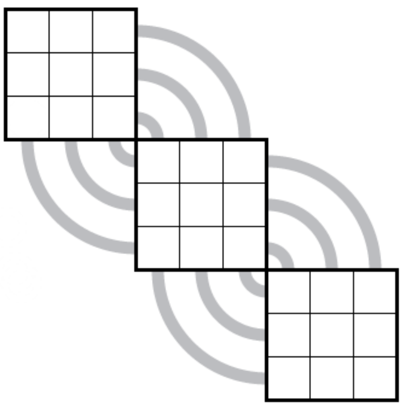

Figure 1: Empty Cube Sudo-Kurve grid.

Here we provide a sample Cube Sudo-Kurve puzzle (Figure 2) and solve it. This puzzle appeared on GMPuzzles on February 12, 2013 [4].

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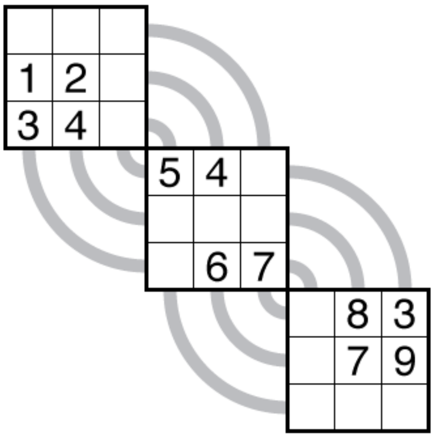

Figure 2: A sample Cube Sudo-Kurve puzzle from the GMPuzzles blog.

Our first step is to notice that there are two instances of the symbol 3 and two of the symbol 7. Let us first consider the 3. The 3 in the block on the upper-left prevents the left column of the center block from containing a 3. Similarly, the 3 in the lower-right block prevents the right column of the center block from containing a 3. We therefore know the 3 must go in the center of the center block.

More generally, we note intuitively that given any two occurrences of a symbol we can determine the position of the third. This will be proven later. For now, this also means we can fill in the third occurrence of the 7, see Figure 3.

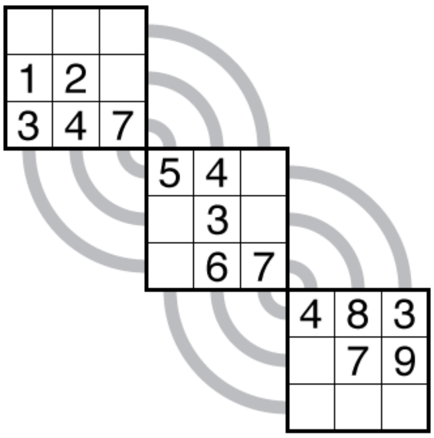

Figure 3: Determining positions of 3 and 7.

Now consider the middle row of the upper-left block, which is also the middle column of the center block and the middle row of the lower-right block. This row is missing the digits 8 and 5. However, we know from the 5 in the center

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block that the 5 cannot go into the upper-left block, so the 5 must belong in the lower-right block. This also fixes the value of 8 in the upper-left block, as in Figure 4.

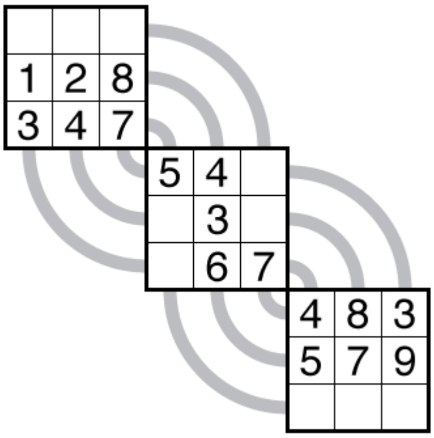

Figure 4: Filling in a row.

Now we have two occurrences each of 5 and 8 so we can fill in their third occurrences (see Figure 5).

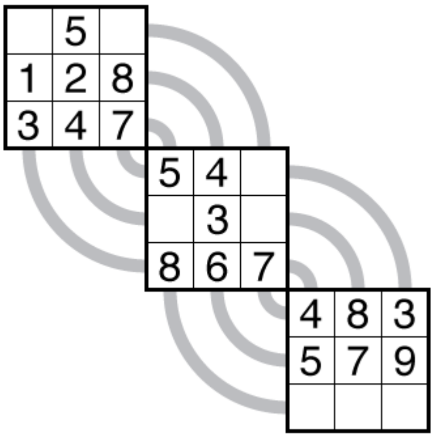

Figure 5: Determining positions of 5 and 8.

The column that starts from the left column of the upper-left block is missing a 2 and a 9. As there is a 9 in the lower-right block, the 9 in that column cannot be in that block, so it must be in the upper-left. We now can place the missing 2, as in Figure 6.

DOI: https://doi.org/10.2478/rmm-2018-0005

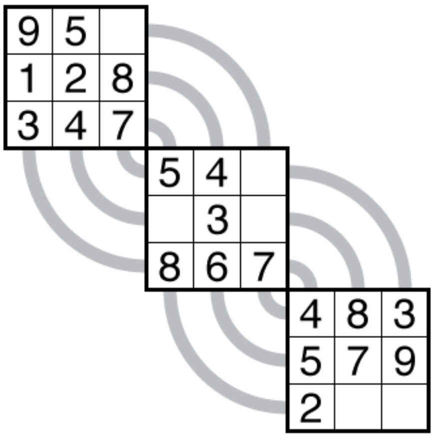

Figure 6: Filling in a column.

Now we can fill in the remaining 2 and 9. Additionally, there is one digit missing in the upper-left block — a 6. We fill these in Figure 7.

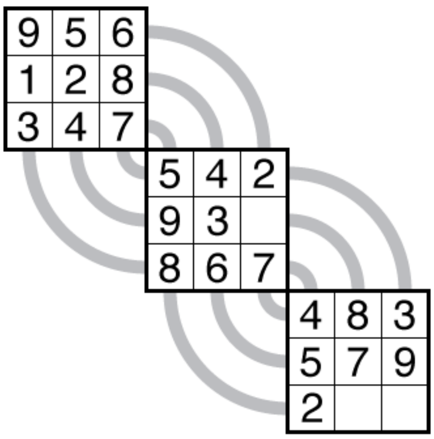

Figure 7: Filling in a block.

We can now easily fill in the rest of the Cube Sudo-Kurve, see Figure 8.

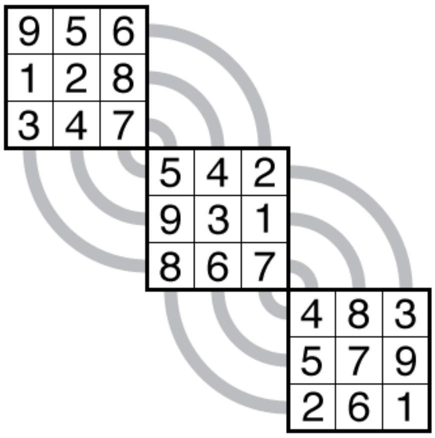

Figure 8: Complete Cube Sudo-Kurve.

3 Sudo-Cube

We now introduce another variation of Sudoku, which we later prove to be isomorphic to Cube Sudo-Kurve. We call it a *Sudo-Cube*. This interpretation will make it easier to compute the total number of solution grids, with or without symmetry. The grid is a $3 \times 3 \times 3$ cube. The digits 1 though 9 are placed in the cells of the cube so that the nine digits in each layer perpendicular to one of the axis are all distinct. To represent the Sudo-Cube in this paper we arrange horizontal layers of the cube on a plane next to each other as in Figure 9. We can assume that the bottom layer is on the left and the top layer on the right.

We denote the left block which represents the bottom layer of the cube as $B1$, the center block representing the middle layer of the cube as $B2$, and the right block representing the top layer of the cube as $B3$.

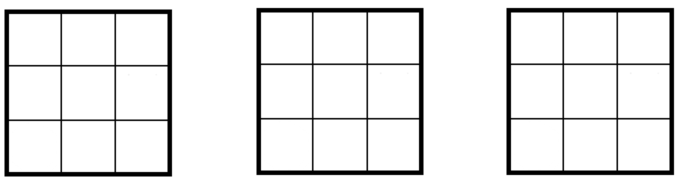

Figure 9: Empty Sudo-Cube grid.

Now we prove that these two grids are equivalent.

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Theorem 1. *A filled Cube Sudo-Kurve is isomorphic to a filled Sudo-Cube.*

Proof. We match the top left block of the Cube Sudo-Kurve to $B1$ — the bottom layer of the Cube Sudo-Cube. We match the bottom right square of the Cube Sudo-Kurve, to the top layer of the Sudo-Cube; that is, to $B3$. We flip the middle square of the Cube Sudo-Kurve with respect to the anti-diagonal and match the result to the middle layer of the Sudo-Cube, aka $B2$.

We can see that the bent row corresponding to the first row of the top left block of the Cube Sudo-Kurve, together with the last column of the middle block and the first row of the bottom right block of the Cube Sudo-Kurve, becomes the first row of each of the blocks of the Sudo-Cube. In other words, the rows of the Cube Sudo-Kurve correspond to front-facing squares in the Sudo-Cube. Similarly, the columns of the Cube Sudo-Kurve correspond to the side-facing squares. □

Note that the bent rows and columns now correspond to actual rows and corresponding columns of all blocks in a Sudo-Cube.

We now denote each square in the Sudo-Cube with a coordinate. The square in the kth row in the mth column in the nth block shall be denoted with the triple (k, m, n).

Visualizing this Sudo-Kurve as a cube helps us prove other results.

Lemma 1. Knowing two instances of a symbol uniquely defines the location of the third instance of the symbol.

Proof. Indeed, assume two instances of a symbol are found at (k_1, m_1, n_1) and (k_2, m_2, n_2). We try to find all triples (k_3, m_3, n_3) where the third instance of the symbol can be located. Now since, there are only three values of the first coordinate possible, knowing k_1 and k_2 leaves us with k_3. Similarly, we can determine m_3 and n_3, so there is only one possible position for the third instance of the symbol to be found. □

4 Number of solution grids

We start this section by discussing the situations where we cannot finish placing a particular symbol. Suppose we fill blocks $B1$ and $B2$ with digits. Then each digit has a unique place in $B3$ it has to go to. But it could happen that two different digits need to be in the same cell in $B3$. We call this situation an *obstruction*. An obstruction only happens if the two digits use the same two rows and the same two columns in $B1$ and $B2$.

We have two cases here. The first case is that the digits a and b swap places when moving from $B1$ to $B2$. We call this a *swap*. The second case is when the digits a and b form opposite corners of a rectangle after projection. We call this a *cross*.

We can now count the number of ways to complete this Cube Sudo-Kurve grid.

Theorem 2. *The number of ways to fill the Cube Sudo-Kurve grid is* 14515200.

Proof. Note that the number of ways to fill in $B2$ and $B3$ does not depend on which of the 9! ways we fill in $B1$. We therefore assume the grid is filled in like in Figure 10.

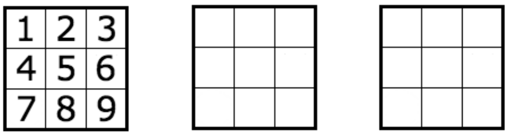

Figure 10: Standard Starting Position.

Later will multiply by 9! to get the total number of solutions to this Sudo-Kurve.

We now have 3 cases to contend with, as described below.

Case 1. The first row of $B2$ is comprised entirely of the elements of either the second or the third row of $B1$.

Let us assume the first row of $B2$ is comprised entirely of the elements of the second row of $B1$, that is $\{4, 5, 6\}$ in some order. This forces the second row of $B2$ to be $\{7, 8, 9\}$ in some order, and forces the third row of $B2$ to be $\{1, 2, 3\}$ in some order.

We see that the 4 can really only go in 2 spots: the second or third box in the first row of $B2$. Similarly, the 5 can only go in 2 spots, as can the 6, and it is easy to see that overall, this gives 2 ways to order the first row (this is equivalent to the fact that there are 2 derangements of 3 objects), which also locks in the positions of the 4, 5, and 6 in $B3$. Similarly, there are 2 ways to arrange the elements of the second row of $B2$ and 2 ways to arrange the third, which gives a total of $2^3 = 8$. Similarly, there are 8 ways to fill the grid when the first row of $B2$ is $\{7, 8, 9\}$ making the total of 16.

Case 2. The first row of $B2$ contains three elements from different columns and not all from the same row.

Assume that the first row of $B2$ contains 4 and 5. Then the third element of the first row of $B2$ would need to be among $\{7, 8, 9\}$. But since 7 is in the same column as 4, and 8 is in the same column as 5, the third element of the first row of $B2$ is 9. All other ways of choosing digits for the first row of $B2$ are equivalent to this one as we can reshuffle rows and columns.

It can be seen that there are $2 \cdot \binom{3}{2} = 6$ ways to choose the elements for the top row. Namely, there are 2 ways to choose which row we will have two elements from and $\binom{3}{2}$ ways to pick the two elements from that row, which determines the third number.

We now assume the top row is $\{4, 5, 9\}$ in some order. We use braces to indicate that the order is not known yet. We can then deduce the following about what elements are in what rows as seen in Figure 11.

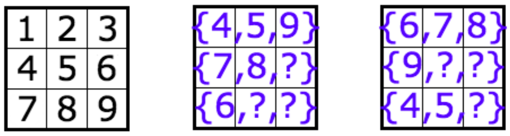

Figure 11: Deductions.

We know that 7 and 8 must be in the second and 6 in the third row of $B2$.

As it turns out, once we have the information about what elements are in the first row, there are only two ways to finish the puzzle.

To prove this, we simply solve the puzzle. Assume the first row is $5, 9, 4$ in that order. We can then immediately fill in more cells as in Figure 12.

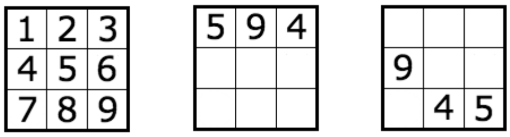

Figure 12: Subcase 1 of Case 2.

But in fact, we can uniquely determine the rest of the grid. Note that in $B2$, the bottom-right square cannot be filled with a 6, 7, or 8. It therefore contains instead a 1, 2, or 3. It cannot be 3, as we already have a 3 in the right column. It cannot be a 2 as then 2 is swapped with 9. It has to be 1. After that, 2 has to go in the first column. It cannot be in the middle row, as then 2 needs to be in the bottom right corner of $B3$ which is occupied. That means 2 has to be in the bottom left corner. Therefore, number 3 is in the middle row in $B2$. We know 3 cannot be in the first column as it clashes (forms a swap) with 4. Therefore, the only way is as shown in Figure 13.

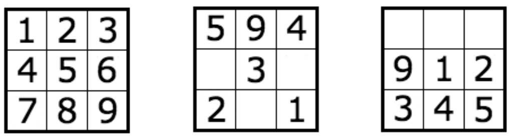

Figure 13: Subcase 1 of Case 2 Deductions.

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After that the rest is uniquely defined.

Thus, for each of the $6 \cdot 2 = 12$ ways to choose and fix the first row, we only get one solution, which implies there are actually 12 total solutions in this case.

Case 3. The first row of $B2$ contains exactly two elements of the same row of $B1$. Similarly, there are exactly two elements from the same column.

This case is similar to Case 2, except we have $\{4, 5, 7\}$ in the first row, for example, instead of $\{4, 5, 9\}$. Note that 4 and 5 are elements of the second row of $B1$, while 7 is an element of the third row of $B1$, and happens to also be in the same column as 4 in $B1$. In total, there are $2 \cdot \binom{3}{2} \cdot 2 = 12$ ways to choose the elements in the first row of $B2$. If we assume the first row is $\{4, 5, 7\}$, we can determine the following information about what rows contain what numbers as in Figure 14.

1	2	3
4	5	6
7	8	9

{4,5,7}
{8,9,?}
{6,?,?}

{6,8,9}
{7,?,?}
{4,5,?}

Figure 14: Case 3.

However, we can actually determine more information. It turns out the order of the numbers in the first row is fixed. Note that 4 and 7 must go in the last two cells of the first row of $B1$. Integer 5 must therefore be in the first cell of the first row of $B1$, and 5 must be in the bottom-right-most box of $B3$. But now notice 4 cannot be the second element in the first row of $B2$, or else the 4 must occupy the same position as the 5 in $B3$. The position of 4, and subsequently of the 7, is therefore fixed.

There is now enough information in this problem to completely solve this puzzle now, see Figure 15.

1	2	3
4	5	6
7	8	9

5	7	4
8	9	2
6	3	1

9	6	8
3	1	7
2	4	5

Figure 15: Case 3 Solved.

All other ways of choosing digits for the first row of $B2$ are equivalent to this one as we can reshuffle rows and columns. Since there is only one solution for each initial arrangement, there are 12 solution grids in this case.

Therefore the total number of solution grids with $B1$ fixed is $16 + 12 + 12 = 40$, implying there are $40 \cdot 9! = 14515200$ total solution grids. □

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5 Alternative count

We can also exploit symmetries to determine the total number of solution grids. The symmetries of the Sudo-Cube are similar to those of regular Sudoku. They are, swapping two parallel layers of the Sudo-Cube, rotating or reflecting the entire cube, and relabeling the digits of the cube. Note that swapping layers manifests in the display of the Sudo-Cube as either like swapping blocks, rows in the blocks, or columns in the blocks.

Now assume, without loss of generality, that we fix the number 5 in the center of $B1$. Then the number 5 in a solution to this Cube Sudo-Kurve must be in row 1 or row 3 and column 1 or column 3 of $B2$. We can switch these columns and rows arbitrarily, until the 5 is in the top left corner of $B2$. We then relabel the rest of the cells so that $B1$ is in standard configuration.

Thus, after counting the number of solutions after fixing $B1$ and the top left corner of $B2$ as 5, we can multiply it by $4 \cdot 9!$. There happen to be 10 ways, and we get the same number of solution grids as before.

5.1 Explicit Cases

Let us call each of these 10 ways *sudo-cases*. We can list out all 10 sudo-cases with the 5 in the upper-left of the middle square. The first four correspond to Case 1, when the first row of $B2$ is comprised entirely of the elements of either the second or the third row of $B1$. Together with the condition that 5 is in the upper left corner of $B2$ we get that the second row of $B2$ must be 5, 6, and 4. Here are sudo-cases 1 through 4.

1	2	3
4	5	6
7	8	9

5	6	4
8	9	7
2	3	1

9	7	8
3	1	2
6	4	5

1	2	3
4	5	6
7	8	9

5	6	4
9	7	8
2	3	1

8	9	7
3	1	2
6	4	5

1	2	3
4	5	6
7	8	9

5	6	4
8	9	7
3	1	2

9	7	8
2	3	1
6	4	5

1	2	3
4	5	6
7	8	9

5	6	4
9	7	8
3	1	2

8	9	7
2	3	1
6	4	5

Now we consider the second case when the first row of $B2$ contains three elements from different columns, not all from the same row. There are three possibilities for the first row of $B2$. Sudo-cases 5, 6, and 7 are presented below.

1	2	3	5	9	4	6	7	8
4	5	6	8	3	7	9	1	2
7	8	9	2	6	1	3	4	5

1	2	3	5	6	7	9	4	8
4	5	6	8	9	1	3	7	2
7	8	9	2	3	4	6	1	5

1	2	3	5	9	7	6	4	8
4	5	6	8	3	1	9	7	2
7	8	9	2	6	4	3	1	5

In the Case 3, the first row of $B2$ contains exactly two elements of the same row of $B1$ and exactly two elements from the same column. There are three sudo-cases 8, 9, and 10 below.

1	2	3	5	7	4	9	6	8
4	5	6	8	9	2	3	1	7
7	8	9	6	3	1	2	4	5

1	2	3	5	6	8	9	7	4
4	5	6	3	9	7	8	1	2
7	8	9	2	4	1	6	3	5

1	2	3	5	7	8	9	6	4
4	5	6	3	9	2	8	1	7
7	8	9	6	4	1	2	3	5

6 The number of different solution grids up to symmetries

We want to calculate the number of different solution grids up to symmetries. The symmetries are:

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- Relabeling of the digits.
- Swapping different layers of the cube. In other words, swapping blocks, rows, and columns.
- Movements of the cube: rotations and reflections.

We already have 10 sudo-cases. But we did not account for all possible symmetries. We only accounted for relabeling and two types of swaps: swapping top and bottom rows and swapping the first and the last column. The last two swaps are equivalent to reflections of the cube.

We did not account for permuting blocks $B1$, $B2$, and $B3$. We also did not yet account for swapping the left and right layers with the middle layer. We did not yet account for swapping the front and back layers with the middle layer. We also did not try every move of the cube.

Many of these ten sudo-cases are really equivalent. We start by showing that sudo-case 1 is not equivalent to any other sudo-case.

Lemma 2. Sudo-case 1 will remain sudo-case 1 under all transformations.

Proof. Suppose we pick a direction on the cube that is up. Then the blocks $B1$, $B2$, and $B3$ are uniquely defined. Consider a row of 9 digits that spans all three blocks. Each row is a partition of digits 1-9 into three sets of three elements. Sudo-case 1 is the only sudo-case where for all initial direction this partition is the same for every row.

Relabeling, shuffling layers, and movements of the cube do not change this property. Thus sudo-case 1 remains equivalent to sudo-case 1 under all transformations. □

Now we continue with our main result of classifying Cube Sudo-Kurve solution grids up to symmetry.

Theorem 3. *There are only two distinct sudo-cases under symmetry: sudo-case 1 and sudo-case 2. Sudo-cases 3 through 10 are equivalent to sudo-case 2.*

Proof. We show below that using a reflection with respect to the main diagonal in each of the blocks $B1$, $B2$, and $B3$, sudo-cases 5, 6, 7, and 9 are equivalent to sudo-cases 2, 3, 4, and 8 correspondingly. By swapping $B1$ and $B2$, sudo-case 4 becomes sudo-case 2. By swapping $B2$ and $B3$, sudo-case 3 becomes sudo-case 2. By swapping the top and the middle layer in each block, sudo-case 10 becomes sudo-case 8. By rotation sudo-case 8 becomes sudo-case 2.

Sudo-cases 2-9 are thus equivalent. □

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6.1 Reflection with respect to the main diagonal in each block

To start, we consider a symmetry that keeps the 5 in place: the reflection of each block with respect to the main diagonal. We then need to relabel our digits so that the first block is in the standard form. That is, we swap the digits in the following pairs: (2,4), (3,7), and (6,8).

Sudo-case 2, which is

1	2	3		5	6	4		8	9	7
4	5	6		9	7	8		3	1	2
7	8	9		2	3	1		6	4	5

becomes, upon reflection,

1	4	7		5	9	2		8	3	6
2	5	8		6	7	3		9	1	4
3	6	9		4	8	1		7	2	5

which itself becomes, upon relabeling,

1	2	3		5	9	4		6	7	8
4	5	6		8	3	7		9	1	2
7	8	9		2	6	1		3	4	5

which is sudo-case 5.

Sudo-case 3, or,

1	2	3		5	6	4		9	7	8
4	5	6		8	9	7		2	3	1
7	8	9		3	1	2		6	4	5

becomes, upon reflection,

1	4	7		5	8	3		9	2	6
2	5	8		6	9	1		7	3	4
3	6	9		4	7	2		8	1	5

which, upon relabeling, is

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1	2	3
4	5	6
7	8	9

5	6	7
8	9	1
2	3	4

9	4	8
3	7	2
6	1	5

or sudo-case 6.

Sudo-case 4 is

1	2	3
4	5	6
7	8	9

5	6	4
9	7	8
3	1	2

8	9	7
2	3	1
6	4	5

which becomes, upon reflection,

1	4	7
2	5	8
3	6	9

5	9	3
6	7	1
4	8	2

8	2	6
9	3	4
7	1	5

and upon relabeling becomes

1	2	3
4	5	6
7	8	9

5	9	7
8	3	1
2	6	4

6	4	8
9	7	2
3	1	5

which is sudo-case 7.

Finally, **Sudo-case 8** is

1	2	3
4	5	6
7	8	9

5	7	4
8	9	2
6	3	1

9	6	8
3	1	7
2	4	5

which becomes, upon reflection,

1	4	7
2	5	8
3	6	9

5	8	6
7	9	3
4	2	1

9	3	2
6	1	4
8	7	5

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which, upon relabeling, becomes

1	2	3		5	6	8		9	7	4
4	5	6		3	9	7		8	1	2
7	8	9		2	4	1		6	3	5

which is sudo-case 9.

6.2 Swap B1 and B2

We now try another operation: switching $B1$ and $B2$. With this we show that sudo-case 2 is equivalent to sudo-case 4.

Sudo-case 2 is

1	2	3		5	6	4		8	9	7
4	5	6		9	7	8		3	1	2
7	8	9		2	3	1		6	4	5

.

Now we swap $B1$ and $B2$ and get

5	6	4		1	2	3		8	9	7
9	7	8		4	5	6		3	1	2
2	3	1		7	8	9		6	4	5

.

Now, if we were to relabel this configuration, the 7's would become 5's. Unfortunately, this would mean the 5 would not be in the upper-left corner of $B2$. To fix this, we swap the first and last row to get

2	3	1		7	8	9		6	4	5
9	7	8		4	5	6		3	1	2
5	6	4		1	2	3		8	9	7

.

After relabeling we get

1	2	3		5	6	4		8	9	7
4	5	6		9	7	8		2	3	1
7	8	9		3	1	2		6	4	5

which is sudo-case 4.

6.3 Swap B2 and B3

We now swap $B2$ and $B3$. This allows us to show that sudo-case 2 and sudo-case 3 are the same.

Sudo-case 2 is

1	2	3		5	6	4		8	9	7
4	5	6		9	7	8		3	1	2
7	8	9		2	3	1		6	4	5

and after swapping $B2$ and $B3$ we get

1	2	3		8	9	7		5	6	4
4	5	6		3	1	2		9	7	8
7	8	9		6	4	5		2	3	1

.

In order to get the 5 from the bottom-right of $B2$ to the top-left, we need to swap the bottom and top rows, then swap the left and right columns:

9	8	7		5	4	6		1	3	2
6	5	4		2	1	3		8	7	9
3	2	1		7	9	8		4	6	5

.

And relabeling gives

1	2	3		5	6	4		9	7	8
4	5	6		8	9	7		2	3	1
7	8	9		3	1	2		6	4	5

which is sudo-case 3.

6.4 Swap top and middle row in each block

We now swap the top and middle rows of each block. This will show that sudo-cases 8 and 10 are equivalent.

Sudo-case 8 is

1	2	3	5	7	4	9	6	8
4	5	6	8	9	2	3	1	7
7	8	9	6	3	1	2	4	5

and swapping the top and middle rows gives us

4	5	6	8	9	2	3	1	7
1	2	3	5	7	4	9	6	8
7	8	9	6	3	1	2	4	5

.

We now need the 2 in $B2$ to be in the upper-left, so we swap the left- and rightmost columns of each block

6	5	4	2	9	8	7	1	3
3	2	1	4	7	5	8	6	9
9	8	7	1	3	6	5	4	2

.

Finally, relabeling gives us

1	2	3	5	7	8	9	6	4
4	5	6	3	9	2	8	1	7
7	8	9	6	4	1	2	3	5

which is sudo-case 10.

6.5 Rotation

Now we start with sudo-case 8

1	2	**3**	5	7	4	9	6	8
4	5	6	8	**9**	2	3	1	7
7	8	9	6	3	1	**2**	4	5

.

We rotate it along the 3-9-2 space diagonal (in bold above and below):

DOI: https://doi.org/10.2478/rmm-2018-0005

9	6	**3**
1	2	4
5	7	8

8	5	2
3	**9**	7
4	1	6

7	4	1
6	8	5
2	3	9

.

We then need a 2 in the upper-left of $B2$, so we swap the left and right columns:

3	6	9
4	2	1
8	7	5

2	5	8
7	9	3
6	1	4

1	4	7
5	8	6
9	3	2

.

And finally, relabeling gives

1	2	3
4	5	6
7	8	9

5	9	7
8	3	1
2	6	4

6	4	8
9	7	2
3	1	5

which is sudo-case 7. Sudo-case 8 is therefore equivalent to sudo-case 2.

7 Other Cube Sizes

It is interesting to determine similar properties for cubes of smaller sizes.

For a $1 \times 1 \times 1$ cube, there is clearly only 1 way to fill the grid whether or not we consider symmetries.

For a $2 \times 2 \times 2$ cube, there are 24 distinct ways to arrange the numbers in the first layer. After that, the only way to arrange the top layer is to put everything in the diametrically opposite place. The total number of solutions grids without considering symmetry for a cube of size 2 is 24.

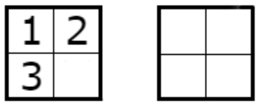
Since all solution grids of this cube are derived from relabeling the numbers in the first layer, all solution grids of the $2 \times 2 \times 2$ cube are isomorphic to each other under relabeling.

Thus, the sequence of the number of different solution grids as a function of the grid size starts as 1, 24, 14515200. The sequence of the number of distinct solution grids under symmetry starts as 1, 1, 2.

8 The minimum number of clues

It is known that in "regular" Sudoku the minimum number of clues needed to uniquely determine a solution grid is 17 [3].

We can now determine the minimum number of clues required to force a unique solution in a Cube Sudo-Kurve. For a Cube Sudo-Kurve of size 1, we do not need any symbols. For a Cube Sudo-Kurve of size 2, we must need at least three symbols so we can differentiate between all symbols. Here is an example of such a minimal Cube Sudo-Kurve:



For a Cube Sudo-Kurve of size 3, we must have at least eight different symbols. Otherwise, we would not be able to tell the difference between the two or more missing symbols. To prove that 8 is the required minimum, we created the following two puzzle grids in Figure 16 and Figure 17, which have only one solution. We encourage readers to solve them both.

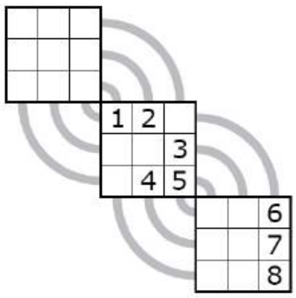


Figure 16: Puzzle with Minimum Number of Clues (easier).

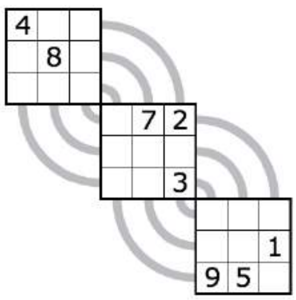


Figure 17: Puzzle with Minimum Number of Clues (harder).

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9 Placing single digits

We now count the number of ways to place a particular digit in a cube of size n.

Lemma 3. For $n \times n \times n$ cube, the number of ways to place any given digit is $(n!)^2$.

Proof. There are n^2 locations to place the symbol in the bottom block. In the next layers from the bottom, one column and one row is forbidden. Therefore, we have leftover spaces isomorphic to $(n-1)$ cube, so we can inductively compute that there are $((n-1)!)^2$ ways to place the symbol in the rest of the cube, and the final total is $n^2 \cdot ((n-1)!)^2 = (n!)^2$. □

In particular, for $n = 2$ we get 4, and for $n = 3$, we get 36.

We can also look at the placement of the same digit up to isomorphisms of the cube. That is we are looking at the shape that is formed inside a cube by the same symbol. Looking at shapes allows us to ignore the actual digits, that is we are studying the shape interaction up to relabeling. The idea to use shapes of symbols was used by Conway and Ryba to describe different Latin squares of size 4 up to movements of the plane and relabeling [1].

For $n = 2$, the only way we can place a given digit is to place it along a main diagonal of the cube.

For $n = 3$, if our digit occupies the center, then it has to use up one of the four main diagonals. Up to rotations, all of these shapes are the same.

Suppose a digit takes up a corner and does not use the center. Without loss of generality, we can say that the corner has coordinates $(1, 1, 1)$. Then the other two points must have coordinates $(2, 3, 3)$, $(3, 2, 2)$, up to movements of the cube. In other words, one symbol is at a vertex of a cube, another symbol is in the center of a face not adjacent to the vertex, and the third symbol is in the middle of an edge that is neither adjacent to the vertex nor to the face from which the center is used.

Thus, for each corner, there are three possibilities. There are total of 24 arrangements of a single digit in this case. Up to rotations and reflections all these shapes are the same, and they all form a scalene triangle.

If there is no symbol in a corner, then all three of them must be in the middles of edges no two of which share a vertex. For example the digits could be at $(1, 2, 1)$, $(2, 3, 3)$, and $(3, 1, 2)$.

There are eight cases like this. All such shapes are isomorphic to an equilateral triangle.

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A given symbol in a 3 by 3 by 3 cube could only be in one of the three shapes described above. Exactly one digit, the one in the center, corresponds to the diagonal, six digits have to use a triangle passing through a corner, and two digits form equilateral triangles.

Let us consider sudo-case 1. We can recognize it by the shapes formed by each of the symbols. For instance, the digit 7 forms a diagonal. We consider planes that go through this diagonal and two opposite edges. There are 3 such planes and they are listed below.

- The plane that is formed by the main diagonals of each block contains only digits 1, 5, and 9.
- The plane formed by the last row in $B1$, the middle row in $B2$ and the top row in $B3$ contains only digits 7, 8, and 9.
- The plane formed by the last column in $B1$, the middle column in $B2$, and the first column in $B3$ contains only digits 3, 6, and 9.

By checking all sudo-cases, we see that the only case where the three planes that include two opposite edges and the main diagonal formed by the same digit all have exactly three different symbols is sudo-case 1. This property is invariant under relabeling and movements of the cube. Given that the we get to the standard form of sudo-case 1 by relabeling and reflections of the cube, that means this property stays before relabeling and reflections. That means we can recognize the sudo-case 1 by this property before any action.

10 Acknowledgments

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Ellipse, hyperbola and their conjunction

Arkadiusz Kobiera
Warsaw University of Technology
arkadiusz.kobiera@itc.pw.edu.pl

Abstract: *This article presents a simple analysis of cones which are used to generate a given conic curve by section by a plane. It was found that if the given curve is an ellipse, then the locus of vertices of the cones is a hyperbola. The hyperbola has foci which coincidence with the ellipse vertices. Similarly, if the given curve is the hyperbola, the locus of vertex of the cones is the ellipse. In the second case, the foci of the ellipse are located in the hyperbola's vertices. These two relationships create a kind of conjunction between the ellipse and the hyperbola which originate from the cones used for generation of these curves. The presented conjunction of the ellipse and hyperbola is a perfect example of mathematical beauty which may be shown by the use of very simple geometry. As in the past the conic curves appear to be very interesting and fruitful mathematical beings.*

Keywords: Geometry, conics, ellipse, hyperbola.

Introduction

The conical curves are mathematical entities which have been known for thousands years since the first Menaechmus' research around 250 B.C. [2]. Anybody who has attempted undergraduate course of geometry knows that ellipse, hyperbola and parabola are obtained by section of a cone by a plane. Every book dealing with the this subject has a sketch where the cone is sectioned by planes at various angles, which produces different kinds of conics. Usually authors start with the cone to produce the conic curve by section. Then, they use it to prove some facts about the conics. Many books focus on the curves themselves and their features. Even books which describe the conics theory in a quite comprehensive way [2, 4, 1] abandon the cone after the first couple paragraphs or go to quite complex analysis of quadratics. We may find hundreds of theorems about the curves but the relation between the cone and the conics is left to the exercise section at best [4] or authors quickly go to more complex systems of conics in three-dimensional space [2]. Probably the cone seems to be too simple to spent time on this topic, however we will show that the cone (strictly speaking family of cones) may have interesting properties as well. Apart of pure geometry, celestial mechanics is the second field where conics are important – the orbits are conic curves. Unfortunately, the books about celestial mechanics say only a few words about the the cone if any at al. [3, 5]. In this short paper we would like to focus on the cone and its relation to conic curves which is surprisingly omitted in books, but interesting.

Ellipse and the cones

Let us consider following problem: Given is an ellipse $\mathcal{E}$ defined by two focus points F_1 and F_2 and vertex A. This ellipse is created by section of the cone **S** by plane ρ. It is shown in Figure 1.

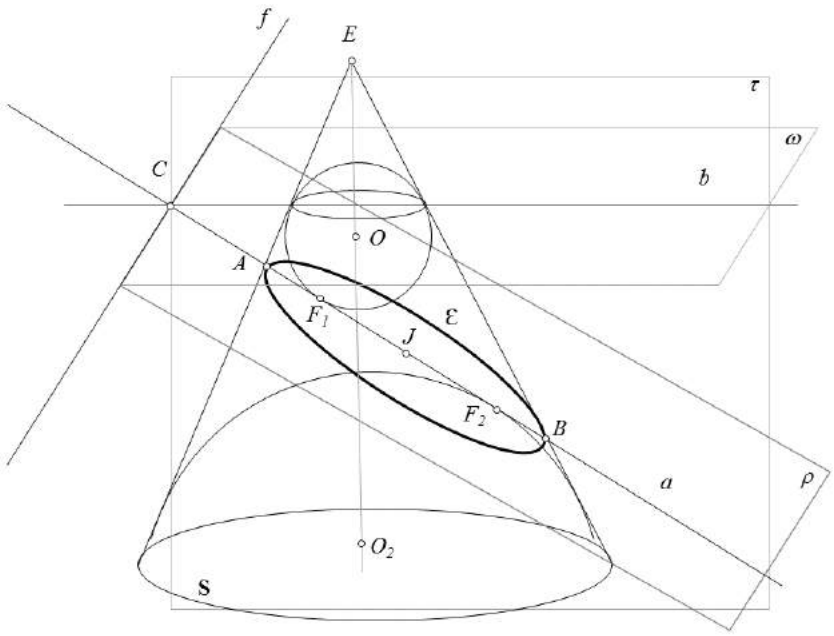

Figure 1: Ellipse and the cone.

Our task is to find the vertex E of the cone **S**. Apart from the foci, the ellipse has also two characteristics points: the vertices A and B. The distances from the vertices to one of the focus e.g. F_1 will be noted as $r_a = |F_1A|$, and $r_b = |F_1B|$. The semi-axes of this ellipse are $a = |AJ|$ and $b = |H_1J|$ where J is the center of the ellipse. The distance between foci is $c = |F_1F_2|$. The radii r_a and r_b define the eccentricity:

$$e = \frac{r_a - r_b}{r_a + r_b} = \frac{c}{a}. \quad (1)$$

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Obviously, we may use any set of these parameters to define the ellipse $\mathcal{E}$, however we will prefer the radii and focus F_1.

The first question is about the cone: *"Is the cone **S** unique?"* The answer is in the following lemma:

Lemma 1. *If the ellipse $\mathcal{E}$ lies on plane ρ and it is defined by two vertices A, B, and focus F_1 (or foci F_1, F_2 and vertex A) then it may be generated by infinite number of cones **S** sectioned by the plane ρ.*

Proof:
The proof will be explained in a rather quite informal manner. To solve this exercise let's reduce the three-dimensional problem to a two-dimensional problem by considering plane τ which is defined by cone's axis and foci (or vertices) of the ellipse. It is shown in Figure 2. We put line a on plane τ. The line coincidences with the ellipse vertices A and B and the foci F_1, F_2 as well. The line a is also an intersection of planes τ and ρ. Note that the focus points (e.g. F_1) are points of tangency of a sphere of center O with the plane ρ. This sphere is called Dandelin's sphere and it is simultaneously tangent to the cone. The tangency points of the sphere and the cone create a circle which defines plane ω [2]. The intersection of planes ω and ρ is line *f*. We create also an additional line b on plane ω which is perpendicular to f and goes through the axis of the cone. The intersection of the Dandelin's sphere by the plane τ is a circle with center O. The circle is tangent to lines t_1 and t_2 which are two elements of the cone. These lines are obtained by cutting the cone by plane τ. They meet line a at points A and B.

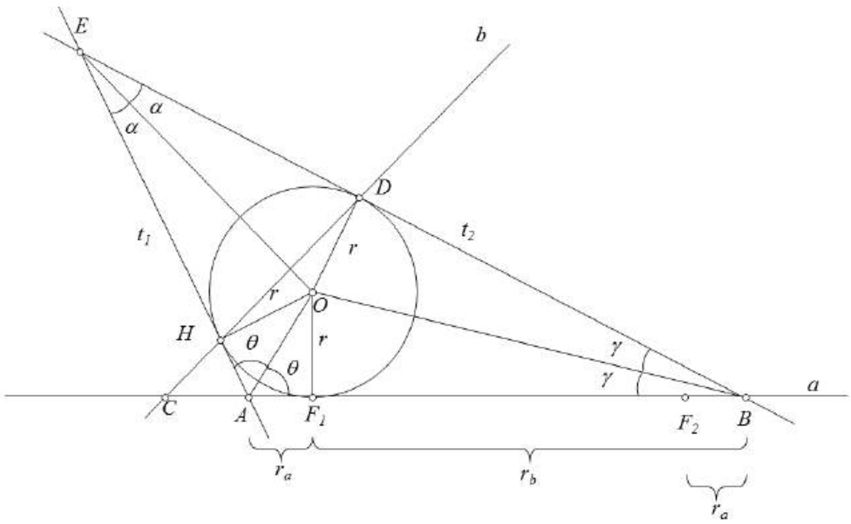

Figure 2: Section of cone **S** by plane τ.

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The problem was reduced to a problem of finding the point E which is vertex of triangle ABE that inscribes circle O. Since the ellipse $\mathcal{E}$ is given, the three points A, B and F_1 are fixed. Point E is a point of intersection of lines t_1 and t_2. These lines are defined by points A and B and the circle O which is tangent to the lines. If the radius r is smaller than a certain limit r_{max} then the two lines meet at point E (this fact seems to be quite obvious so we skip this part of the proof). The maximum radius r_{max} is determined by the case when the lines t_1 and t_2 are parallel. In such case the lines t_1 and t_2 become element lines of a cylinder as it is a limiting case of the cone when the point E goes to infinity. In this case the ellipse $\mathcal{E}$ is obtained as a section of the cylinder. One may show that the limiting radius is equal to the minor semi-axes of the ellipse

$$r_{max} = a\sqrt{1-e^2}. \tag{2}$$

If the radius r can be of any length between 0 and r_{max} then the location of point E is not unique and its position depends on radius r. Hence, one can construct an infinite number of cones which may be used to generate the ellipse $\mathcal{E}$. □

If the cone S is not unique, the next question is: *"What is the locus of the cone vertices E?"* First, we calculate the distance from the cone vertex E to the ellipse vertex B

$$|EB| = |BD| + |ED| = r_b + |ED| \tag{3}$$

The second equality results from the fact that BD and BF_1 are tangent to circle O and they have common endpoint B. Obviously, the angles $\angle F_1BO$ and $\angle DBO$ are equal and right triangles OF_1B and ODB are congruent. Then segments F_1B and BD are of the same length r_b. One can write similar equations for segment EA

$$|EA| = |HA| + |EH| = r_a + |ED|. \tag{4}$$

Here we use the fact that triangles HOE and DOE are congruent and triangles HOA and F_1OA are congruent as well. Comparison of the above equation leads to following proposition:

Proposition 1. *If the ellipse $\mathcal{E}$ defined by two vertices A, B and focus F_1 (or foci F_1, F_2 and vertex A) is generated by section of the cones $\boldsymbol{S}$ by the plane ρ then the locus of vertices E of the all possible cones $\boldsymbol{S}$ is a hyperbola $\mathcal{H}$. The foci of the hyperbola $\mathcal{H}$ are points A and B, vertices are points F_1 and F_2 (ellipse foci).*

Proof:
Let us calculate the difference of length of two segments EB and EA

$$|EB| - |EA| = r_b + |ED| - (r_a + |ED|) = r_b - r_a = const, \tag{5}$$

$$|EB| - |EA| = |F_1F_2|. \tag{6}$$

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This difference is a constant number because r_a and r_b are constant as ellipse parameters, also the distance $|F_1F_2|$ is obviously constant. This directly agrees with definition of a hyperbola which foci are located in points A and B (see Figure 3). It is also clear that vertices of this hyperbola $\mathcal{H}$ are points F_1 and F_2. □

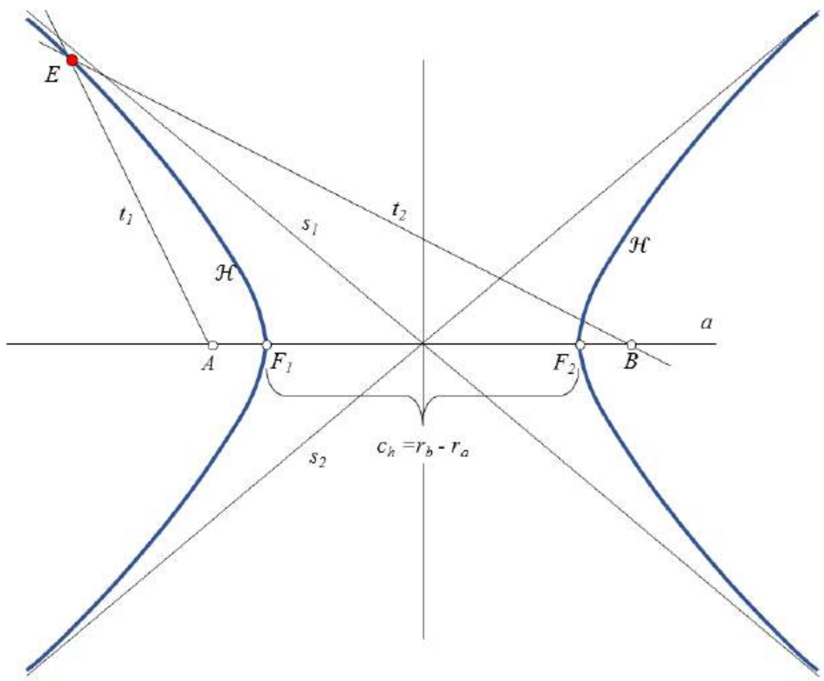

Figure 3: Hyperbola $\mathcal{H}$.

Indeed, the hyperbola $\mathcal{H}$ contains all the possible locations of vertices E. The left branch contains vertices where the Dandelin's sphere is tangent to focus F_1. The upper part of the branch represents the case when the sphere is above the plane ρ, lower part is for opposite position of the sphere. The right branch is for the case where the sphere is tangent at point F_2. If the radius r of the sphere vanishes to 0, the point E goes toward foci F_1 or F_2. If the sphere's radius r goes to the maximum value r_{max} the point E goes to infinity on the hyperbola's branches. Asymptotic lines s_1, s_2 are the axes of cylinders which are limiting cases of the cones with vertex in infinity.

Hyperbola and the cones

Now we can ask reversed question: *What is the locus of vertices G of cones $\boldsymbol{Z}$ which generate the given hyperbole $\mathcal{H}$.* One can consider the hyperbola $\mathcal{H}$ which was found in the previous part. This will not reduce generality of our reasoning. We will keep same plane τ where four points are defined A, B, F_1 and F_2. They also define the hyperbola $\mathcal{H}$ on the plane τ. Figure 4 shows the situation where the hyperbola is created by sectioning the cone **Z** by plane τ. We state the following lemma by analogy to the case of ellipse:

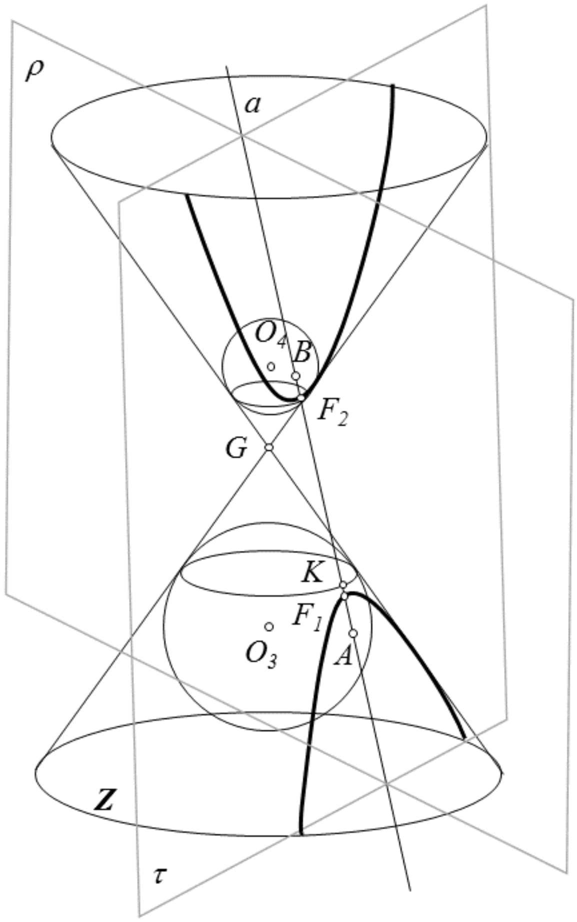

Figure 4: Hyperbola $\mathcal{H}$ created by section of the cone **Z**.

Lemma 2. *If the hyperbola $\mathcal{H}$ lies on plane τ and it is defined by two foci A, B, and vertex F_1 (or two vertices F_1, F_2 and focus A) then it may be generated by infinite number of cones $\boldsymbol{Z}$ sectioned by plane τ.*

Proof:
The proof is analogous to proof of Lemma 1. First, we reduce the problem to planimetry by considering the plane ρ (see Figure 5).

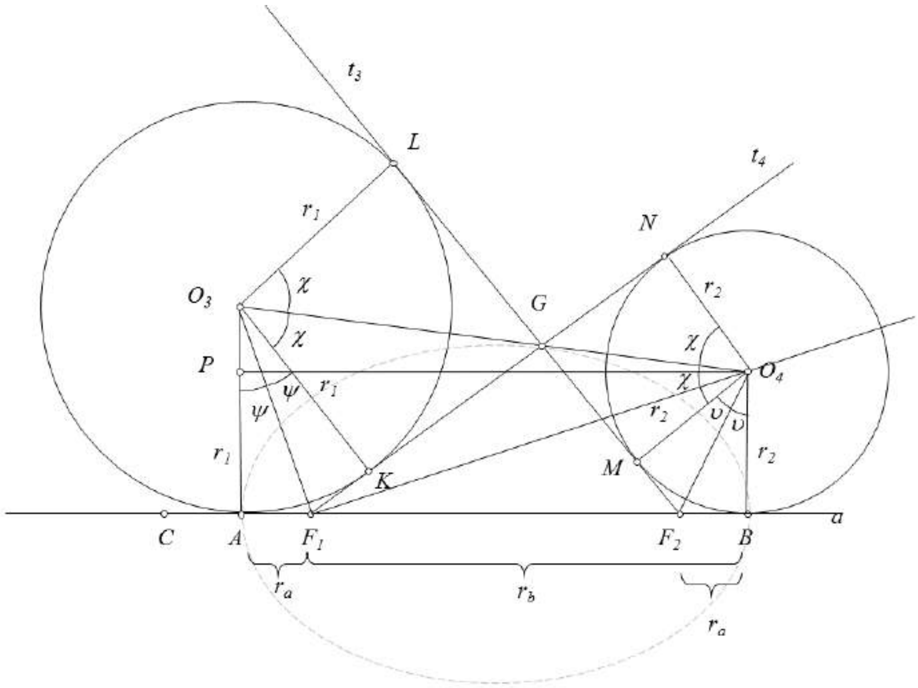

Figure 5: Section of the cone **Z** by plane ρ.

By lemma's assumption we have points A, B and F_1 given, then also the point F_2 is established because it is the vertex of the hyperbola. The vertex G of the cone is defined by section of two lines t_3 and t_4 which are elements of cone **Z**. The lines lie on plane ρ and go through points F_1 and F_2 and are tangent to two circles O_3 and O_4 respectively. The circles are sections of Dandelin's spheres (Figure 4) which are tangent to the plane τ. They are also tangent to line a at points A and B which are foci of the hyperbola $\mathcal{H}$. Let r_1 be the radius of the circle O_3. Then the point G is not unique and its position depends on the radius r_1. The radius r_1 may vary from zero to infinity: $0 < r_1 < \infty$. Hence, there exists infinite number of cones Z which generate the hyperbola if they are sectioned by the plane τ. □

The next step is finding the locus of vertices G. By analogy to the Proposition 1 we write following proposition:

Proposition 2. *If the hyperbola $\mathcal{H}$ defined by two vertices F_1, F_2 and focus A (or foci A, B and vertex F_1) is generated by section of the cones **Z** by plane τ then the locus of vertices G of the cones **Z** is an ellipse $\mathcal{E}$. The foci of the ellipse $\mathcal{E}$ are points F_1 and F_2, vertices are points A and B (ellipse foci).*

Proof:
We will look for the relationship between the distances from the vertex G to the points F_1 and F_2. First, we will consider the right triangle O_3PO_4 (Figure 5). Point P is the normal projection of point O_4 onto segment AO_3. By using

Pythagoras theorem we have

$$|O_3O_4|^2 = |AB|^2 + (r_1 - r_2)^2. \tag{7}$$

where r_2 is radius of the circle O_3. The triangles GKO_3 and GMO_4 are also right triangles because the points K and N are points of tangency of the lines t_3 and t_4 to the circles O_3 and O_4. Hence, one can write

$$|O_3G|^2 = |GK|^2 + r_1^2, \tag{8}$$

$$|O_4G|^2 = |GM|^2 + r_2^2. \tag{9}$$

Recalling that

$$|O_3O_4| = |O_3G| + |O_4G| \tag{10}$$

and substituting equations (8), (9) and (10) to equation (7) the following equation is obtained

$$\begin{aligned}(|KG| + |MG|)^2 - 2|KG||MG| + 2|O_3G||O_4G| = |AB|^2 - 2r_1r_2 = \\ = |AB|^2 - 2|O_4M||O_3K|\end{aligned} \tag{11}$$

When both sides of equation (11) are divided by $|O_4M||O_3K|$ we get

$$\frac{(|GK| + |GM|)^2}{2r_1r_2} - \left(\frac{|KG|}{|O_3K|}\frac{|GM|}{|O_4M|} - \frac{|O_3G|}{|O_3K|}\frac{|O_4G|}{|O_4M|}\right) = \frac{|AB|^2}{2r_1r_2} - 1. \tag{12}$$

The triangles O_3KG and O_4MG are similar because they are right triangles and and angles $\angle KGO_3$ and $\angle MGO_4$ are equal. The second statement is true because the triangles O_4MG and O_4NG are congruent and angles $\angle KGO_3$ and $\angle NGO_4$ are congruent as well (points G, O_3 and O_4 lie on the axis of the cone, hence the segments O_3G and O_4G are co-linear). Let the measure of angles $\angle KO_3G$ and $\angle MO_4G$ be χ. Simple trigonometrical relations based on Figure 5 yield:

$$\frac{|KG|}{|O_3K|} = \tan\chi = \frac{|GM|}{|O_4M|}, \tag{13}$$

$$\frac{|O_3G|}{|O_3K|} = \frac{1}{\cos\chi} = \frac{|O_4G|}{|O_4M|}. \tag{14}$$

The second term of left hand side of equation (12) can be simplified by use of the two relationship stated above

$$\left(\frac{|KG|}{|O_3K|}\frac{|GM|}{|O_4M|} - \frac{|O_3G|}{|O_3K|}\frac{|O_4G|}{|O_4M|}\right) = (\tan\chi)^2 - \frac{1}{(\cos\chi)^2} = -1. \tag{15}$$

Finally, we get the simple equation:

$$(|GK| + |GM|)^2 = |AB|^2 - 4r_1r_2. \tag{16}$$

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The next step is finding the product r_1r_2. Let's note that the angle $\angle AF_1O_3$ is equal to $\angle AF_1O_3$ and it is $\pi/2-\psi$. We may say the same about $\angle KF_1O_3$. This fact leads to conclusion that the angle $\angle NF_1B$ is equal to

$$\angle NF_1B = \pi - 2\angle AF_1O_3 = \pi - 2(\pi/2-\psi) = 2\psi. \tag{17}$$

Obviously, the line F_1O_4 is the bisector of this angle. Hence, the angle $\angle O_4F_1B$ is equal to ψ. Triangles F_1AO_3 and F_1BO_4 are similar and we may write the following proportion:

$$\frac{|O_3A|}{|F_1A|} = \frac{|F_1B|}{|O_4B|}. \tag{18}$$

Length of segment O_3A is r_1, lenght of segment O_4B is r_2. We may rewrite the above equation as

$$\frac{r_1}{r_a} = \frac{r_b}{r_2}. \tag{19}$$

Hence, $r_1r_2 = r_ar_b$.

We successfully arrived to the conclusion that the sum of length of segments GK and GM is constant

$$(|GK|+|GM|)^2 = |AB|^2 - 4r_ar_b = const. \tag{20}$$

$|AB|$ is equal to $r_a + r_b$ then

$$(|GK|+|GM|)^2 = (r_a+r_b)^2 - 4r_ar_b = (r_b-r_a)^2. \tag{21}$$

The fact that $r_a = |AF_1| = |F_1K| = |F_2M| = |BF_2|$ and equation (21) allows us to calculate the sum of the distances between vertex G and the foci F_1 and F_2

$$|GF_1|+|GF_2| = |GK|+r_a+|GM|+r_a = (|GK|+|GM|)+2r_a = r_b+r_a. \tag{22}$$

Hence, the sum of distances of the vertex G from foci F_1 and F_2 is constant ($|AB| = r_a + r_b$)

$$|GF_1|+|GF_2| = |AB|. \tag{23}$$

This equation is the simplest form of definition of the ellipse and we proved the proposition. □

Conclusions

We have shown the existence of a very interesting relationship between ellipse and hyperbola by use of very simple geometry. It was shown that ellipse and hyperbola are conjugate. This conjunction is created by locus of vertices of cones which generate the two conics. Although it seems to be a very basic property of the conics, surprisingly it is not mentioned even in some books devoted to conics geometry only. On the other hand it is wonderful that such simple mathematics may lead to such interesting results and express the beauty of geometry that is imperfectly shown in Figure 6.

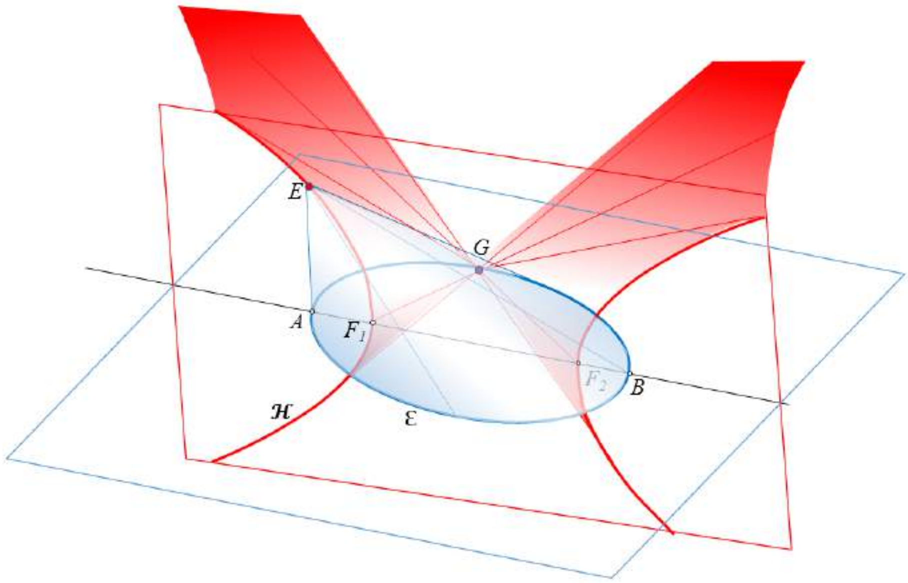

Figure 6: Conjugate ellipse $\mathcal{E}$ and hyperbola $\mathcal{H}$ as curves generated by section of cones whose vertices are located on these curves.

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Games and Puzzles

Reflections on the $n + k$ dragon kings problem

Doug Chatham
Morehead State University, Department of Mathematics and Physics
d.chatham@moreheadstate.edu

Abstract: *A* dragon king *is a shogi piece that moves any number of squares vertically or horizontally or one square diagonally but does not move through or jump over other pieces. We construct infinite families of solutions to the $n + k$* dragon kings problem *of placing k pawns and $n + k$ mutually nonattacking dragon kings on an $n \times n$ board, including solutions symmetric with respect to quarter-turn or half-turn rotations, solutions symmetric with respect to one or two diagonal reflections, and solutions not symmetric with respect to any nontrivial rotation or reflection. We show that an $n + k$ dragon kings solution exists whenever $n \geqslant k + 5$ and that, given some extra conditions, symmetric solutions exist for $n \geqslant 2k + 5$.*

Keywords: shogi, combinatorics, symmetry, n-queens problem.

Introduction

There are many problems in the mathematics literature that involve placing chess pieces on a board under given constraints. For example, the classic *n queens problem* asks for placements of n queens on an $n \times n$ board so that no two queens are on the same row, column, or diagonal [1]. The *$n + k$ queens problem* asks for placements of k pawns and $n + k$ queens on an $n \times n$ board so that each pair of queens on the same row, column, or diagonal has at least one pawn between them [5]. The n queens problem has a solution for $n = 1$ and each $n \geqslant 4$ [1]. For $k = 1, 2, 3$, the $n + k$ queens problem has a solution for each $n \geqslant k + 5$. For $k \geqslant 4$, the $n + k$ queens problem is known to have solutions for $n > 25k$, but it is suspected that $25k$ is much larger than the true lower bound [5].

In this paper we consider a simpler variation of the $n + k$ queens problem where each queen is replaced by a "dragon king", which is a shogi piece that moves any number of squares vertically and horizontally (but not through or over other pieces) and one square diagonally; i.e., like a combination of the chess king and rook [2]. So we consider the $n+k$ *dragon kings problem*: On an $n \times n$ board, can we place k pawns and $n+k$ dragon kings on the board so that no two dragon kings "attack" each other (i.e., no dragon king is one move away from any other dragon king)? We show in Section 1 that for each $k \geqslant 0$ and $n \geqslant k+5$, we can construct a solution to the $n+k$ dragon kings problem.

Many placements of pieces on a square board are symmetric with respect to rotation or reflection. In Section 1 we explore symmetric solutions to the $n+k$ dragon kings problem, and show that, given some extra conditions, solutions in each of the possible symmetry classes exist for $n \geqslant 2k+5$.

We conclude in Section 2 with a discussion of open questions and other avenues for further study.

In this paper all boards are square, with rows numbered, from top to bottom, $0, 1, \ldots n-1$ and columns numbered, from left to right, $0, 1. \ldots n-1$. The square in row r and column c of a board is denoted as *square* (r, c). In the figures of this paper, dragon kings are represented by ☖, pawns are represented by ☗, and squares that we wish to emphasize are empty are marked by $\circ$.

1 Results

In [5], a connection was established between the $n+k$ queens problem and *alternating sign matrices (ASMs)*, which are matrices consisting of 0s, 1s, and -1s where the nonzero elements alternate in sign and the first and last nonzero element of each row and column is a 1 [3]. We note a similar connection for the $n+k$ dragon kings problem. Suppose we have an arrangement of $n+k$ mutually nonattacking dragon kings on an $n \times n$ board with k pawns. If there are no pawns in a row (or column), that row (column) can have at most one dragon king. With each additional pawn on a row (column), the capacity of that row (column) increases by at most one. With k pawns, we can place at most $n+k$ mutually nonattacking dragon kings. Furthermore, to get that capacity of $n+k$ dragon kings, each available segment of each row and column must be occupied by a dragon king. So, the pieces in each row must alternate between dragon king and pawn and the first and last piece in each row or column must be a dragon king. Take a solution and create an $n \times n$ matrix A so that the entry a_{ij} in the i^{th} row and j^{th} column is 1 if there is a dragon king in the i^{th} row and j^{th} column of our solution, -1 if there is a pawn in the i^{th} row and j^{th} column of our solution, and 0 if the i^{th} row and j^{th} column of our solution is empty. That matrix is an alternating sign matrix. Solutions to the $n+k$ dragon kings problem correspond to ASMs for which no two 1s are diagonally adjacent.

In [5, Theorem 2] it was proved that for $n > 1$, no arrangement of k pawns and $n+k$ mutually nonattacking queens is symmetric with respect to reflection. The argument made against vertical and horizontal reflection works for dragon kings as well as queens, and we repeat that argument here.

Proposition 1. *(c.f. [5, Theorem 2]) No solution to the $n+k$ dragon kings problem (where $n > 1$) is symmetric with respect to vertical or horizontal reflection.*

Proof: Suppose we have a solution that is symmetric with respect to reflection across a vertical mirror. If n is even, then the dragon kings in the central columns must be adjacent and therefore attacking. If n is odd, then each square of the central column must be occupied; otherwise, the pieces closest to an empty square in the central column must be identical, contradicting the fact that the pieces in each row must alternate. The central column must have a dragon king on its first, third, and every other subsequent row. But then every square in the columns adjacent to the central column are attacked, and we cannot place a dragon king in those columns, contradicting the fact that each column must have at least one dragon king.

Switching each "row" and "column" in the previous argument gives us the argument against symmetry with respect to reflection across a horizontal mirror. ■

However, the argument against diagonal reflection fails. In Figure 4 we present an example of a $7+1$ dragon kings arrangement that is symmetric with respect to diagonal reflection.

We can therefore partition the set of $n+k$ dragon kings problem solutions into five symmetry classes:

1. *ordinary* arrangements that are not symmetric with respect to any (nontrivial) rotation or reflection. Figure 1 is ordinary.

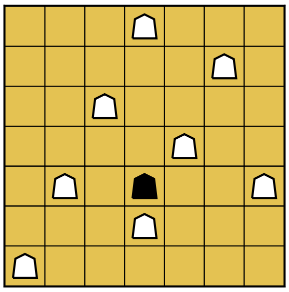

Figure 1: An ordinary $7+1$ dragon kings arrangement.

2. *centrosymmetric* arrangements that are symmetric with respect to half-turn rotation, but not quarter-turn rotations or any reflections. Figure 2 is centrosymmetric.

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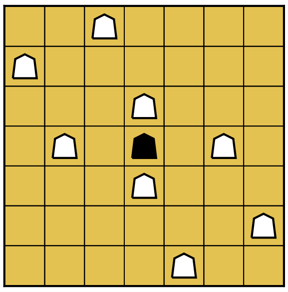

Figure 2: A centrosymmetric $7 + 1$ dragon kings arrangement.

3. *doubly centrosymmetric* arrangements that are symmetric with respect to quarter-turn rotations, but not with respect to reflections. Figure 3 is doubly centrosymmetric.

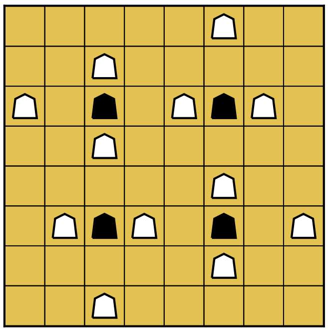

Figure 3: A doubly centrosymmetric $8 + 4$ dragon kings arrangement.

4. *monodiagonally symmetric* arrangements that are symmetric with respect to reflection across either the "main diagonal" that goes from the upper-left corner of the board to the lower-right corner or the "main antidiagonal" that goes from the lower-left corner of the board to the upper-right corner, but not both, and are also not symmetric with respect to nontrivial rotations. Figure 4 is monodiagonally symmetric.

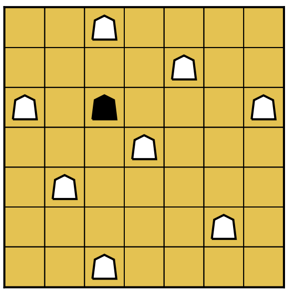

Figure 4: A monodiagonally symmetric $7 + 1$ dragon kings arrangement.

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5. *bidiagonally symmetric* arrangements that are symmetric with respect to reflection across both the main diagonal and main antidiagonal (and are therefore also symmetric with respect to half-turn rotations, but not quarter-turn rotations). Figure 5 is bidiagonally symmetric.

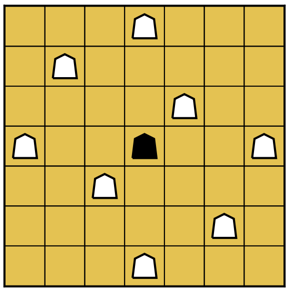

Figure 5: A bidiagonally symmetric $7+1$ dragon kings arrangement.

For each of these symmetry classes we construct families of solutions to the $n+k$ dragon kings problem and demonstrate that for each $k \geqslant 0$, (given extra conditions for the centrosymmetric, doubly centrosymmetric and bidiagonally symmetric classes) solutions exist for n large enough.

1.1 Ordinary solutions

We show by induction that there is at least one ordinary solution for each $k \geqslant 0$ and $n \geqslant k+5$. We start with two lemmas that will be used for the induction steps.

Lemma 2. *Given some* $n \geqslant 5$ *and* $k \geqslant 0$*, suppose we have a solution to the* $n+k$ *dragon kings problem with unoccupied squares* $(0,0), (n-1,0), (n-1,n-2), (n-1,n-1), (n-2,n-1),$ *and* $(n-3,n-1)$*. Then we can construct a solution to the* $(n+1)+(k+1)$ *dragon kings problem, with unoccupied squares* $(0,0), (n,0), (n,n-1), (n,n), (n-1,n),$ *and* $(n-2,n)$*.*

Proof: Add a new row n and column n to the given board and then place a pawn in square $(n-2,n-1)$ and dragon kings in squares $(n-2,n)$ and $(n,n-1)$, as illustrated in Figure 6. The new dragon kings do not attack each other or any of the previously placed dragon kings. We have an $(n+1)\times(n+1)$ board with $k+1$ pawns and $n+k+2$ mutually nonattacking dragon kings.

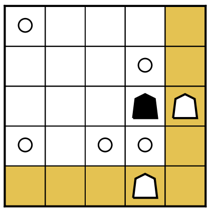

Figure 6: Illustration of first steps of the construction in Lemma 2. We add the shaded row and column to an ordinary arrangement and add two dragon kings and a pawn as shown.

Rotate the board a quarter-turn counterclockwise and relabel the squares so for each $0 \leqslant x, y \leqslant n$, square (x, y) becomes $(n-y, x)$, as illustrated in Figure 7. The empty squares that were $(0, n)$, $(0, 0)$, $(n-1, 0)$, $(n, 0)$, $(n, 1)$ and $(n, 2)$ are now labeled $(0,0)$,$(n,0)$,$(n,n-1)$, (n,n), $(n-1,n)$, and $(n-2,n)$, respectively. ■

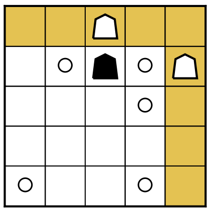

Figure 7: Board from Figure 6 after a quarter-turn rotation counterclockwise.

Lemma 3. *Given some $n \geqslant 1$ and $k \geqslant 0$, suppose we have an $n \times n$ board with an ordinary arrangement of k pawns and $n+k$ mutually nonattacking dragon kings. Then we can construct an ordinary arrangement of k pawns and $(n+1)+k$ mutually nonattacking dragon kings on an $(n+1) \times (n+1)$ board.*

Proof: First we show that at least two diagonally opposite corner squares are empty. Recall from the beginning of the section that our solution corresponds to an alternating sign matrix. In an alternating sign matrix, the first row (respectively, last row, first column, last column) has at most one 1 and no -1s [3]. So either the first or last entry in the first row (column) is a 0 and the corresponding square is empty. So at least two diagonally opposite corner squares must be empty.

Without loss of generality, suppose squares $(0, 0)$ and $(n-1, n-1)$ are empty. Then add a row n and column n and place a dragon king in the new square (n, n). We now have an $(n+1) \times (n+1)$ board with k pawns and $(n+1)+k$ mutually nonattacking dragon kings. Squares $(0, 0)$, $(0, n)$, and $(n, 0)$ are empty, so the arrangement is not symmetric with respect to quarter-turn or half-turn rotations or reflections across the diagonal from $(n, 0)$ to $(0, n)$. Also, the arrangement is not symmetric with respect to reflection across the diagonal from $(0, 0)$ to (n, n) since the upper-left $n \times n$ block is ordinary. ■

Proposition 4. *For $k \geqslant 0$ and $n \geqslant k+5$, we can construct an ordinary solution to the $n+k$ dragon kings problem.*

Proof: First we show by induction on k that for $k \geq 0$ and $n = k+5$, we can place k pawns and $n+k$ mutually nonattacking dragon kings on an $n \times n$ board. For the base case of $k = 0$, take a 5×5 board and place dragon kings on squares $(2,0)$,$(1,2)$,$(0,4)$,$(3,3)$, and $(4,1)$. These pieces do not attack each other, and squares $(0,0)$, $(4,0)$, $(4,3)$, $(4,4)$, $(3,4)$ and $(2,4)$ are empty. We use Lemma 2 for the induction step.

Next we show that these solutions are all ordinary. For $n \geqslant 5$, let B_n be the $n \times n$ solution generated in the previous paragraph. By inspection we can verify that B_5, B_6, and B_7 are ordinary. For $n \geqslant 8$, by induction we can show that the first and last row and column form a "hoop" with no pawns and four dragon kings on squares $(0, n-3)$, $(1, n-1)$, $(3, 0)$, $(n-1, 3)$. This hoop has no nontrivial rotational or reflective symmetries, so the arrangement on the whole board is ordinary.

Next we show by induction on $m \geqslant 0$ that for $n = k+5+m$ we can make an ordinary arrangement of k pawns and $n+k$ mutually nonattacking dragon kings on an $n \times n$ board. The base case is the previous paragraph. We use Lemma 3 for the induction step. ■.

The lower bound in Proposition 4 is not tight. For example, consider the ordinary $12+12$ dragon kings problem solution we get on an 12×12 board by placing pawns on squares $(1,4)$, $(3,7)$, $(4,2)$, $(5,6)$, $(5,9)$, $(6,1)$, $(6,3)$, $(7,10)$, $(8,2)$, $(8,5)$, $(9,8)$, and $(10,6)$ and dragon kings on squares $(0,4)$, $(1,2)$, $(1,7)$, $(2,5)$, $(3,3)$, $(3,9)$, $(4,1)$, $(4,6)$, $(5,4)$, $(5,8)$, $(5,10)$, $(6,0)$, $(6,2)$, $(6,6)$, $(7,9)$, $(7,11)$, $(8,1)$, $(8,3)$, $(8,7)$, $(9,5)$, $(9,10)$, $(10,2)$, $(10,8)$, and $(11,6)$. Using that example and arguments similar to those in the proof of Proposition 4, we can prove

Proposition 5. *For $k \geqslant 12$ and $n \geqslant k$, we can construct an ordinary solution to the $n+k$ dragon kings problem.*

1.2 Centrosymmetric solutions

We first note a restriction on the existence of centrosymmetric $n+k$ dragon kings problem solutions.

Proposition 6. *(c.f. [6, Proposition 2.1]) If n is even and k is odd, there are no centrosymmetric or bidiagonally symmetric $n+k$ dragon king arrangements.*

Proof. If n is even, the number of pawns in the left half must be equal to that in the right half. Therefore, the number of pawns must be even. ■

We next show that, if n and k are both even or n is odd, then for each $k \geqslant 0$ there are centrosymmetric $n+k$ dragon kings problem solutions for n sufficiently large.

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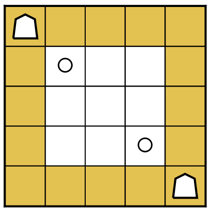

Figure 8: Illustration of the construction in Lemma 7. Given a centrosymmetric (or, respectively, monodiagonally symmetric, bidiagonally symmetric) solution in the unshaded region, we add two rows, two columns, and two dragon kings as shown to get a solution of the same symmetry class on a larger board.

Lemma 7. *Given a centrosymmetric (or, respectively, monodiagonally symmetric, bidiagonally symmetric) placement of p pawns and $n+k$ mutually nonattacking dragon kings on an $n \times n$ board, we can make a centrosymmetric (resp., monodiagonally symmetric, bidiagonally symmetric) placement of p pawns and $(n+2)+k$ mutually nonattacking dragon kings on an $(n+2) \times (n+2)$ board.*

Proof: On the $(n+2) \times (n+2)$ board, place the given arrangement in the middle $n \times n$ block. If squares $(1,1)$ and (n,n) are empty, place dragon kings on squares $(0,0)$ and $(n+1,n+1)$, as shown in Figure 8. Otherwise, place dragon kings on $(0,n+1)$ and $(n+1,0)$. In the hoop formed by the first and last row and column, the pieces are centrosymmetric and bidiagonally symmetric, and they do not attack each other or any dragon king in the middle $n \times n$ block. Since the middle block is centrosymmetric (resp., monodiagonally symmetric, bidiagonally symmetric), the entire arrangement is centrosymmetric (resp., monodiagonally symmetric, bidiagonally symmetric). ■

Lemma 8. *Given a centrosymmetric placement of p pawns and $n+k$ mutually nonattacking dragon kings on an $n \times n$ board, we can make a centrosymmetric placement of $p+2$ pawns and $(n+6)+k$ mutually nonattacking dragon kings on an $(n+4) \times (n+4)$ board.*

Proof: On the $(n+4) \times (n+4)$ board, place pawns on squares $(1,2)$ and $(n+2,n+1)$ and dragon kings on squares $(0,2)$, $(1,0)$, $(1,n+2)$, $(n+2,1)$, $(n+2,n+3)$, and $(n+3,n+1)$. Then, rotating if necessary, place the given arrangement in the middle $n \times n$ block of the $(n+4) \times (n+4)$ board so that squares $(2,n+1)$ and $(n+1,2)$ are empty. See Figure 9 for an illustration. Since the pieces in the hoop formed by the first two and last two rows and columns is centrosymmetric and attacks no piece in the center block, we conclude that the arrangement on the whole board satisfies the claimed properties. ■

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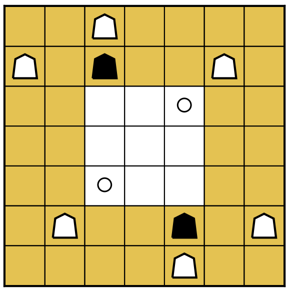

Figure 9: Illustration of the construction in Lemma 8. Given a centrosymmetric solution in the unshaded region, we add four rows, four columns, two pawns, and six dragon kings as shown to get a centrosymmetric solution on a larger board.

Lemma 9. *Suppose we have a centrosymmetric (or, respectively, doubly centrosymmetric, bidiagonally symmetric) $n+k$ dragon kings problem solution, where n is even. Then we can construct a centrosymmetric (resp., doubly centrosymmetric, bidiagonally symmetric) $(n+1)+k$ dragon kings problem solution.*

Proof: Suppose $n = 2m$. The central four squares of the given $n \times n$ board ($(m-1, m-1)$, $(m, m-1)$, $(m-1, m)$ and (m, m)) do not have any dragon kings in them, for if one of the squares had a dragon king, then another of the squares would have a dragon king by symmetry and those pieces would be mutually attacking.

On an $(n+1) \times (n+1)$ board, copy the upper-left (and respectively, lower-left, upper-right, lower-right) $m \times m$ block of the given arrangement onto the upper-left (resp.,lower-left, upper-right, lower-right) $m \times m$ block of the $(n+1) \times (n+1)$ board. Then place a dragon king on square (m, m) of the $(n+1) \times (n+1)$ board. This dragon king is on its own row and column and is not diagonally adjacent to any dragon king. We can check that the resulting arrangement (as illustrated in Figure 10) is an $(n+1)+k$ dragon kings solution in the claimed symmetry class. ■

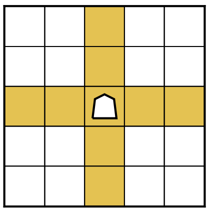

Figure 10: Illustration of Lemma 9. Given a centrosymmetric, doubly centrosymmetric, or bidiagonally symmetric solution on a $2m \times 2m$ board, we add a new central row and column and place a dragon king in the new central square to obtain a solution of the same symmetry class on a $(2m+1) \times (2m+1)$ board.

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Proposition 10. *If k is even, there is a centrosymmetric $n+k$ dragon kings problem solution for all $n \geqslant 2k+6$. If k is odd, there is a centrosymmetric $n+k$ dragon kings problem solution for all odd $n \geqslant 2k+5$.*

Proof: First we show by induction on k that if k is even, a centrosymmetric $n+k$ dragon kings problem solution exists for $n = 2k+6$. For the base case, we use the 6×6 board with dragon kings on squares $(0,1)$, $(1,3)$, $(2,5)$, $(3,0)$, $(4,2)$, and $(5,4)$. Then we use Lemma 8 for the inductive step. To show that a centrosymmetric solution exists for even $n = 2k+6+2m$, we use Lemma 7 for an induction on m. To show that a centrosymmetric solution exists for odd $n = 2k+7+2m$, apply Lemma 9 to the solutions generated in the previous sentence.

Now suppose k is even. We show that a centrosymmetric solution exists for odd $n = 2k+5$ by using as a base case ($k=1$) the 7×7 board with a pawn on square $(3,3)$ and dragon kings on squares $(0,1)$, $(1,3)$, $(2,6)$, $(3,2)$, $(3,4)$, $(4,0)$, $(5,3)$, and $(6,5)$ and Lemma 8 for the inductive step. To show that a centrosymmetric solution exists for odd $n = 2k+5+2m$, we use Lemma 7 for another induction on m. ■.

The bounds in Proposition 10 are not tight, as we can see with two examples:

1. the centrosymmetric $8+4$ dragon kings solution with pawns on squares $(2,2)$, $(3,6)$, $(4,1)$, and $(5,5)$ and dragon kings on squares $(0,2)$, $(1,6)$, $(2,1)$, $(2,3)$, $(3,5)$, $(3,7)$, $(4,0)$, $(4,2)$, $(5,4)$, $(5,6)$, $(6,1)$, and $(7,5)$, and
2. the centrosymmetric $9+5$ dragon kings solution with pawns on squares $(2,5)$, $(3,2)$, $(4,4)$, $(5,6)$, and $(6,3)$ and dragon kings on squares $(0,5)$, $(1,2)$, $(2,4)$, $(2,6)$, $(3,1)$, $(3,8)$, $(4,3)$, $(4,5)$, $(5,0)$, $(5,7)$, $(6,2)$, $(6,4)$, $(7,6)$, and $(8,3)$.

Using these examples and arguments similar to those in Proposition 10, we can prove

Proposition 11. *If $k \geqslant 4$ is even, there is a centrosymmetric $n+k$ dragon kings problem solution for all $n \geqslant 2k$. If $k \geqslant 5$ is odd, there is a centrosymmetric $n+k$ dragon kings problem solution for all odd $n \geqslant 2k-1$.*

1.3 Monodiagonally symmetric solutions

In this subsection we show that for each $k \geqslant 0$ we can construct a monodiagonally symmetric $n+k$ dragon kings problem solution for n large enough.

Lemma 12. *Given a monodiagonally symmetric placement of p pawns and $n+k$ mutually nonattacking dragon kings on an $n \times n$ board with empty corners, we can construct a monodiagonally symmetric placement of $p+2$ pawns on $n+6$ mutually nonattacking dragon kings on an $(n+4) \times (n+4)$ board with empty corners.*

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Proof: Without loss of generality, suppose the given arrangement is symmetric with respect to reflection across the main diagonal but is not symmetric with respect to reflection across the main antidiagonal. On an $(n+4)\times(n+4)$ board, place pawns on squares $(1, n+1)$ and $(n+1, 1)$ and dragon kings on squares $(0, n+1)$, $(1, 1)$, $(1, n+3)$, $(n+1, 0)$, $(n+2, n+2)$, and $(n+3, 1)$. In the central $n\times n$ block place the given arrangement. See Figure 11 for an illustration. The pieces in the hoop formed by the first two and last two rows and columns is symmetric with respect to reflection across the main diagonal but not the main antidiagonal, and the pieces attack nothing in the central block. Also the corners are empty. We conclude that the resulting arrangement on the full board satisfies the desired conditions. ■

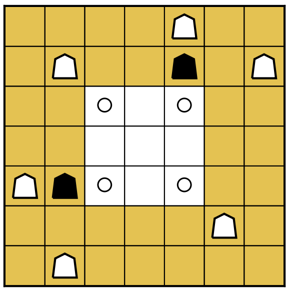

Figure 11: Illustration of Lemma 12. Given a monodiagonally symmetric solution in the central unshaded region, we add four rows, four columns, two pawns, and six dragon kings as shown to get a solution with the same symmetry on a larger board.

Proposition 13. *For every $k \geqslant 0$ there is a monodiagonally symmetric $n+k$ dragon kings problem solution for every $n \geqslant 2k+5$.*

Proof: We first show that the statement is true for $n = 2k+5$. For $k = 0$, consider the 5×5 board with dragon kings on squares $(0, 3)$, $(1, 1)$, $(2, 4)$, $(3, 0)$, and $(4, 2)$. For $k = 1$, consider the 7×7 board with a pawn on square $(2, 2)$ and dragon kings on squares $(0, 2)$, $(1, 4)$, $(2, 0)$, $(2, 6)$, $(3, 3)$, $(4, 1)$, $(5, 5)$, and $(6, 2)$. Then we use Lemma 12 for an induction on k.

Next we show that the statement is true for $n = 2k+6$. For $k = 0$, take the 6×6 board with dragon kings on squares $(0, 3)$, $(1, 5)$, $(2, 2)$, $(3, 0)$, $(4, 4)$ and $(5, 1)$. For $k = 1$, take the 8×8 board with a pawn on square $(5, 5)$ and dragon kings on squares $(0, 5)$, $(1, 1)$, $(2, 4)$, $(3, 6)$, $(4, 2)$, $(5, 0)$, $(5, 7)$, $(6, 3)$, and $(7, 5)$. We use Lemma 12 for another induction on k.

To show the assertion is true for $n = 2k+5+m$, use the solutions generated in the previous two paragraphs as base cases and use Lemma 7 for induction on m. ■

The bound in Proposition 13 is not tight, as we can see in the monodiagonally symmetric $9+3$ dragon kings problem solution with pawns on squares $(4, 4)$, $(5, 6)$, and $(6, 5)$ and dragon kings on squares $(0, 4)$, $(1, 7)$, $(2, 5)$, $(3, 3)$, $(4, 0)$, $(4, 6)$, $(5, 2)$, $(5, 8)$, $(6, 4)$, $(6, 6)$, $(7, 1)$, and $(8, 5)$.

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1.4 Doubly centrosymmetric solutions

We note some necessary conditions for doubly centrosymmetric $n + k$ dragon kings problem solutions.

Proposition 14. *(c.f. [6, Proposition 2.3]) Let $n \geqslant 1$ and $k \geqslant 0$ be integers for which there is a doubly centrosymmetric $n + k$ dragon kings arrangement. Then n and k must satisfy one of the following conditions:*

1. $n \equiv 0$ *(mod 4) and* $k \equiv 0$ *(mod 4)*
2. $n \equiv 1$ *(mod 4) and* $k \equiv 0$ *(mod 4)*
3. $n \equiv 3$ *(mod 4) and* $k \equiv 1$ *(mod 4)*

Proof: Since a doubly centrosymmetric arrangement is invariant under rotations of any integer multiple of quarter-turn rotations, if such an arrangement has a piece at square (a, b), it must also have pieces of the same type at squares $(b, n-1-a)$, $(n-1-a, n-1-b)$, $0 \leqslant a, b < n$, and $(n-1-b, a)$. Unless $(a, b) = (\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil)$ and n is odd, the four squares listed above are distinct. So, the number of dragon kings is congruent to either 0 or 1 (mod 4), and the number of pawns is also congruent to either 0 or 1 (mod 4).

The number of pawns in an $n + k$ dragon kings arrangement is k, so we must have $k \equiv 0$ (mod 4) or $k \equiv 1$ (mod 4). The number of dragon kings in an $n + k$ dragon kings arrangement is $n+k$, so either $n+k \equiv 0$ (mod 4) or $n+k \equiv 1$ (mod 4). If n is even, clearly $n + k \equiv 0$(mod 4) and $k \equiv 0$ (mod 4), so $n \equiv 0$(mod 4), and we have condition 1. If n is odd, and the middle square is empty, we have an even number of pawns and an even number of dragon kings, which leads to a contradiction with the parity of n. If n is odd and the middle square has a dragon king, then $n + k \equiv 1$ (mod 4) and $k \equiv 0$ (mod 4), so $n \equiv 1$ (mod 4) and we have condition 2. If n is odd and the middle square has a pawn, then $k \equiv 1$ (mod 4) and $n + k \equiv 0$ (mod 4), so $n \equiv 3$(mod 4) and we have condition 3. ■

If n and k follow the restrictions above, and n is sufficently large, then we can construct a doubly centrosymmetric $n + k$ dragon kings problem solution.

Lemma 15. *Given a doubly centrosymmetric placement of p pawns and $n + k$ mutually nonattacking dragon kings on an $n \times n$ board, we can make a doubly centrosymmetric placement of p pawns and $(n + 4) + k$ mutually nonattacking dragon kings on an $(n + 4) \times (n + 4)$ board.*

Proof: On the $(n+4) \times (n+4)$ board, place dragon kings on squares $(0, n+2)$, $(1, 0)$, $(n + 2, n + 3)$, and $(n + 3, 1)$. Then place the given arrangement on the $n \times n$ board obtained by removing the first two and last two rows and columns. We can check that the resulting arrangement, as illustrated in Figure 12, is a doubly centrosymmetric placement of k pawns and $(n+4)+k$ mutually nonattacking dragon kings on an $(n + 4) \times (n + 4)$ board. ■

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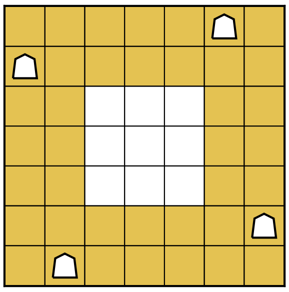

Figure 12: Illustration of the construction in Lemma 15. Given a doubly centrosymmetric solution in the central unshaded region, we add four rows, four columns, and four dragon kings as shown to get a doubly centrosymmetric solution on a larger board.

Lemma 16. *Given a doubly centrosymmetric placement of p pawns and $n+k$ mutually nonattacking dragon kings on an $n \times n$ board, we can make a doubly centrosymmetric placement of $p+4$ pawns and $(n+8)+(k+4)$ mutually nonattacking dragon kings on an $(n+8) \times (n+8)$ board.*

Proof: On the $(n+8) \times (n+8)$ board, place pawns on squares $(2,2)$, $(2,n+5)$, $(n+5,2)$ and $(n+5,n+5)$ and dragon kings on squares $(0,2)$, $(1,n+5)$, $(2,1)$, $(2,3)$, $(2,n+7)$, $(3,n+5)$, $(n+4,2)$, $(n+5,0)$, $(n+5,n+4)$, $(n+5,n+6)$, $(n+6,2)$, and $(n+7,n+5)$. Then place the given arrangement on the $n \times n$ board obtained by removing the first four and last four rows and columns. We can check that the resulting arrangement (illustrated in Figure 13) satisfies the desired conditions. ■

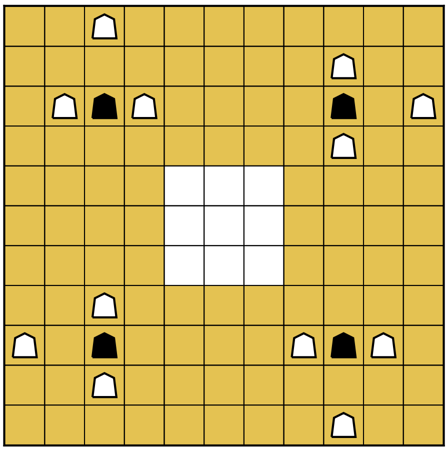

Figure 13: Illustration of the construction in Lemma 16. Given a doubly centrosymmetric solution in the central unshaded region, we add eight rows, eight columns, four pawns, and twelve dragon kings as shown to get a doubly centrosymmetric solution on a larger board.

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Proposition 17. *There is a doubly centrosymmetric $n+k$ dragon kings problem solution if n and k satisfy one of the following:*

1. $n \equiv 0$ *(mod 4), $n \geqslant 4$, and $k = 0$*

2. $n \equiv 0$ *(mod 4), $k \equiv 0$ (mod 4), $k \geqslant 4$, and $n \geqslant 2k$*

3. $n \equiv 1$ *(mod 4), $k \equiv 0$ (mod 4), and $n \geqslant 2k+1$*

4. $n \equiv 3$ *(mod 4), $k = 1$, and $n \geqslant 11$*

5. $n \equiv 3$ *(mod 4), $k \equiv 1$ (mod 4), $k \geqslant 5$, and $n \geqslant 2k+1$*

Proof: We consider the cases in order.

1. For $n = 4$ and $k = 0$ we use the 4×4 board with dragon kings on squares $(0,2)$, $(1,0)$, $(2,3)$, and $(3,1)$. We finish this case by induction using Lemma 15

2. Let $k = 4i$ for $i \geqslant 1$. We show there is a solution for $n = 2k = 8i$. For $i = 1$, we use the 8×8 board shown in Figure 3 with pawns on squares $(2,2)$, $(2,5)$, $(5,2)$, and $(5,5)$ and dragon kings on squares $(0,5)$, $(1,2)$, $(2,0)$, $(2,4)$, $(2,6)$, $(3,2)$, $(4,5)$, $(5,1)$, $(5,3)$, $(5,7)$, $(6,5)$, and $(7,2)$. For $i > 1$ we use Lemma 16 for an induction. We complete this case by inductively constructing solutions for $n = 8i + 4m$ for $m \geqslant 0$ using Lemma 15

3. For $n = 1$ and $k = 0$ we use the 1×1 board with a dragon king placed on the only square. For all other values of n and k in this case, take the solutions generated in the previous two cases and apply Lemma 9.

4. In this case, for $n = 11$ consider the 11×11 board with a pawn on square $(5,5)$ and dragon kings on squares $(0,8)$, $(1,6)$, $(2,0)$, $(3,5)$, $(4,1)$, $(5,3)$, $(5,7)$, $(6,9)$, $(7,5)$, $(8,10)$, $(9,4)$, and $(10,2)$. For $n = 11 + 4j$ use Lemma 15 inductively.

5. In this case, $n = 4i + 3$ for some $i \geqslant 1$ and $k = 4j + 1$ for some $j \geqslant 1$. We first show that a solution exists for $n = 2k+1 = 8j+3$. For $j = 1$ consider the 11×11 board with pawns on squares $(3,3)$, $(3,7)$, $(5,5)$, $(7,3)$, and $(7,7)$ and dragon kings on squares $(0,4)$, $(1,7)$, $(2,3)$, $(3,1)$, $(3,5)$, $(3,8)$, $(4,10)$, $(5,3)$, $(5,7)$, $(6,0)$, $(7,2)$, $(7,5)$, $(7,9)$, $(8,7)$, $(9,3)$, and $(10,6)$. For larger j we use induction with Lemma 16 for the inductive step. We complete the case by showing that a solution exists for $n = 2k + 1 + 4m$ by induction on m using Lemma 15. ■

1.5 Bidiagonally symmetric solutions

Recall that Proposition 6 restricts the values of n and k for which there are bidiagonally symmetric solutions. We show that for each $k \geqslant 0$, if n is sufficiently large and n and k satisfy the necessary conditions given in Proposition 6, then we can construct a bidiagonally symmetric $n + k$ dragon kings solution.

Lemma 18. *Given a bidiagonally symmetric placement of* p *pawns and* $n + k$ *mutually nonattacking dragon kings on an* $n \times n$ *board (with* $n \geqslant 3$*), we can make a bidiagonally symmetric placement of* $p + 4$ *pawns and* $(n + 8) + (k + 4)$ *mutually nonattacking dragon kings on an* $(n + 8) \times (n + 8)$ *board.*

Proof: On the $(n + 8) \times (n + 8)$ board place pawns on squares $(2, 5)$, $(5, 2)$, $(n+2, n+5)$, and $(n+5, n+2)$ and dragon kings on squares $(0, 2)$, $(1, 5)$, $(2, 0)$, $(2, n + 5)$, $(3, 3)$, $(5, 1)$, $(n + 2, n + 6)$, $(n + 4, n + 4)$, $(n + 5, 2)$, $(n + 5, n + 7)$, $(n + 6, n + 2)$, and $(n + 7, n + 5)$. Then place the given arrangement, or its horizontal reflection, on the central $n \times n$ block so that squares $(4, 4)$ and $(n + 3, n + 3)$ remain empty. We can check that the resulting arrangement (as illustrated in Figure 14) satisfies the desired properties. ■

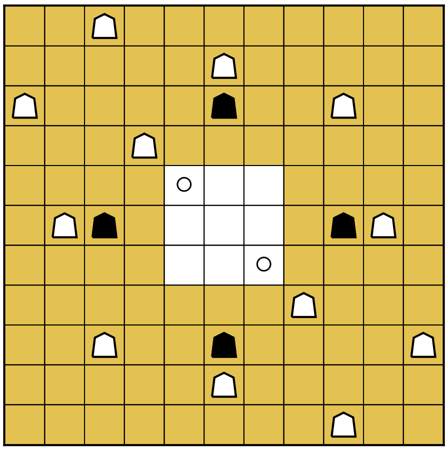

Figure 14: Illustration of the construction in Lemma 18. Given a bidiagonally symmetric solution in the unshaded central region, we add eight rows, eight columns, four pawns, and twelve dragon kings as shown to get a bidiagonally symmetric solution on a larger board.

Proposition 19. *For* k *even and* $n \geqslant 2k + 6$*, there is a bidiagonally symmetric* $n + k$ *dragon kings problem solution. For* n *and* k *odd and* $n \geqslant 2k + 5$*, there is a bidiagonally symmetric* $n + k$ *dragon kings problem solution.*

Proof: Suppose first that k is even. We show there is a bidiagonally symmetric $n+k$ dragon kings problem solution for $n = 2k+6$. For $k = 0$, consider the 6×6 board with dragon kings on squares $(0, 2)$, $(1, 4)$, $(2, 0)$, $(3, 5)$, $(4, 1)$, and $(5, 3)$. For $k = 2$, consider the 10×10 board with pawns in squares $(3, 6)$ and $(6, 3)$ and dragon kings in squares $(0, 3)$, $(1, 6)$, $(2, 4)$, $(3, 0)$, $(3, 8)$, $(4, 2)$, $(5, 7)$,$(6, 1)$, $(6, 9)$, $(7, 5)$, $(8, 3)$, and $(9, 6)$. We use Lemma 18 for the inductive step.

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Next we apply Lemma 7 to show there is a solution for $n = 2k + 6 + 2m$, where $m \geqslant 0$. To construct solutions for $n = 2k + 7 + 2m$, apply Lemma 9 inductively.

Now suppose k is odd. We next show there is a solution for $n = 2k + 5$. For $k = 1$, take the 7×7 board with a pawn in square $(3, 3)$ and dragon kings in squares $(0, 3)$, $(1, 1)$, $(2, 4)$, $(3, 0)$, $(3, 6)$, $(4, 2)$, $(5, 5)$, $(6, 3)$. For $k = 3$, take the 11×11 board with pawns in squares $(3, 7)$, $(5, 5)$ and $(7, 3)$ and dragon kings in squares $(0, 10)$,$(1, 7)$, $(2, 5)$, $(3, 3)$, $(3, 9)$, $(4, 6)$, $(5, 2)$, $(5, 8)$, $(6, 4)$, $(7, 1)$, $(7, 7)$, $(8, 5)$, $(9, 3)$, and $(10, 0)$. We again use Lemma 18 for the inductive step.

We again apply Lemma 7 to show there is a solution for $n = 2k + 5 + 2m$, where $m \geqslant 0$. ■

$n \backslash k$	0	1	2	3	4	5	6
4	2	0	0	0	0	0	0
5	14	0	0	0	0	0	0
6	90	32	0	0	0	0	0
7	646	762	124	0	0	0	0
8	5242	14412	9056	1688	94	0	0
9	47622	250326	380776	216678	48374	3540	0
10	479306	4252504	12538132	16006424	9629406	2790292	389100

Table 1: Number of $n + k$ dragon kings problem solutions for $4 \leqslant n \leqslant 10$ and $0 \leqslant k \leqslant 6$

2 Conclusion and Open Problems

We have shown that ordinary solutions to the $n + k$ dragon kings problem exist for $n \geqslant k+5$. If k is even or n is odd, centrosymmetric solutions and bidiagonally symmetric solutions exist for $n \geqslant 2k + 5$. Monodiagonally symmetric solutions exist for $n \geqslant 2k + 5$. If $k \geqslant 4$, doubly centrosymmetric solutions exist under any of the following three conditions:

1. $n \equiv 0 \pmod 4$, $k \equiv 0 \pmod 4$, and $n \geqslant 2k$
2. $n \equiv 1 \pmod 4$, $k \equiv 0 \pmod 4$, and $n \geqslant 2k + 1$
3. $n \equiv 3 \pmod 4$, $k \equiv 1 \pmod 4$, and $n \geqslant 2k + 1$

There are many unanswered questions provoked by the results in this paper, such as

1. Many of the lower bounds we have found are not tight. How much can we tighten the bounds? Furthermore, can we determine all the pairs (n, k) for which there are solutions in each of the symmetry classes to the $n + k$ dragon kings problem?

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2. Given n and k, how many solutions are there to the $n + k$ dragon kings problem?

   We know the number of solutions to the $n + 0$ dragon kings problem (originally called the "n kings problem" by Kaplansky [7]); it is given by sequence A002464 of the On-Line Encyclopedia of Integer Sequences [8]. Taking Kosters' ASM algorithm for counting solutions of the $n+k$ queens problem [6, Figure 6] and adapting it to the $n + k$ dragon kings problem, we obtain the numbers in Table 1.

3. What happens if we consider different types of board, such as cylinders, tori, or three-dimensional boards?

4. What solution patterns can we construct if we replace the dragon king with another piece, such as the k-step dragon king. defined in [4], which moves k squares diagonally or any number of squares vertically or horizontally?

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