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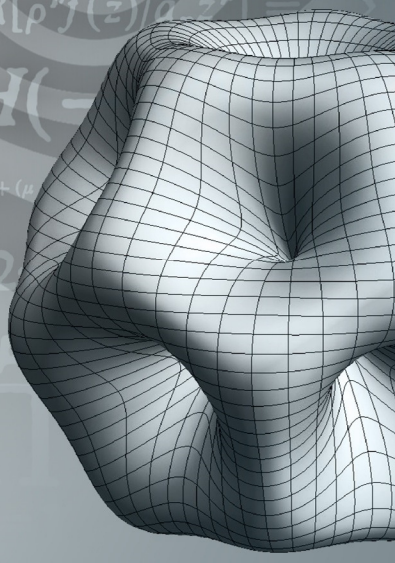

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Digital root patterns of three-dimensional space

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Abstract: *In this study, we define vedic cube as the layout of each digital root in a three-dimensional multiplication table. In order to discover the geometric patterns in vedic cube, we adopt two methods to analyze the digital root in a three-dimensional space. The first method is floor method, which divides vedic cube into several X-Y planes according to different Z values (floors) to analyze the geometric characteristics on each floor. The second method is symmetric plane method, which decomposes vedic cube by its main and secondary symmetric planes.*

Key-words: Digital root, pattern, vedic square, vedic cube, symmetry.

1 Introduction

1.1 Digital root

Exploring regularity and pattern of numbers is the most fascinating experience in mathematical research. The discovery of digital root is one of those surprises. The digital root of a positive integer is the summation of each digit ranging from one to nine. For example, the digital root of 19 is the addition of 1 and 9, repeat the procedure until the final number ranges from one to nine. After calculation, the digital root of 19 is 1, in other words, addition mod 9. $D(N)$ is used to denote the digital root of a positive integer N. Digital root operation of 19 is listed as follows:

$$D(19) = D(1 + 9) = D(10) = D(1) = 1. \tag{1}$$

Vedic square is one of the most famous regularities discovered many centuries ago [10]. It is the layout of digital roots in a typical multiplication table shown in Figure 1, and the digital root of a specific point $D(X, Y)$ in vedic square can be calculated as

$$D(X, Y) = D(X \times Y) = D(D(X) \times D(Y)). \tag{2}$$

X and Y are any positive integers.

1.2 Vedic square and Islamic culture

Quran has profound influence in Islamic culture in perceiving values of life and expressions of art. Quran an prohibits the representation of the human figure in art; therefore Muslims perform their art in a sophisticated way using many geometric patterns to decorate Quran, architecture, pottery, and apparel. Muslims have incorporated vedic square into their mathematical system since 770 AD [10].

1	2	3	4	5	6	7	8	9
2	4	6	8	1	3	5	7	9
3	6	9	3	6	9	3	6	9
4	8	3	7	2	6	1	5	9
5	1	6	2	7	3	8	4	9
6	3	9	6	3	9	6	3	9
7	5	3	1	8	6	4	2	9
8	7	6	5	4	3	2	1	9
9	9	9	9	9	9	9	9	9

Figure 1: vedic square. It is the layout of digital roots in a 2-D multiplication table.

Some geometric characteristics in vedic square are including symmetry and complementation in digits, patterns in columns and rows. Figure 2 shows the patterns formed by each digital root, digital roots with summation of nine mirror each other along vertical line, including digit 1 versus 8, digit 2 versus 7, digit 3 versus 6, and digit 4 versus 5. vedic square can also be used to produce a new pattern, which is composed by a line of numbers from specific column or row in vedic square with a constant angle [7]. Furthermore, there are several literatures discussing geometric patterns of Islamic art [1, 11].

1.3 Element: from 2-D to 3-D space

Designer Cecil Balmond uses many informal concepts in designing architectural structure. In Serpentine Gallery Pavilion 2002 project designed by Toyo Ito, Cecil Balmond and Arup [6], Balmond used a simple element, a square, to develop specific half-to-a-one-third algorithm. The structure form was created by lining up the middle point of one side to the one-third point of adjacent side.

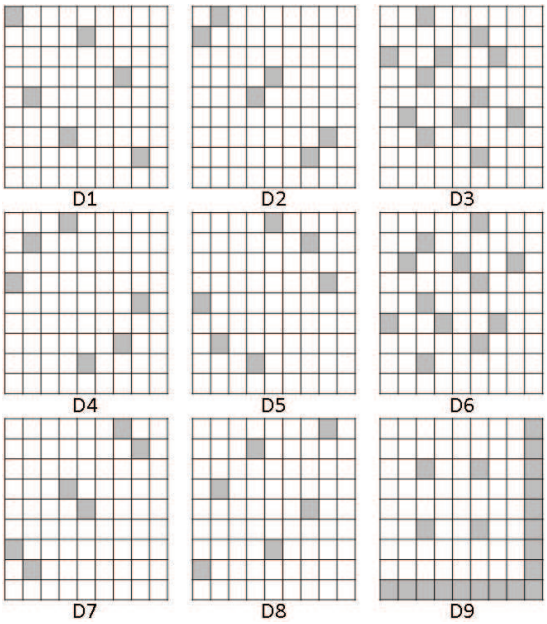


Figure 2: Patterns in vedic square. the patterns formed by each digital root, digital roots with summation of nine mirror each other along vertical line, including digit 1 versus 8, digit 2 versus 7, digit 3 versus 6, and digit 4 versus 5.

Repeat the procedure at four sides of the square and stretch the sides out of the original square, then a smaller square compared to the original one can be generated. Follow this rule, the result footprint is shown in Figure 3 [5]. Part of the footprint is selected as the constructing element of structures roof, the corners are cut, and the remaining part is folded as buildings supporting structure to form a box. Serpentine Gallery Pavilion 2002 is a great project developed from a simple element to a delicate masterpiece and provides the insight from 2-D to 3-D space. Balmond's works are often inspired by nature element, Islamic art, and numbers [3, 4].

So far, many studies have discussed about the digital root patterns in 2-D plane, and their characteristics in vedic square [2, 7, 9, 10]. However, there have been little discussions about digital root patterns in 3-D space. This study, inspired by the beauty of Islamic art [12] and Serpentine Gallery Pavilion 2002, aims to explore those undiscovered characteristics of digital roots, and the corresponding patterns in the 3-D space.

2 Vedic cube

Inspired by both Islamic art and Serpentine Gallery Pavilion 2002, this study assumes that there are more interesting digital root patterns in 3-D space yet to be uncovered. We create a three-dimensional multiplication table by adding Z-axis on the original one for further discussion, and also define a vedic cube

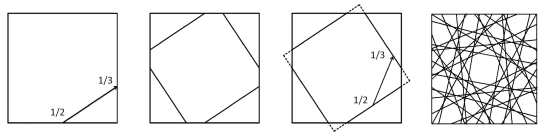


Figure 3: The half-to-a-one-third algorithm concept of Serpentine Gallery Pavilion 2002 (modified) [5].

as the digital root in a 3-D space created in this study. The side length of vedic cube is 9, and there are 729 points in it. Let $D(X, Y, Z)$ denote the digital root of a specific point (X, Y, Z) in vedic cube, which

$$D(X, Y, Z) = D(X \times Y \times Z) = D(D(X) \times D(Y) \times D(Z)). \tag{3}$$

As same as ´vedic square, X, Y and Z are any positive integers. An example of digital root operation is listed as follows: the value of point (2,3,5) in vedic cube is

$$D(2 \times 3 \times 5) = D(30) = D(3) = 3. \tag{4}$$

In this study, digital root M, digit M and $D(X, Y, Z) = M$ all denote digital root operational result.

In order to find out the digital root distribution in a 3-D space, Matlab software is used to calculate the digital root value of each number in vedic cube and plot specific digital roots. Figure 4 shows all points of digital root 1 scatter in vedic cube. Distribution of digital root 2, 3, 4 and 9 are shown in the Figure A-1 of appendices. Due to the difficulty in recognizing digital root patterns in 3-D space, we adopt two methods in this study to analyze and simplify the complexity of digital root patterns in vedic cube.

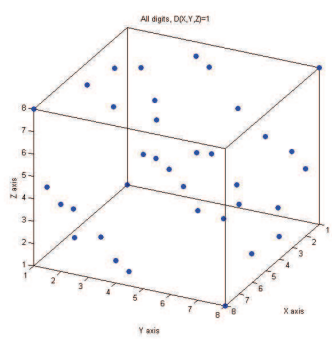


Figure 4: All points of digital root 1 in vedic cube. The pattern is too intricate to be recognized immediately.

3 Analysis methods, findings and explanations

3.1 Floor method: dividing vedic cube into floors

The first method is dividing vedic cube into several layers according to different Z value (X value or Y value is also feasible), which is similar to the floors of a building ranging from 1 to 9. Figure 5 illustrates all the digital roots on each floor. The order of the first floor ($F1$) to the ninth floor ($F9$) is defined from left to right, top to bottom. For example, the upper middle square in Figure 5 is the second floor with corresponding code defined as $F2$. The rest of the floors also possess exactly the same rule.

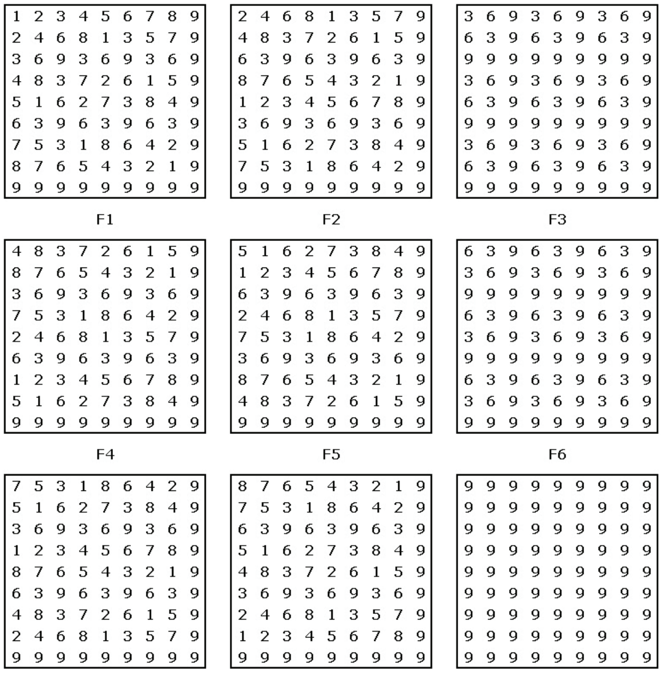
According to Figure 5, the digital root distribution can be categorized into three groups of digit. The group A of digit contains digit 1, 2, 4, 5, 7 and 8. Use digit 1 as example, each floor of $F1$, $F2$, $F4$, $F5$, $F7$, and $F8$ (Group α of floor) contains 6 points of digit 1. There are total 36 points of digit 1, six digits occupy 216 points in vedic cube within the defined 3-D space. The group B of digit contains digit 3 and 6. Use digit 3 as example, each floor of group α contains 12 points of digit 3. Each floor of $F3$ and $F6$ (Group β) contains 18 points of digit 3. There are total 108 points of digit 3. The group B of digit occupies 216 points in vedic cube within the defined 3-D space. The group C of digit is digit 9. Each floor of group α contains 21 points of digit 9. Each floor of group β contains 45 points of digit 9, and $F9$ (Group γ) contains 81 points. The group C of digit occupies 297 points in vedic cube. In total, the summation of all points of each digit is 729, which is equal to the total number of points in vedic cube.

3.2 Code definition of basic patterns and columns

In order to easily explain specific patterns, we create a code system to represent each basic pattern. The basic pattern has certain connection with digit's order of position occurring in vedic square. For example, pattern consisted of digit 1 on vedic square is defined as $D1F1$ with abbreviation as $D1$. Same rule is also used for digit 2 to digit 9 as $D2F1$, $D3F1$, $D4F1$, $D5F1$, $D6F1$, $D7F1$, $D8F1$ and $D9F1$, respectively. However, except the first floor ($F1$), the code of floor cannot be ignored in other floors. For example, the pattern of digit 1 on the second floor ($F2$) is defined as $D1F2$, pattern of digit 1 on third floor ($F3$) is defined as $D1F3$, etc. Furthermore, the column and row on vedic cube are also defined for further discussion. $C2F1$ denotes the second column ($C2$) of $F1$, which is consisting of nine numbers 2, 4, 6, 8, 1, 3, 5, 7, 9, as shown in Figure 5. Numbers in column and row are identical on vedic cube. Definition of Ch represents for both basic h^{th} column and row in vedic square, which is similar to the rule of basic patterns.

3.3 Characteristics of Digital root patterns in 3-D space

Several geometric characteristics of same digit can be found on the different floors. Take digit 1 for example, $D1F2$ (pattern on the second floor) is identical to $D5F1$, and $D1F7$ (pattern on the seventh floor) is identical to $D4F1$ (Figure 5). They mirror each other along the vertical line of projection onto the $X-Y$



1	2	3	4	5	6	7	8	9
2	4	6	8	1	3	5	7	9
3	6	9	3	6	9	3	6	9
4	8	3	7	2	6	1	5	9
5	1	6	2	7	3	8	4	9
6	3	9	6	3	9	6	3	9
7	5	3	1	8	6	4	2	9
8	7	6	5	4	3	2	1	9
9	9	9	9	9	9	9	9	9

F1

2	4	6	8	1	3	5	7	9
4	8	3	7	2	6	1	5	9
6	3	9	6	3	9	6	3	9
8	7	6	5	4	3	2	1	9
1	2	3	4	5	6	7	8	9
3	6	9	3	6	9	3	6	9
5	1	6	2	7	3	8	4	9
7	5	3	1	8	6	4	2	9
9	9	9	9	9	9	9	9	9

F2

3	6	9	3	6	9	3	6	9
6	3	9	6	3	9	6	3	9
9	9	9	9	9	9	9	9	9
3	6	9	3	6	9	3	6	9
6	3	9	6	3	9	6	3	9
9	9	9	9	9	9	9	9	9
3	6	9	3	6	9	3	6	9
6	3	9	6	3	9	6	3	9
9	9	9	9	9	9	9	9	9

F3

4	8	3	7	2	6	1	5	9
8	7	6	5	4	3	2	1	9
3	6	9	3	6	9	3	6	9
7	5	3	1	8	6	4	2	9
2	4	6	8	1	3	5	7	9
6	3	9	6	3	9	6	3	9
1	2	3	4	5	6	7	8	9
5	1	6	2	7	3	8	4	9
9	9	9	9	9	9	9	9	9

F4

5	1	6	2	7	3	8	4	9
1	2	3	4	5	6	7	8	9
6	3	9	6	3	9	6	3	9
2	4	6	8	1	3	5	7	9
7	5	3	1	8	6	4	2	9
3	6	9	3	6	9	3	6	9
8	7	6	5	4	3	2	1	9
4	8	3	7	2	6	1	5	9
9	9	9	9	9	9	9	9	9

F5

6	3	9	6	3	9	6	3	9
3	6	9	3	6	9	3	6	9
9	9	9	9	9	9	9	9	9
6	3	9	6	3	9	6	3	9
3	6	9	3	6	9	3	6	9
9	9	9	9	9	9	9	9	9
6	3	9	6	3	9	6	3	9
3	6	9	3	6	9	3	6	9
9	9	9	9	9	9	9	9	9

F6

7	5	3	1	8	6	4	2	9
5	1	6	2	7	3	8	4	9
3	6	9	3	6	9	3	6	9
1	2	3	4	5	6	7	8	9
8	7	6	5	4	3	2	1	9
6	3	9	6	3	9	6	3	9
4	8	3	7	2	6	1	5	9
2	4	6	8	1	3	5	7	9
9	9	9	9	9	9	9	9	9

F7

8	7	6	5	4	3	2	1	9
7	5	3	1	8	6	4	2	9
6	3	9	6	3	9	6	3	9
5	1	6	2	7	3	8	4	9
4	8	3	7	2	6	1	5	9
3	6	9	3	6	9	3	6	9
2	4	6	8	1	3	5	7	9
1	2	3	4	5	6	7	8	9
9	9	9	9	9	9	9	9	9

F8

9	9	9	9	9	9	9	9	9
9	9	9	9	9	9	9	9	9
9	9	9	9	9	9	9	9	9
9	9	9	9	9	9	9	9	9
9	9	9	9	9	9	9	9	9
9	9	9	9	9	9	9	9	9
9	9	9	9	9	9	9	9	9
9	9	9	9	9	9	9	9	9
9	9	9	9	9	9	9	9	9

F9

Figure 5: All digits from first floor ($F1$) to ninth floor ($F9$) in floor method analysis. The order of floors is from left to right, top to bottom.

plane (also for $Y-Z$ and $X-Z$ planes). Also, three pairs of patterns: $D1F1$ and $D1F8$, $D1F3$ and $D1F6$, $D1F4$ and $D1F5$, all possess exactly the same characteristics.

Since patterns of $F5$ to $F8$ can be mirrored along the vertical line onto $X-Y$ plane from $F1$ to $F4$, this part of the study focuses on analyzing $F1$ to $F4$. Patterns of digit 9 are easily to predict. For group α of floor, they occur on positions either when both X and Y are multiples of 3, or one of X and Y is 9. For group β, they occur when either X or Y is multiple of 3. For group γ ($F9$), all digits are 9 shown in Figure A-1(d) of appendices.

The study finds that patterns of specific digit on different floors in 3-D space follow special rule which can be indicated from vedic square (Figure 5 and Figure 6). Here is an example of digit 1, rule of digit 1 varying according to different floors is shown in Figure 6(a). Pattern in Figure 6(a) refers to which basic pattern (Figure 2) is identical to pattern of digit 1 on the Xth floor (FX).

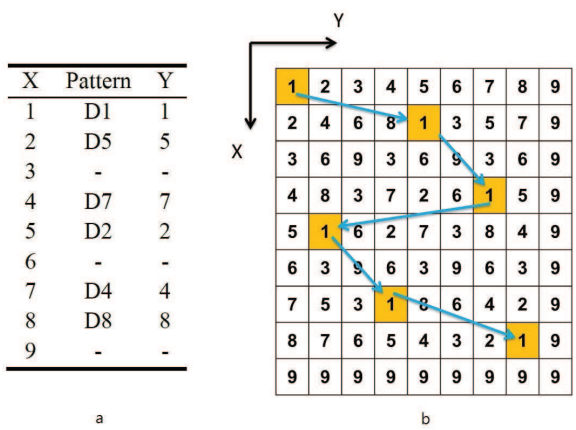
In Figure 6(b), when X is 2, digit 1 occurs on the fifth position (Y=5). X refers to the second floor, while Y represents the order of patterns is 5 in Figure 6(a). It is concluded that $D1F2$ pattern is identical to $D5F1$ pattern. From Figure 6(b), the patterns of digit 1 on $F4$, $F5$, $F7$, and $F8$ can be indicated from $D7$, $D2$, $D4$, and $D8$ in vedic square.

The explanation of this can be traced back to digit 5 in vedic square, because only the product of 5 and 2 can result in digit 1, it perfectly explains why $D1F2$ is identical to $D5F1$. Other patterns on different floors also follow the same principle.

To check whether pattern of digit p on the qth floor ($DpFq$) is the same as pattern of digit r on the sth floor ($DrFs$), the following equation is used to identify:

$$\text{If } D(p \times s) = D(q \times r), \text{ then } DpFq = DrFs. \tag{5}$$

When s=1, Dr, which represents the corresponding basic pattern of $DpFq$, can be identified. Digits occupying Dr's position on Fq is digit p. In other words, digital root patterns of 3-D space can be identified from 2-D plane. vedic square is not just a part of vedic cube, it is the fundamental basis of digital root patterns in 3-D space.



X	Pattern	Y
1	D1	1
2	D5	5
3	-	-
4	D7	7
5	D2	2
6	-	-
7	D4	4
8	D8	8
9	-	-

a

1	2	3	4	5	6	7	8	9
2	4	6	8	1	3	5	7	9
3	6	9	3	6	9	3	6	9
4	8	3	7	2	6	1	5	9
5	1	6	2	7	3	8	4	9
6	3	9	6	3	9	6	3	9
7	5	3	1	8	6	4	2	9
8	7	6	5	4	3	2	1	9
9	9	9	9	9	9	9	9	9

b

Figure 6: Rule of digit 1 is varying according to different floors. (a) Pattern refers to which basic pattern (Figure 2) is identical to pattern of digit 1 on the Xth floor (FX). (b) Patterns of specific digit on different floors in 3-D space follow special rule which can be indicated from vedic square. When X is 2, digit 1 occurs on the fifth position (Y=5). X refers to the second floor, while Y represents the order of patterns is 5 in Figure 6(a).

The pattern in certain floor might include one or more basic patterns. For example, $D3F3$ is composed of three different basic patterns shown in Figure 7. When X equals 3, the corresponding Y includes 1, 4 and 7 as shown in Figure 7(a). It can be concluded that $D3F3$ is composed by $D1$ (yellow), $D4$ (green) ,and $D7$ (blue) patterns shown in Figure 7(b). It can be expressed as $D3F3 = D1F1{+}D4F1{+}D7F1$. When X equals 6, the corresponding Y includes 2, 5 and 8, the equation can be expressed as $D3F6 = D2F1 + D5F1 + D8F1$. Similar composition in basic patterns also occurs in $D3F6$, $D6F3$, $D6F6$, $D9F3$, $D9F6$, and $D9F9$. $D9F9$ is composed by all basic patterns with digit 9.

Vedic square indicates not only the variation rule of digits patterns but also column and row on different floors in 3-D space. For example, the order of

column and row order on $F2$ are multiplying $F1$ by 2, which is equal to the number on the second column ($C2$) of $F1$, which is 2, 4, 6, 8, 1, 3, 5, 7, 9. First column ($C1$) on $F2$ is equal to $C2$ on $F1$, the relation can be expressed as $C1F2 = C2F1$. Columns and rows on specific floors follow this characteristic. To check whether h^{th} column (Ch) on i^{th} floor (Fi) is identical to j^{th} column (Cj) on k^{th} floor (Fk), the following equation is used to identify:

$$\text{If } D(hi) = D(jk), \text{ then } ChFi = CjFk. \tag{6}$$

When k=1, Cj, which represents the corresponding basic column of $ChFi$, can be identified.

This rule can also be applied to identify the order of floors in multi-dimensional space. For instance, there are 9 digital root cubes assumed in 4-D space. These 9 cubes are composed of multiplying vedic cube by 1 to 9. The value of second cube is multiplying digits of vedic cube by 2 and processes to digital root operation. The order of floors in second cube is $F2$, $F4$, $F6$, $F8$, $F1$, $F3$, $F5$, $F7$, $F9$ in vedic cube. The rule order of floors can be indicated from the column number of vedic square. The numbers of column on vedic square are very important basis for analyzing multi-dimensional space. In fact, variations from point to point, column to column, plane to plane, and cube to cube all possess similar rule. Noted that rules are different in the equation (5) of digital root patterns on specific floors.

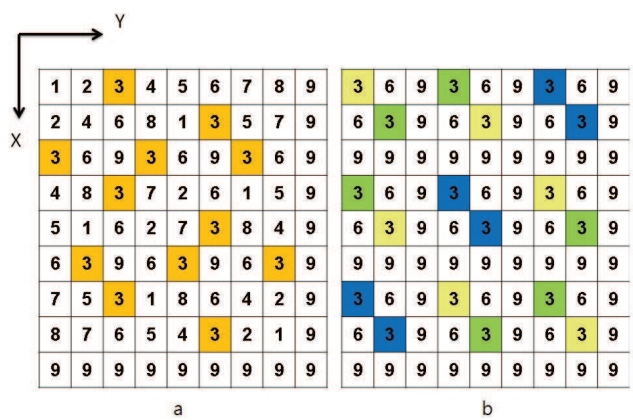


Figure 7: $D3F3$ is composed of three different basic patterns. (a) When X equals to 3, the corresponding Y is 1, 4 and 7, that means $D3F3$ is composed of $D1$, $D4$ and $D7$. And when X equals to 6, the corresponding Y is 2, 5 and 8. (b) $D3F3$ is composed of $D1$ (yellow), $D4$ (green) and $D7$ (blue) patterns.

3.4 Compensation of Patterns

This paragraph explains why patterns mirror each other along vertical line when digits summation in 9. There is a common geometric center for digit 1 to digit 8 (group A and B), that is point (4.5, 4.5, 4.5). In order to explain the symmetric characteristics of patterns, this part only discusses from digit 1 to digit 8 on both X and Y-axis. Figure 8(a) shows a diagonal inclining from left to right, which is

the main diagonal $X = Y$. Similarly, Figure 8(b) shows the other symmetric line inclining from right to left, which is the secondary diagonal $X + Y = 9$. Digits mirror each other along the main diagonal. In multiplication table, multiplying X by Y equals to multiplying Y by X, which can explain why digits mirror each other along the main diagonal. Furthermore, digits also mirror each other along secondary diagonal shown in Figure 8(b).
Before explaining the secondary diagonal, the basic operational rule of digital root should be clarified. In general, there are several basic operational rules of digital root, including addition rule:

$$D(X + Y) = D(D(X) + D(Y)). \tag{7}$$

Whether or not it can be applied in subtraction rule is another interesting topic. For example, 22 minus 15 equals 7, if their individual digital roots are calculated first, the result will be -2, which is a negative digital root. Calculation result for two equations should result in the same calculation outcome as $D(-2) = D(7) = 7$, because both -2 and 7 have the same remainder 7 when divided by 9 in modular arithmetic, which can be written as follow :

$$-2 \equiv 7 \pmod 9. \tag{8}$$

Ghannam (2012) assumes that $D(-51) = D(6) = 6$, and suggests ignoring the negative sign of the integer [9]. The digital root of a negative integer is calculated by adding it with multiple of 9 until the new number ranges from 1 to 9. The reason is because the digital root of an integer will remain the same if it adds or minuses the multiple of 9. Therefore, the digital root of -51 should be rewrite as follows:

$$D(-51) = D(-6) = D(3) = 3. \tag{9}$$

To explain the concept of the digital root of negative integer, digit 1 to 8 in vedic square are replaced by -8 to -1 in Y-axis, while X remains the same, see Figure 8(b). The original secondary diagonal becomes main diagonal. The characteristic of digit 8 mirroring each other along secondary diagonal is exactly like digit 1 mirroring along main diagonal. That can explain why two patterns mirror each other along vertical line when the digits summation is 9. It is important to note that this research only focuses on discussing digital root of positive integers. Negative digital root is used to help readers understand the symmetry in patterns, and to clarify the proper operation of negative digital root.

Here is another explanation why value of new coordinate point is identical to the original coordinate point $P(X, Y)$ from algebraic perspective. Value of any new coordinate point $P^*(9-Y, 9-X)$, which is the projection point of (X, Y) to secondary diagonal, is $D(P)$ in vedic square, and the difference of value between $D(P^*)$ and $D(P)$ is multiple of 9, which is shown as follows:

$$D(P^*) - D(P) = (81 - 9X - 9Y + XY) - (XY) = 81 - 9X - 9Y = 9k, k \in Z. \tag{10}$$

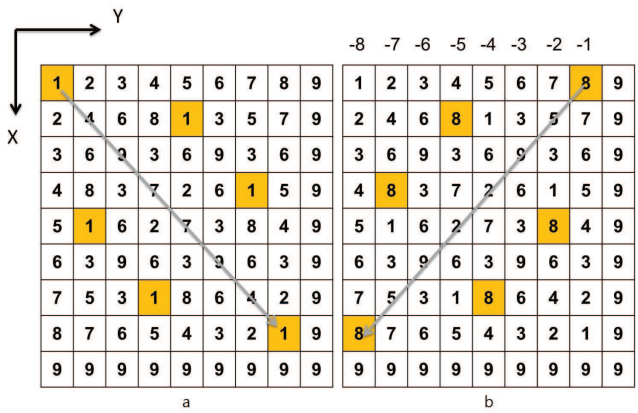


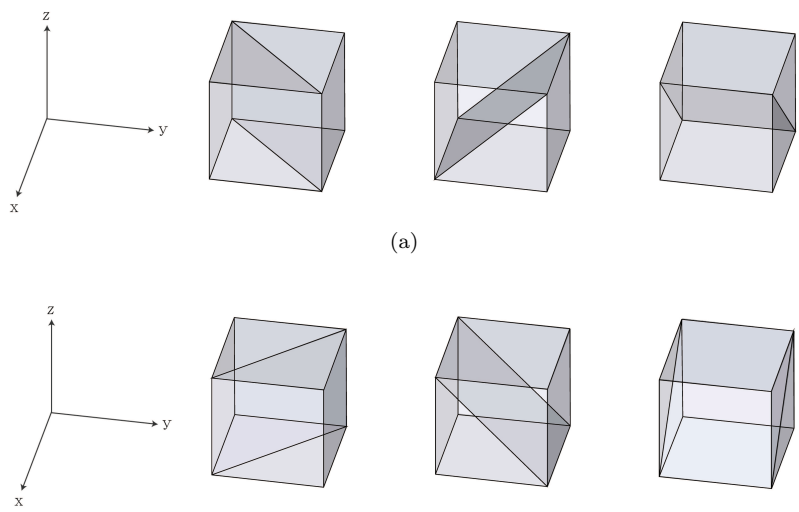
Figure 8: Digits mirror each other along the main and secondary diagonals on the vedic square. (a) Digit 1 mirrors each other along the main diagonal. (b) Digit 1 to 8 in vedic square is replaced by -8 to -1 in Y-axis, while X remains the same. The original secondary diagonal becomes main diagonal. The characteristic of digit 8 mirroring each other along secondary diagonal is exactly like digit 1 mirroring along main diagonal.

If value of the ninth column ($C9$) and the ninth row mirror along the secondary diagonal, the new points will become $(9-Y, 0)$, and $(0, 9-X)$, respectively. The two group of points are outside the range of vedic square. In order to maintain the consistency and symmetric characteristic of patterns, this research mainly considers from digit 1 to digit 8 on all three axes in 3-D space. However, for the recreational reason, we still provide digit 9 scattering plot with different boundary conditions in the appendices.

Floor method is used to analyze the characteristics on each floor and to identify the correlation with vedic square. However, it is important to discover those unseen pattern from a 3-D perspective. Therefore, we develop another methodology to analyze vedic cube and the digital root patterns in 3-D space.

3.5 Symmetric plane method

We use the second method to decompose vedic cube by two symmetric planes: Main symmetric planes and Secondary symmetric planes. Main symmetric planes (MSP) contain plane $X = Y$, $Y = Z$, and $X = Z$, each of them divides vedic cube into two grey blocks as shown in Figure 9(a). Secondary symmetric planes (SSP) contain $X + Y$=9, $Y + Z$=9, and $X + Z$=9, each of them also divides vedic cube into two grey blocks as shown in Figure 9(b). In both Figure 9(a) and 9(b), every grey block can mirror once to become vedic cube. It has nothing to do with choosing which side of the block as symmetric plane. Also, vedic cube can be decomposed into four prisms by MSP and SSP on $X - Y$, $Y - Z$, and $X - Z$ planes. The base of the prism is an isosceles right triangle with area equal to one-fourth of the square. To observe the digital root patterns in 3-D space, these blocks are intersected and analyzed using MSP and SSP within boundary conditions.



(a)

(b)

Figure 9: Decomposing vedic cube by main and secondary symmetric planes. (a) Main symmetric planes (MSP) contains plane $X = Y$, $Y = Z$, and $X = Z$, each of them divides vedic cube into two grey blocks (b) Secondary symmetric planes (SSP) contains $X + Y = 9$, $Y + Z = 9$, and $X + Z = 9$, each of them also divides vedic cube into two grey blocks.

Figure 10 shows segmentation result of MSP contains six congruent tetrahedrons (orthoscheme), each of them is also called birectangular tetrahedron which includes two right angles. In this study, birectangular tetrahedron is named MSP unit tetrahedron. The volume of a tetrahedron unit is one-sixth of the cube. It can mirror three times along $X = Y$, $Y = Z$, and $X = Z$ plane to become the initial cube. The intersection of MSP is found out to be space diagonal line $X = Y = Z$, it is reasonable to conclude that vedic cube is separated into six blocks by MSP.

The segmentation results of SSP are two congruent hexahedra and six congruent tetrahedrons. They are named as SSP unit hexahedron and SSP unit tetrahedron, respectively. Volume of a SSP unit hexahedron and a tetrahedron are one-fourth of the cube, and one-twelve of the cube, respectively. The intersection of SSP is a point, the central point of vedic cube (4.5, 4.5, 4.5). The intersection point indicates the reason why vedic cube is separated by SSP into eight blocks as shown in Figure 11. It is important to note that volume of each block is not exactly the same.

The final step is to analyze the intersection separated by both MSP and SSP, and the segmentation result is 24 congruent tetrahedrons. It is known as Goursat tetrahedron, or trirectangular tetrahedron which includes three right angles in a vertex. The Schlfli symbol is

$$\begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

in Coxeter group [8]. It is named unit tetrahedron of vedic cube (UTVC) in this study. The base of UTVC is a right triangle with area occupying one-fourth of the square. The height is half of the length, and the volume is 1/24 of vedic cube as shown in Figure 12.

For the MSP unit tetrahedron, it is separated into four congruent UTVCs by the boundary conditions of SSP. There are mainly two cases for SSP unit hexahedron and SSP unit tetrahedron. The first case is six UTVCs separated by SSP unit hexahedron; the second case is two UTVCs separated by SSP unit tetrahedron, with both within the boundary condition of MSP.

UTVC can mirror six times along MSP and SSP to become the vedic cube, only points of digit 1, 2, 3 and 4 of UTVC have to be listed. Then original complicated digital root patterns in 3-D space can therefore be simplified. The symmetric plane method reduces the complexity of vedic cube and allows the basic element in vedic cube to be easily recognized.

Points in UTVC of digit 1, 2, 3 and 4 within boundary conditions ($X \leq Y$, $Y \leq Z$, $X \leq Z$, $X+Y \leq 9$, $Y+Z \leq 9$, $X+Z \leq 9$) are listed in Table 1. Points on the boundary are shared by several UTVCs. Points in UTVCs can be used to reflect others points by MSP and SSP.

We invite readers to execute the "*digit_3D*" program provided in appendices A.1 to observe the digital root patterns from different boundary conditions, floors and views and have fun.

Figure A-2 to A-6 of appendices show the result of specific digit scatter in vedic cube with corresponding boundary conditions by *digit_3D*. BC1, BC2, BC3 and BC4 represent boundary conditions ($X \leq Y$, $Y \leq Z$, $X \leq Z$, MSP unit tetrahedron), ($X + Y \leq 9$, $Y + Z \leq 9$, $X + Z \leq 9$, SSP unit hexahedron), ($X+Y \leq 9$, $Y+Z \geq 9$, $X+Z \geq 9$, SSP unit tetrahedron) and ($X \leq Y$, $Y \leq Z$, $X \leq Z$, $X+Y \leq 9$, $Y+Z \leq 9$, $X+Z \leq 9$, UTVC), respectively. To be noted, digit 9 is not symmetric along SSP in the vedic cube, which is mentioned in section 3.4.

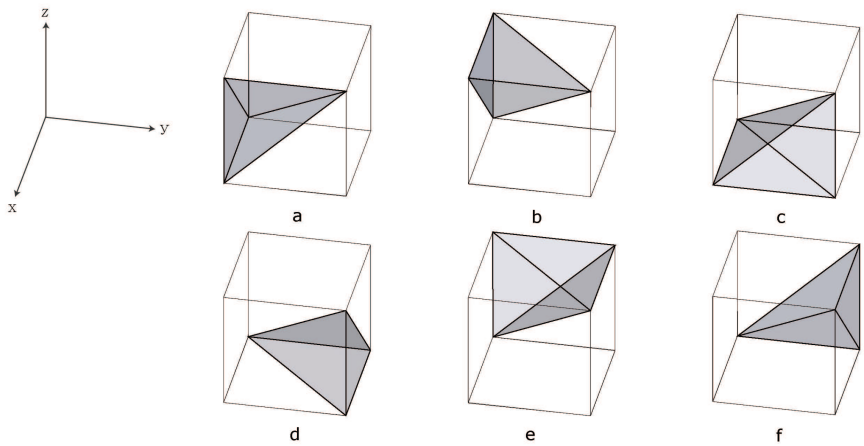


Figure 10: Segmentation result of main symmetric planes (MSP). There are six congruent tetrahedrons (orthoscheme), one of which is also called birectangular tetrahedron including two right angles. Volume of a tetrahedron is one-sixth of the cube. It is called MSP unit tetrahedron in this research. Corresponding boundary conditions of each block are (a) $X \geq Y$, $Y \leq Z$, $X \geq Z$; (b) $X \geq Y$, $Y \geq Z$, $X \leq Z$; (c) $X \geq Y$, $Y \geq Z$, $X \geq Z$; (d) $X \leq Y$, $Y \geq Z$, $X \geq Z$; (e) $X \leq Y$, $Y \leq Z$, $X \leq Z$; (f) $X \leq Y$, $Y \geq Z$, $X \leq Z$.

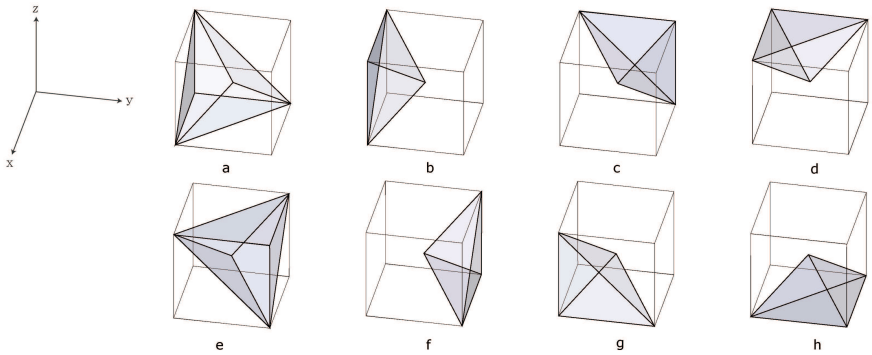


Figure 11: Segmentation result of secondary symmetric planes (SSP). There are two SSP unit hexahedron and six SSP unit tetrahedron, corresponding volume are one-fourth and one-twelve. Corresponding boundary conditions of each block are (a)$X + Y \leq 9$, $Y + Z \leq 9$, $X + Z \leq 9$; (b)$X + Y \leq 9$, $Y + Z \leq 9$, $X + Z \geq 9$; (c)$X + Y \leq 9$, $Y + Z \geq 9$, $X + Z \leq 9$; (d)$X + Y \leq 9$, $Y + Z \geq 9$, $X + Z \geq 9$; (e)$X + Y \geq 9$, $Y + Z \geq 9$, $X + Z \geq 9$; (f)$X + Y \geq 9$, $Y + Z \geq 9$, $X + Z \leq 9$; (g)$X + Y \geq 9$, $Y + Z \leq 9$, $X + Z \geq 9$; (h)$X + Y \geq 9$, $Y + Z \leq 9$, $X + Z \leq 9$.

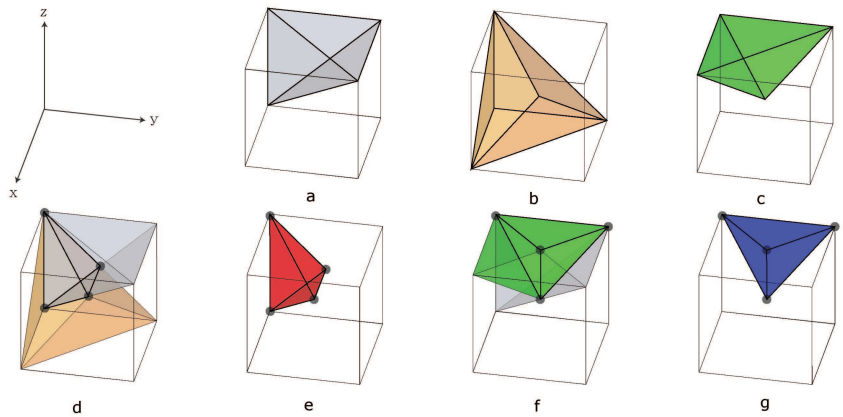


Figure 12: Intersection of the blocks separated by MSP and SSP. (a) MSP unit tetrahedron identical Figure 10(e). (b) SSP unit hexahedron is identical to Figure 11(a). (c) SSP unit tetrahedron is identical to Figure 11(d). (d) Union of part a and b. (e) Intersection of part a and b (f) Union of part a and c. (g) Intersection of part a and c. The final segmentation result is 24 congruent tetrahedrons, part e and g are two of them.

Table 1: Points in the Unit tetrahedron of vedic cube (UTVC).(B.C.: $X \leq Y$, $Y \leq Z$, $X \leq Z$, $X + Y \leq 9$, $Y + Z \leq 9$, $X + Z \leq 9$)

Digit 1	Digit 2	Digit 3	Digit 4
(1,1,1)	(1,1,2)	(1,1,3)	(1,2,2)
(1,2,5)	(1,4,5)	(2,2,3)	(1,1,4)
(2,2,7)	(2,2,5)	(1,3,4)	(2,4,5)
(4,4,4)		(3,4,4)	
		(2,3,5)	
		(1,2,6)	

3.6 Extra findings

In Table 1, there are extra findings in observing points in the UTVC which are not located within the boundary. Those points of digit 1, 2, 4 are (1,2,5), (1,4,5), (2,4,5), respectively. The three points form a right triangle with lengths of 1, 2, and $\sqrt{5}$. Furthermore, points of digit 3 are (1,3,4), (2,3,5), (1,2,6), they also form a right triangle with side length $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$.

3.7 Six coordinate points

Six coordinate points (X, Y), $(2X, 5Y)$, $(4X, 7Y)$, $(5X, 2Y)$, $(7X, 4Y)$, $(8X, 8Y)$ in Figure 6(b) can be used to find other five points with same digit. For example, the digital root of (4,8) is 5 in vedic square , and the other five points would be (8,4), (7,2), (2,7), (1,5) ,and (5,1) due to having the same digital root operation results listed as follows:

$$D(X,Y) = D(2X,5Y) = D(4X,7Y) = D(5X,2Y) = D(7X,4Y) = D(8X,8Y). \tag{11}$$

The same rule can also be applied in 3-D space. For instance, point (5,1) indicates that pattern of digit 5 on $F1$ can be written as $D5F1$, which is also the pattern to the other five points (1,5), (2,7), (7,2), (8,4), (4,8). Take point (4,8) to implement the rule, position of digit 4 on $F1$(vedic square) will be the same as digit 5 on $F8$.

There are two more groups for digit 3 and 6, the first group contains $(X,4Y)$, $(2X,2Y)$, $(4X,Y)$, $(5X,8Y)$, $(7X,7Y)$, and $(8X,5Y)$; the second group contains $(X,7Y)$, $(2X,8Y)$, $(4X,4Y)$, $(5X,5Y)$, $(7X,Y)$, $(8X,2Y)$. For the first group, the digital root operation by multiplying coefficient of X by Y are 4, which yields

$$D(X,4Y) = D(4\times D(X,Y)) = D((3+1)\times D(X,Y)) = D(9+D(X,Y)) = D(X,Y). \tag{12}$$

The equation also applies when the product of X and Y coefficient is 7. However, the six points found from these two groups are identical to the original group. There is a difference between total number of group B (12 numbers of each) and group A (6 numbers) in vedic square, because the points produced by group B mirroring along the diagonal $X = Y$ are different from the six coordinate points. That is the reason why 12 numbers are within each one of group B in vedic square.

The geometric indication of six groups is that we could find a new coordinate point by moving the multiple of coefficient from original X coordinate position on the X axis. The coordinate point on Y axis can also be found by the same rule.

The main advantage of six coordinate points method is convenient to find the other five coordinate points from a single known coordinate point. It is impossible to mirror all points along diagonals from a single point because there must be some points locating on one of two diagonals, at least two coordinate points are required to mirror all the points with the same digital root on a plane.

4 Conclusion

This paper investigates the digital root patterns by adopting two different approaches in the 3-D space. Simplification in vedic cube is very important to understand characteristics of digital root patterns in 3-D space. From floor method, characteristics of digital root patterns of 2-D space can be applied to 3-D space, including distribution regularities of basic pattern, columns on the different floors, and specific digital root patterns on specific floors which might

be composed of several basic patterns. To sum up, 2-D space (vedic square) can be seen as a basic reference for 3-D space (vedic cube) and 4-D space similarly. Symmetric plane method greatly reduces the complexity of vedic cube and it can be used to find out the digital root patterns of 3-D space, reducing the limitation of only analyzing the correlation between 2-D spaces. The discovery pattern of vedic cube expands the geometric patterns from 2-D space to 3-D space, even for multi-dimensional space. Digital root patterns of 3-D space are the mysteries formed by the nature. Except pure mathematics research, this study can be applied as a rule to develop an algorithm to discover textile patterns in 3-D space, molecular crystals, architecture, space, and artistic fields.

5 Acknowledgements

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Appendices

A.1 Digital root patterns visualization "*digit_3D*" program

Please refer to the following url to access the "*digit_3D*" program (current version:1.5).

`http://140.116.77.14/ecowater/member/102_CYLin.htm`

A.2 Points list

Table A-1: Points of digit 1, 2, 3 and 4 in four UTVCs with corresponding boundary conditions.

	$X \le Y$, $Y \le Z$, $X \le Z$			
	$X+Y \le 9$ $Y+Z \le 9$ $X+Z \le 9$	$X+Y \ge 9$ $Y+Z \ge 9$ $X+Z \ge 9$	$X+Y \le 9$ $Y+Z \ge 9$ $X+Z \le 9$	$X+Y \ge 9$ $Y+Z \ge 9$ $X+Z \ge 9$
Digit 1	(1,1,1)	(1,8,8)	(1,8,8)	(1,8,8)
	(1,2,5)	(5,7,8)	(1,4,7)	(2,4,8)
	(2,2,7)	(7,7,7)	(2,2,7)	(2,2,7)
	(4,4,4)	(4,5,5)	(4,5,5)	(4,5,5)
Digit 2	(1,1,2)	(2,8,8)	(1,7,8)	(2,8,8)
	(1,4,5)	(5,5,8)	(1,4,5)	(4,4,8)
	(2,2,5)	(5,7,7)	(2,4,7)	(2,4,7)
Digit 3	(1,1,3)	(3,8,8)	(1,6,8)	(1,6,8)
	(1,2,6)	(6,7,8)	(1,3,7)	(2,3,8)
	(1,3,4)	(4,6,8)	(1,5,6)	(3,5,8)
	(2,2,3)	(3,7,7)	(2,6,7)	(2,6,7)
	(2,3,5)	(5,6,7)	(3,4,7)	(3,4,7)
	(3,4,4)	(4,5,6)	(3,5,5)	(4,5,6)
Digit 4	(1,1,4)	(4,8,8)	(1,5,8)	(1,5,8)
	(1,2,2)	(2,7,8)	(1,7,7)	(2,7,8)
	(2,4,5)	(5,5,7)	(2,4,5)	(4,4,7)

A.3 Scatter plot

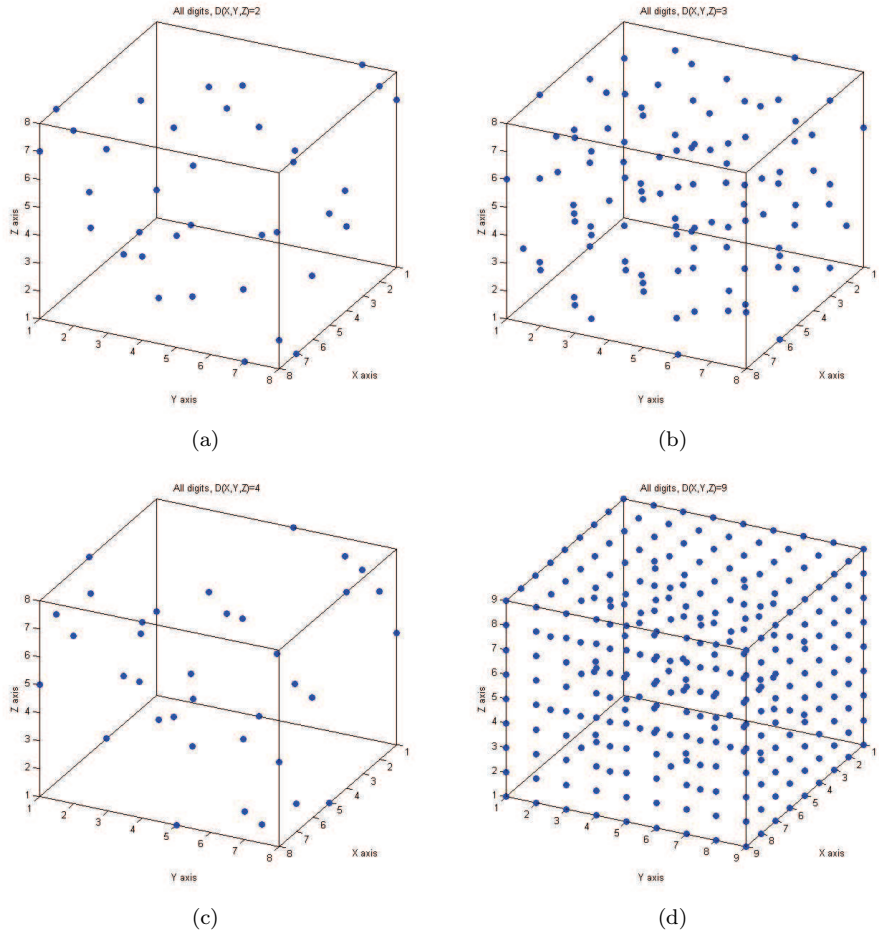

Figure A-1: Points of digit scatter in vedic cube. (a) digit 2 (b) digit 3 (c) digit 4 (d) digit 9.

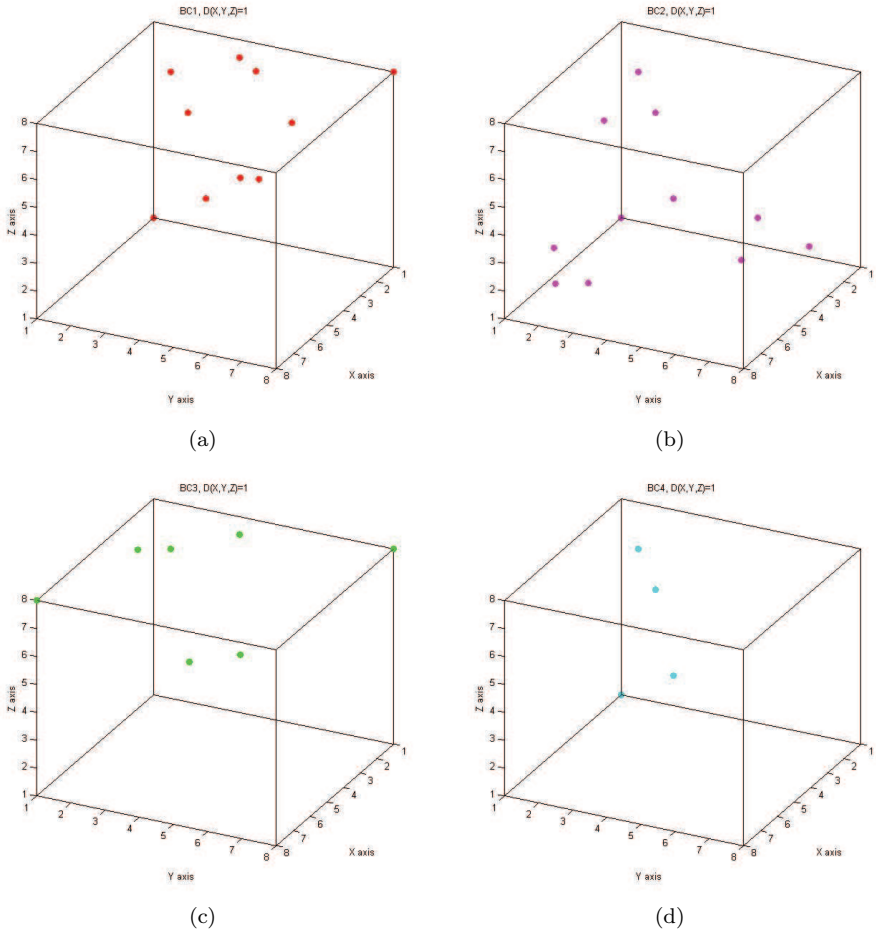

Figure A-2: Points of digit 1 scatter in vedic cube with corresponding boundary conditions (a) BC1 (b) BC2 (c) BC3 (d) BC4.

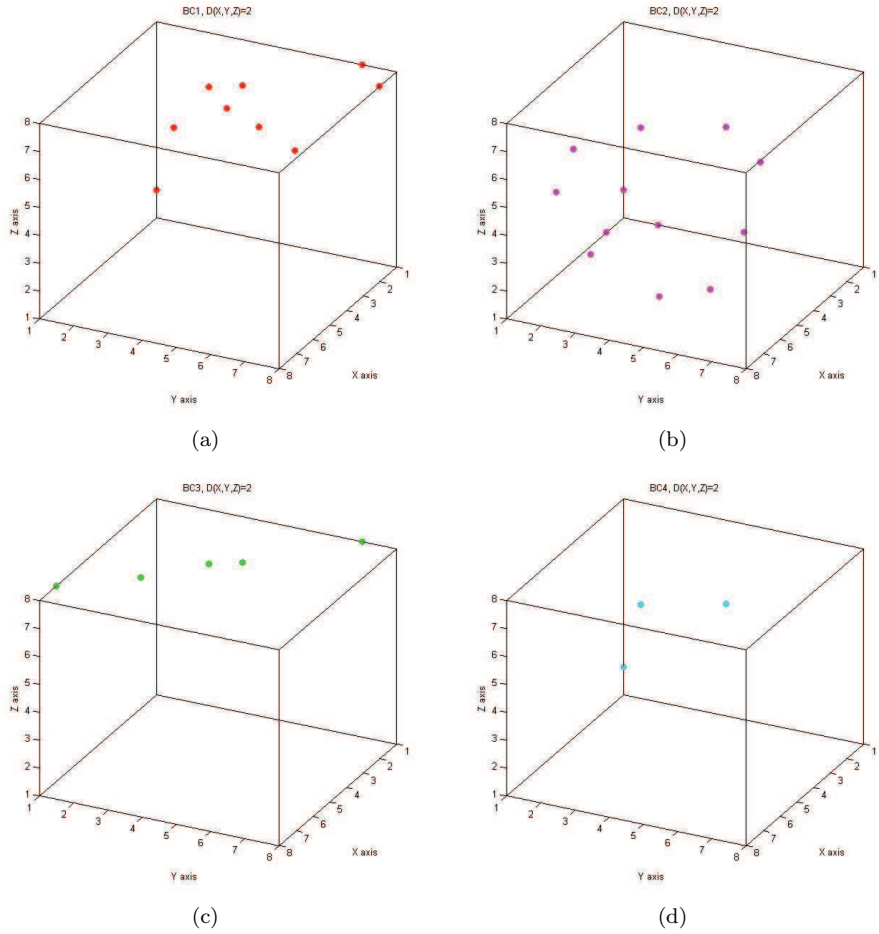

Figure A-3: Points of digit 2 scatter in vedic cube with corresponding boundary conditions (a) BC1 (b) BC2 (c) BC3 (d) BC4.

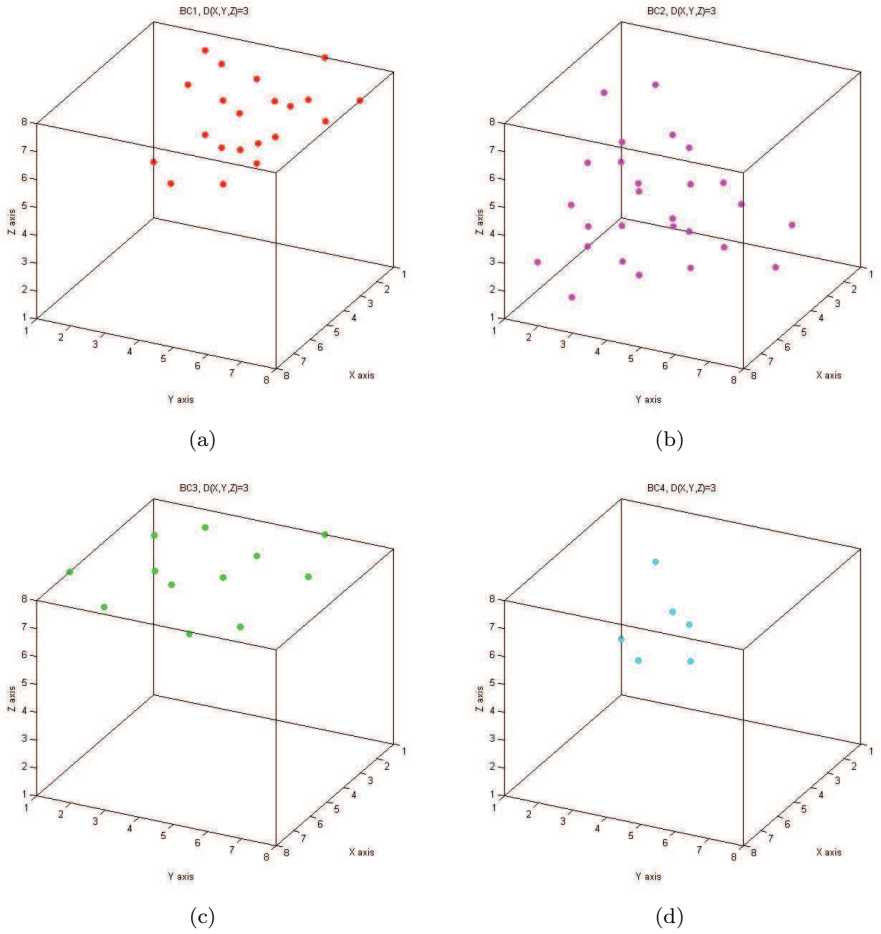

Figure A-4: Points of digit 3 scatter in vedic cube with corresponding boundary conditions (a) BC1 (b) BC2 (c) BC3 (d) BC4.

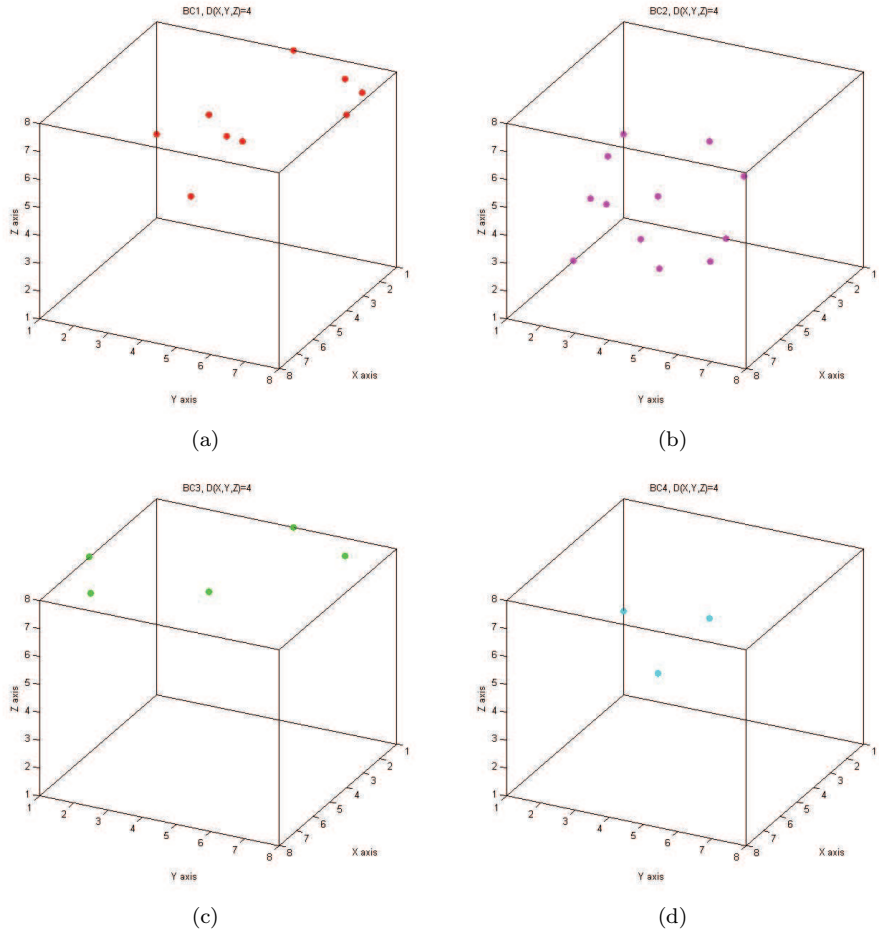

Figure A-5: Points of digit 4 scatter in vedic cube with corresponding boundary conditions (a) BC1 (b) BC2 (c) BC3 (d) BC4.

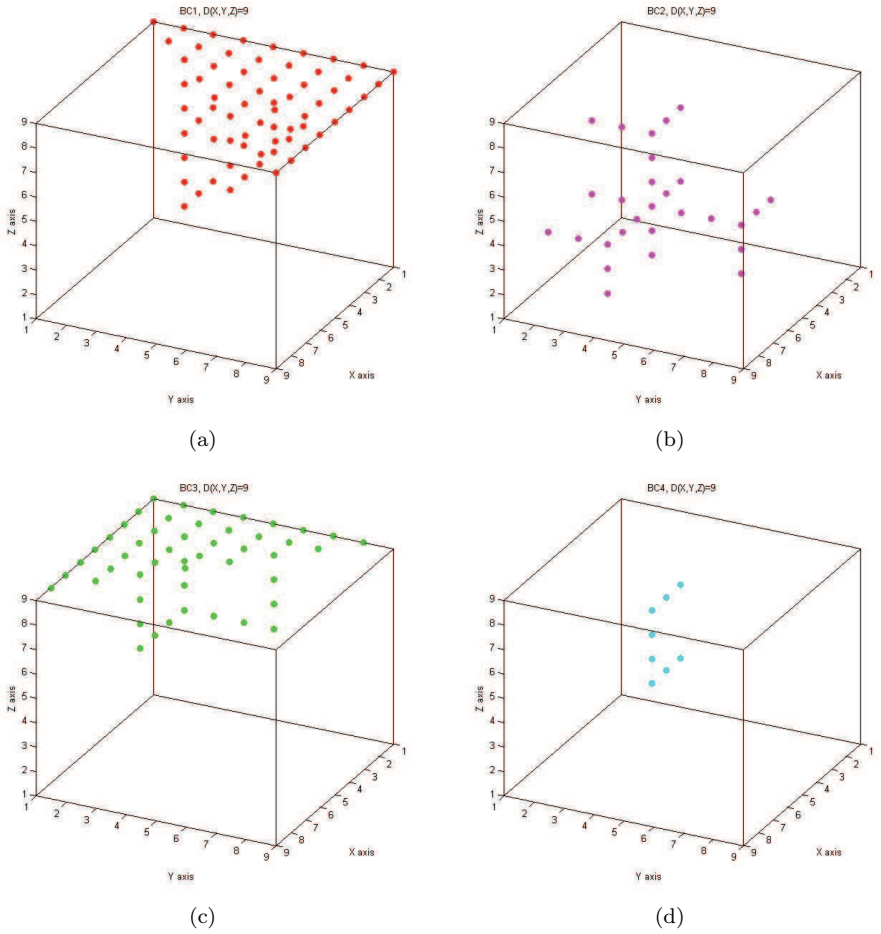

Figure A-6: Points of digit 9 scatter in vedic cube with corresponding boundary conditions (a) BC1 (b) BC2 (c) BC3 (d) BC4.