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Almada Negreiros and the regular nonagon

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Abstract: *Almada Negreiros was a Portuguese artist of the 20th century, one of the most relevant figures of Portuguese modernism. In the course his studies in visual art, he postulated the existence of a* canon*, a set of constants and geometric constructions underlying all art. Among these constructions were the n-th parts of the circle.*

In this paper we consider the regular nonagon. It is known that there is no exact construction for this polygon using just straightedge and compass. Nevertheless, there are some very beautiful approximate constructions, which we describe in the first part of this paper. In the second part, we describe two constructions with a very small error. The last one is due to Almada Negreiros, one that he found in his description of the canon.

Key-words: Regular polygons, nonagon, Almada Negreiros.

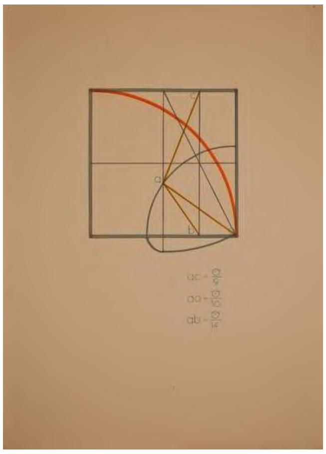
1 Almada Negreiros

José de Almada Negreiros (São Tomé e Príncipe, 1893 – Lisbon, 1970) was a key figure of 20th century Portuguese culture, in both visual arts and literature. He was a quite diverse artist, having produced paintings, murals, drawings, prose, poetry, theatre and critical essays. He is maybe less known for his studies of Portuguese renaissance art. These studies started as a compositional analysis of two famous Portuguese paintings, the *Ecce Homo* and the *Panels of St Vincent*, both on display at the *Museu de Arte Antiga* in Lisbon. Eventually, he thought of extending them to other Renaissance works, at first, and to all art works in general, at a later stage. He postulated the existence of an artistic canon, which could be understood as a set of proportions and constants, underlying all art. He enumerated the canonical elements as follows, in [5]:

> The simultaneous division of the circle in equal and proportional parts is the simultaneous origin of the constants of the relation nine/ten, degree, mean and extreme ratio and casting out nines.

Since the 1930s, Almada produced a very large number of geometric drawings in an attempt to produce these constants and relate them, partially inspired in [3]. In particular, he has many drawings in which he suggests processes for the division of the circle in n parts. He denotes by $\frac{\odot}{n}$ the n-th part of the circle and by $\frac{\bar{\odot}}{n}$ the chord of this arc, which corresponds to the side of the inscribed n-gon.

In this paper we take a look at the construction for the nonagon that is suggested by this drawing, which is part of a series called *The Language of the Square*.



The construction also produces the sides of the decagon and the 14-sided polygon, as we can see, but we will direct our attention especially to the nonagon. Before that, we present other possible artistic constructions for this polygon.

2 Regular polygons

As many of us have learned in high school, there are geometric constructions for some regular polygons, using only straightedge and compass. These constructions usually start off with a circle, which is then divided in n parts, and the points obtained will be the vertices of the polygon. However, this is not possible for not all polygons. There is a well known result of Galois theory that describes completely which polygons are constructible — meaning, which ones admit a straightedge and compass construction. This is Gauss-Wantzel's theorem.

Gauss-Wantzel's Theorem. *The division of the circle in n equal parts with straightedge and compass is possible if and only if*

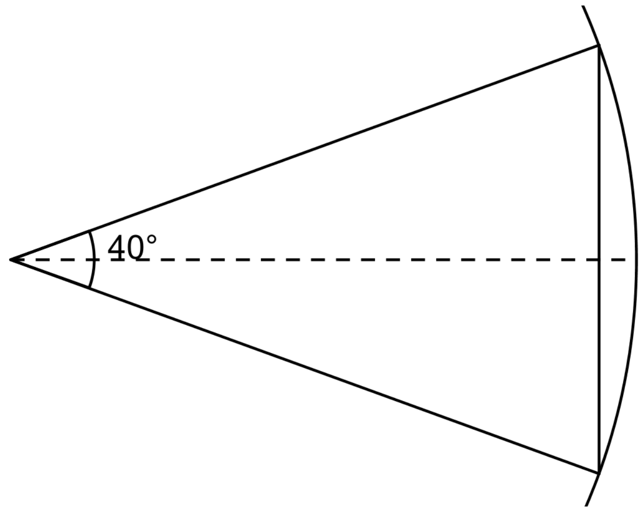
$$n = 2^k p_1 \dots p_t$$

where $p_1, \dots, p_t$ are distinct Fermat primes.

A *Fermat prime* is a prime of the form $2^{2^m} + 1$. Presently, the only known Fermat primes are 3, 5, 17, 257 and 65537. See for instance chapter 19 of [4] for a detailed account.

As we said, this result implies that the nonagon is not constructible with straightedge and compass since $9 = 3 \times 3$, the Fermat prime 3 appears twice in the factorisation of 9.

Therefore we can only hope to have approximate constructions for the nonagon. We will consider two types of construction: ones that provide all the vertices (even if the sides are not exactly equal), and then others that determine good approximations for the side of the nonagon, which then can be translated to the circle using the compass.



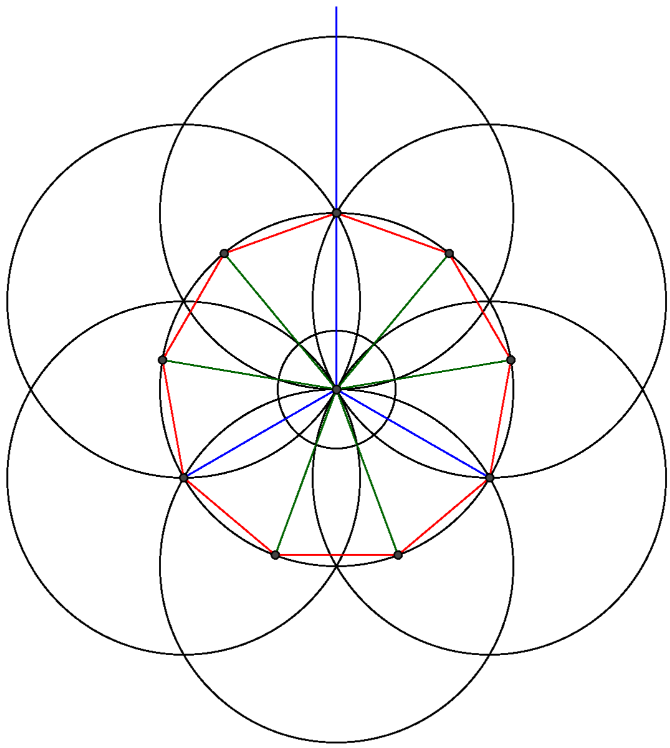
It is possible, nonetheless, to have an exact value for the side of the nonagon. If we take a circle of radius 1, the of the regular nonagon would be

$$2 \sin \frac{40^{\text{o}}}{2} = 2 \sin 20^{\text{o}}.$$

The value up to nine decimal digits is 0,684040287. All approximations of this value that we present in this paper are given by geogebra.

3 The nonagon, all vertices at once

In this section we analyze two constructions for the nonagon. The following one is sometimes associated with Dürer.



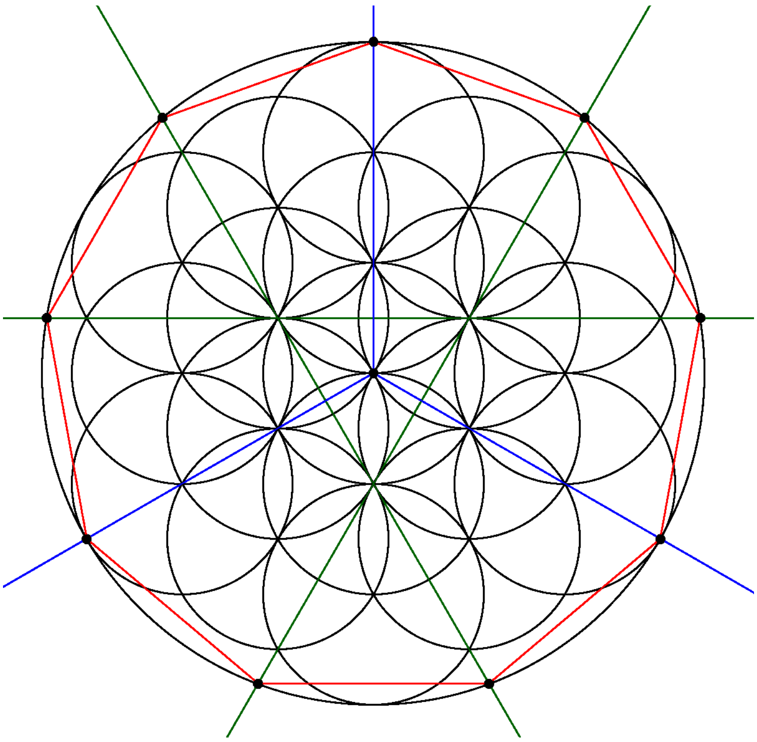
The construction protocol for this figure is rather straightforward. You start with a circle and draw a ray from its center (here we chose a vertical ray, marked blue, going up). Mark a point on this ray, at a distance from the center equal to three times the radius. Now take the distance between these two points as radius (three times the original radius) and draw two circles, one centred at each of the points. We then have two intersection points, which we use as new centres for circles (keeping the same radius). Again, new intersection points appear, and we draw more circles. When we are done, we have a central circle and six other ones around it.

Now we need to draw two types of rays stemming from the center: some are determined by some intersections of the large circles (the blue ones), while others are directed by points of intersection of the large circles with the small one (the green ones). There are three rays of the first type, and they make angles of 120^o between them. The others would have to trisect these angles, which is something that cannot be done with straightedge and compass.

This construction leads to two different lengths for the sides of the nonagon. If we set the radius of the original circle as $1/3$, then the larger circles have radius 1 and the lengths of the sides obtained are as follows.

side	length	error
between two green rays	0,697338	+2%
between a blue ray and a green ray	0,677378	−1%

Here is another construction, known as the flower of life.

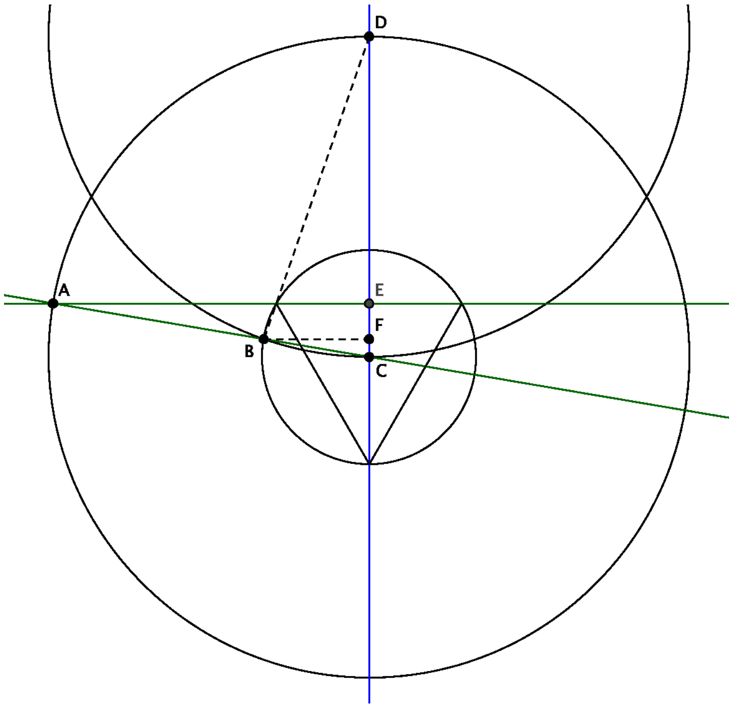


Again, the protocol is long, but rather straightforward. One starts with a circle and a ray from its its center (again we choose one going up). Then, keeping the radius, draw circles centred on the intersection points — over and over again, 19 of them on the whole.

In the end, you draw a big circle, with radius three times the original one. Some points of the nonagon are intersections of this circle with some smaller ones, which are tangent to it, marked by blue rays. Finally, add a regular triangle inside the original circle and draw lines based on its sides. This provides the remaining vertices, marked by green lines.

Is is a bit surprising that the points determined by this construction are exactly the same as the ones determined by the previous one — so the lengths of the sides and the errors are equal.

Here is a proof of this fact. Consider the following figure.



It is easy to see that, in both constructions, the blue lines are in the same position, at angles 120º from each other. For the green lines, we need to prove that A belongs to both lines shown — the one through the centre defines the vertex from point B in the first method, and the horizontal one is used in the second method.

Consider the dilation with center C and ratio 3. It clearly maps the smaller circle to the big one. We now need to prove that it maps point F to point E. Take BC to have unit length, we then have $CE = 1/2$. Thus we need to prove that $CF = 1/6$.

For this we use the Pythagorean theorem twice, applied to triangles BCF and BFD:

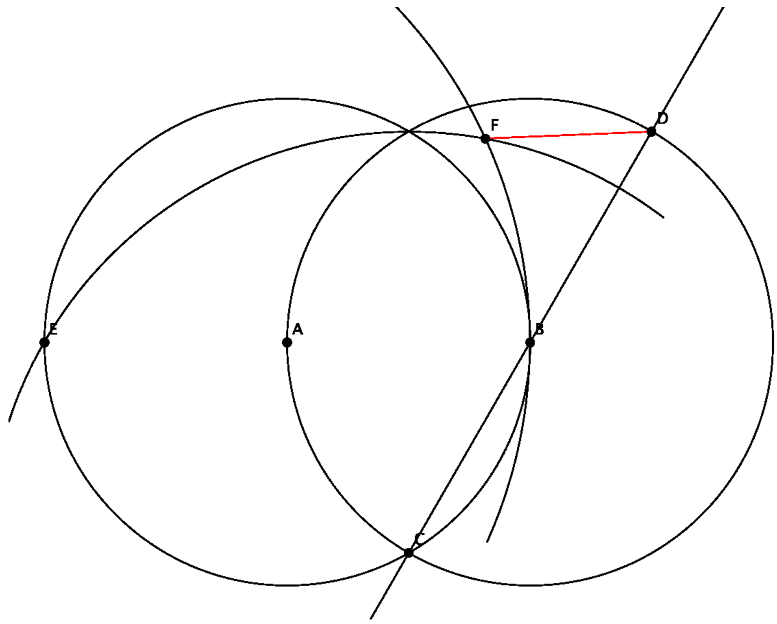
$$CF^2 + FB^2 = 1 \text{ and } FB^2 + FD^2 = 9.$$

Eliminating FB^2 from the equations, we get $1 - CF^2 = 9 - FD^2$. Replacing $FD = 3 - CF$ and working out the computations, we conclude that $CF = 1/6$ as we wished.

4 The nonagon — determining the side

In this last section we present two constructions that determine the side of the nonagon with great precision. By this we mean: if you use any of these methods to draw a nonagon in a sheet of paper of size A4 or US letter, you will not be able to notice the error.

The first method is very simple and elegant, it needs only four circles and a straight line.



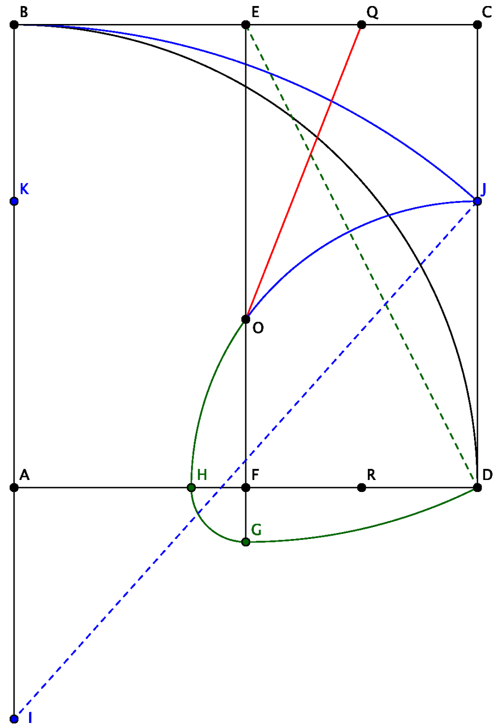
Here is the construction protocol.

- Draw a circle with center A.
- Take a point B on the circle and draw another circle with the same radius, centred at B. Let C be one of the intersection points of the two circles.
- Draw the line CB. Let D be the intersection point of this line with the second circle.
- Open the compass with radius AD and draw a circle with center C. Let E be one of the intersection points of this circle with the original circle, as in the figure.
- Draw a circle with center E and radius EB. Let F be the intersection point of this circle with the previous circle (centred at C).
- The segment FD is approximately the side of the regular nonagon, inscribed in the first circle.

There are two coincidences in this drawing. First, the length AD, used as radius for the third circle, is equal to the distance form C to the other intersection point of the circles centred at A and B. Second, the segment EB is a diameter. These are interesting facts (which the reader can verify easily), but they are not central to the construction.

If we take the first circle to have radius 1, the side obtained has length 0.68412, an error of 0,01%. In a circle of radius 10 cm, for instance, this error would be in the order of the hundredth of milimiter.

Finally, we look at Almada's drawing. The protocol in the drawing is rather simple, but Almada has another one, which we consider even simpler, for the same segment. We sketch it here.



Points E and F are midpoints of the sides, point Q is the midpoint of EC and point R is the midpoint of FD. Point I is on the line AB, such that $IA = AF$ (half the side of the square). All these points appear in Almada's drawing, its positions can be determined just by taking measurements, since the square has side 20 cm.

The aim of the construction is to determine point O, so that OQ is the side of the nonagon, inscribed in the circle with center C and radius AB. The protocol sketched in the drawing is the following one.

- With center E draw arc DG.
- With center F draw arc GH.
- With center D draw arc HO.

This protocol is marked in green. The other one, marked in blue, which appears in another drawing, is the following.

- With center I draw arc BJ.
- With center D draw arc JO.

As we can see, this one is even shorter and avoids an arc of very small radius which might be hard to draw (arc GH).

Nevertheless, one should prove that the point O obtained by both methods is the same. For this we need to see that $DJ = HD$. We consider the square to have unit side.
In the first construction, using the Pythagorean theorem for the triangle DEF, we have

$$ED^2 = \left(\frac{1}{2}\right)^2 + 1,$$

so $EG = ED = \frac{\sqrt{5}}{2}$. Therefore $HF = FG = \frac{\sqrt{5}}{2} - 1$ and

$$HD = HF + FD = \left(\frac{\sqrt{5}}{2} - 1\right) + \frac{1}{2} = \frac{\sqrt{5}-1}{2}.$$

For the second construction we take point K symmetric to J with respect to line EF and use the Pythagorean theorem for the triangle IJK:

$$\left(\frac{3}{2}\right)^2 = 1 + IK^2$$

which yields $IK = \frac{\sqrt{5}}{2}$. Therefore

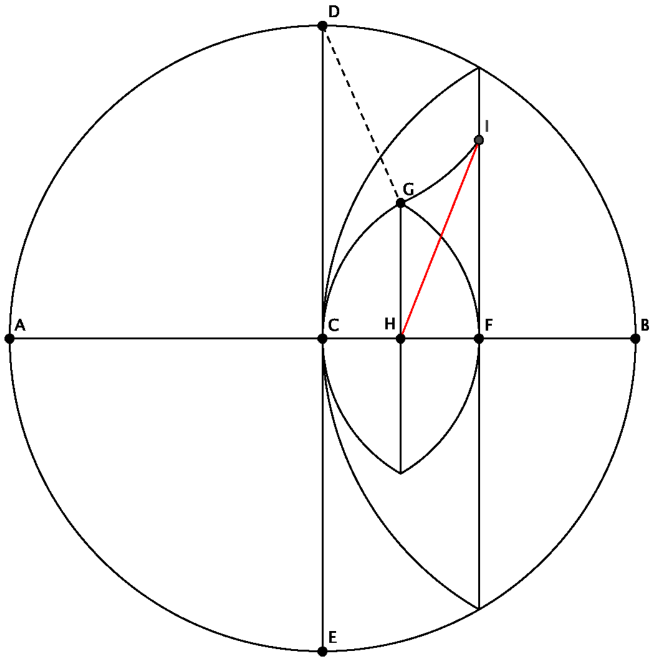
$$DJ = \frac{\sqrt{5}}{2} - \frac{1}{2} = \frac{\sqrt{5}-1}{2}.$$

This proves $DJ = HD$, as we wished. And actually, it proves a bit more. Almada had noted in the drawing that OD should be the side of the decagon, and with this computation we can actually verify this claim. The side of the decagon is $2\sin(360^o/20) = 2\sin 18^o$, and indeed $\sin 18^o = (\sqrt{5}-1)/4$, confirming Almada's claim.[1]

Almada also writes that OR is the side of the 14-sided regular polygon. Like the nonagon, this polygon is not constructible by straightedge and compass, according to Gauss-Wantzel's theorem. The error in this length is 1%.

Finally, we now present two constructions based on this drawing. The second one is the full construction, the first one is a partial one, which yields a reasonable approximation. The reason we show it is just because it is a nice and simple figure, consisting just of two midpoint constructions (well, three, if we count the one necessary to draw the perpendicular diameters, which is not shown).

[1]Consult WolframAlpha, for instance, for the exact value of the sine.



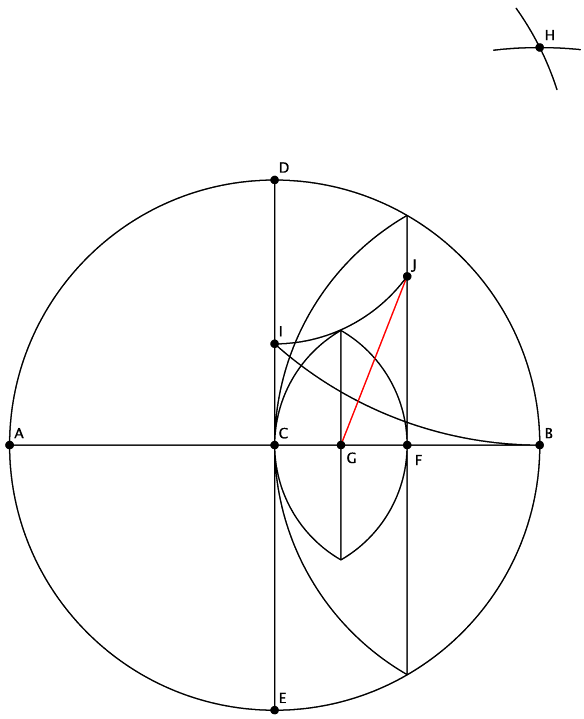
The protocol is quite clear from the figure.

- Draw a circle with center C and two perpendicular diameters, AB and DE.
- Draw a circle with center B and radius BC (as in the original circle). This construction leads to point F (midpoint of BC).
- Draw two arcs of circle with radius CF, marking the midpoint of CF, H and point G.
- Draw an arc with center D and radius DG, leading to point I.
- The segment HI is approximately the side of the regular nonagon.

For a circle with unit radius, this construction leads to a segment of length 0,681486, which represents an error of 0,4%.

We stumbled upon this construction as we tried to find a reasonable protocol for Almada's original drawing. It turned out that point G in the previous construction was close enough to the defining line to justify a shortcut...

We now present the full construction, inspired by Almada's drawing.



The construction protocol is as follows.

- Draw a circle with center C and two perpendicular diameters, AB and DE.
- Use the usual construction to determine the midpoint of CB, F.
- Use the same construction to determine the midpoint of CF, G.
- Draw a circle with center B and radius AF, and another one with center D and radius DF. Let H be the point of intersection of these two circles, as in the figure.
- Draw a circle with center H and the same radius HB and mark point I.
- Draw a circle with center D and radius DI and mark point J.
- The segment GJ is approximately the side of the regular nonagon.

Now we come to the main (and amazing) conclusion: the side of the nonagon obtained this way, for a circle of radius 1, is 0,6840493281, with an error of 0.001%. This corresponds to one thousandth of a millimetre in a circle of radius 10cm, or 1 mm in 100 m!

We believe that Almada reached these conclusions by endlessly drawing geometric constructions, trying to find relations between the elements of the canon. He never presented any calculations. Therefore, we believe he was unaware of the accuracy of his drawings, which makes this approximation, truly, an amazing find.

5 Acknowledgements

The ongoing study of Almada's estate is revealing quite a large number of geometry-based artworks and notebooks, which have only started to be analysed. So far, only the papers [1] and [2] have been published on this subject, but much more work remains to be done.

This work was made possible by the *Modernismo online* project (responsible for the site `http://www.modernismo.pt`), of the Faculdade de Ciências Sociais e Humanas of the Universidade Nova de Lisboa, financed by the Fundação para a Ciência e Tecnologia, that gathers and archives in digital format the material heritage of Portuguese modernism. It now includes a vast archive of Almada's work, which can be consulted online.

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