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The Tetraball Puzzle

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Abstract: *In this paper, the* Tetraball Puzzle, *a spatial puzzle involving tetrahedral arrangements, is presented and discussed.*

Key-words: Spatial puzzles, tetrahedral arrangements, packing problems.

1 The puzzle

A *Tetraball* is made of 4 equal size balls each of which touches the others (Figure 1). This tetrahedral arrangement of balls is the smallest non-trivial case of a tetrahedral stack of balls, which, if there are n balls per edge, has the nth tetrahedral number,

$$T(n) = \frac{n(n+1)(n+2)}{6}$$

balls in all. Figure 2 illustrates the case $T(4) = 20$ balls.

Figure 1: Tetraball.

The spheres of any such arrangement can be colored in 4 colors so as to satisfy the usual coloring condition that any two spheres that touch must be colored differently.

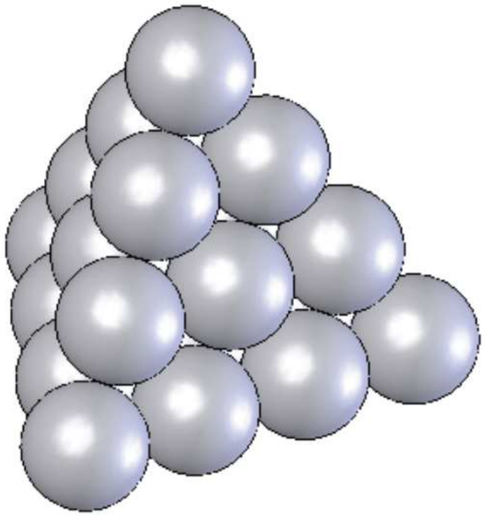

Figure 2: $T(4)$.

The *Tetraball Puzzle* asks to assemble five such 4-colored 2-stacks, or "tetraballs" into a 4-stack so as to satisfy this coloring condition.

The attentive reader will have noticed that we have not uniquely defined our problem, because the tetraball has two enantiomorphous (mirror image) forms, so there are really three problems according to how many of each form there are, namely $5-0$, in which the tetraballs are all the same, $1-4$ in which one is different to the other 4, and $3-2$ in which 3 are different from the other 2. All three problems are uniquely solvable and not extremely hard, although our quick-fire treatment will make them seem much easier than they really are.

2 Discussion

To work! When the puzzle is solved, it will contain one central tetraball and 4 corner ones. We can suppose that the central one is as in Figure 3 with the bottom sphere yellow. The topmost sphere of the upper tetraball is either yellow (Figure 3) or another color (Figure 4). For each possibility there are two cases called:

UNCHANGED
The top sphere of the top piece is the same color as the bottom sphere of the central piece (Figure 3). This implies that the colors of the remaining spheres are completely determined, since each already touches spheres of 3 other colors and must be colored in the fourth, and we find that the two pieces are enantiomorphs of each other. A circled vertex in a geometrical diagram (Figure 8) indicates that the piece at that vertex is unchanged.

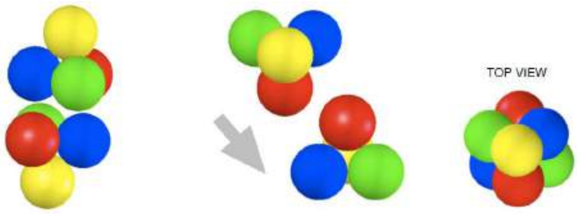

Figure 3: Unchanged.

EXCHANGED

In this second case the bottom sphere of the central piece is yellow (say), while the top sphere of the top piece is another color, say blue. The coloring of the top piece is unique (given this information), and we find the two pieces are identical rather than enantiomorphic. The exchanged coloring is obtained from the unchanged one by interchanging the colors blue and yellow for that piece. We indicate this exchange of colors on a piece at vertex A by a double arrow on the appropriate half of an edge.

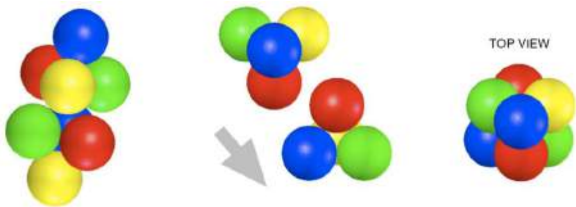

Figure 4: Exchanged.

Now the four balls along any edge belong to two pieces. So when this exchange happens to the first and second spheres along the edge, the third sphere must change, otherwise it would be the same color as the second. The only possibility now exists that the other is exchanged. So in our sketch, the other half of that edge must have a double arrow.

We call this process "trading up" the edge AB. Now there are only two possibilities, either the pieces at CD are unchanged or both exchanged as the pieces in edge AB, i.e., "trading up". We have now proved that, up to symmetry, the tetraball puzzle has just three solutions, classified by the number of edges we have traded up (since we often use square diagrams, we will denote "diagonals" and "boundaries").

We call them 0 *up*, 1 *up* or 2 *up* as follows (Figures 5, 6 and 7):

0 *up* – The 4 vertex pieces are the same as each other and opposite to the central piece so the splitting is $\frac{1}{4}$. In the orbifold notation it is $*322$ [1], the full tetrahedral group, of order 24. All edges have just 2 colors.

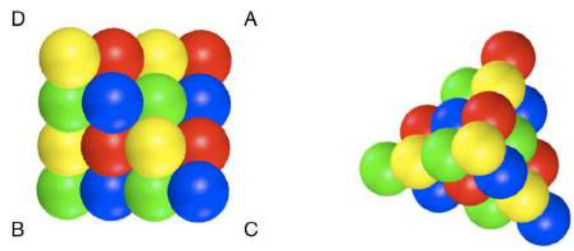

Figure 5: 0 *up*.

1 *up* – The central piece is 1 of 3 that are opposite to the other 2, making the splitting be 3/2. The symmetry group is $*22$ of order 4, the same as that of a rectangular table. The 4 "boundary" edges have been increased to 3 colors and the 2 "diagonal" edges remain 2-colored.

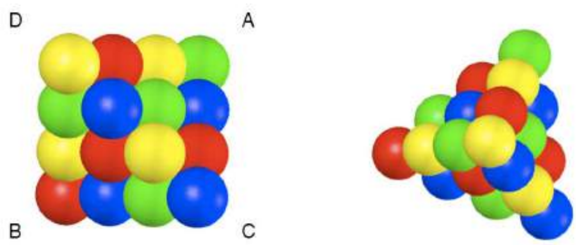

Figure 6: 1 *up*.

2 *up* – All five pieces are the same. (splitting 5/0). The symmetry group is $2*2$ of order 8, the same as that of a tennis ball. The 4 boundary edges are now upped to 4 colors, leaving the two diagonals 2-colored. Some of these properties can be stated uniformly: t *up* – maximum number of colors per edge $= 2 + t$; number of vertex pieces identical to the central piece $= 2t$. The pieces therefore split as $(2t + 1) : (4 - 2t)$.

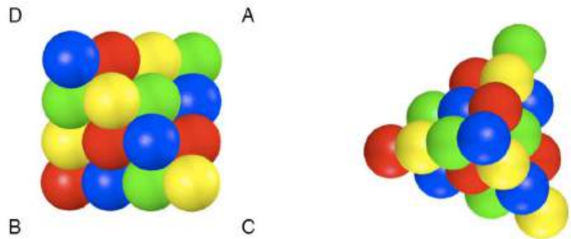

Figure 7: 2 *up*.

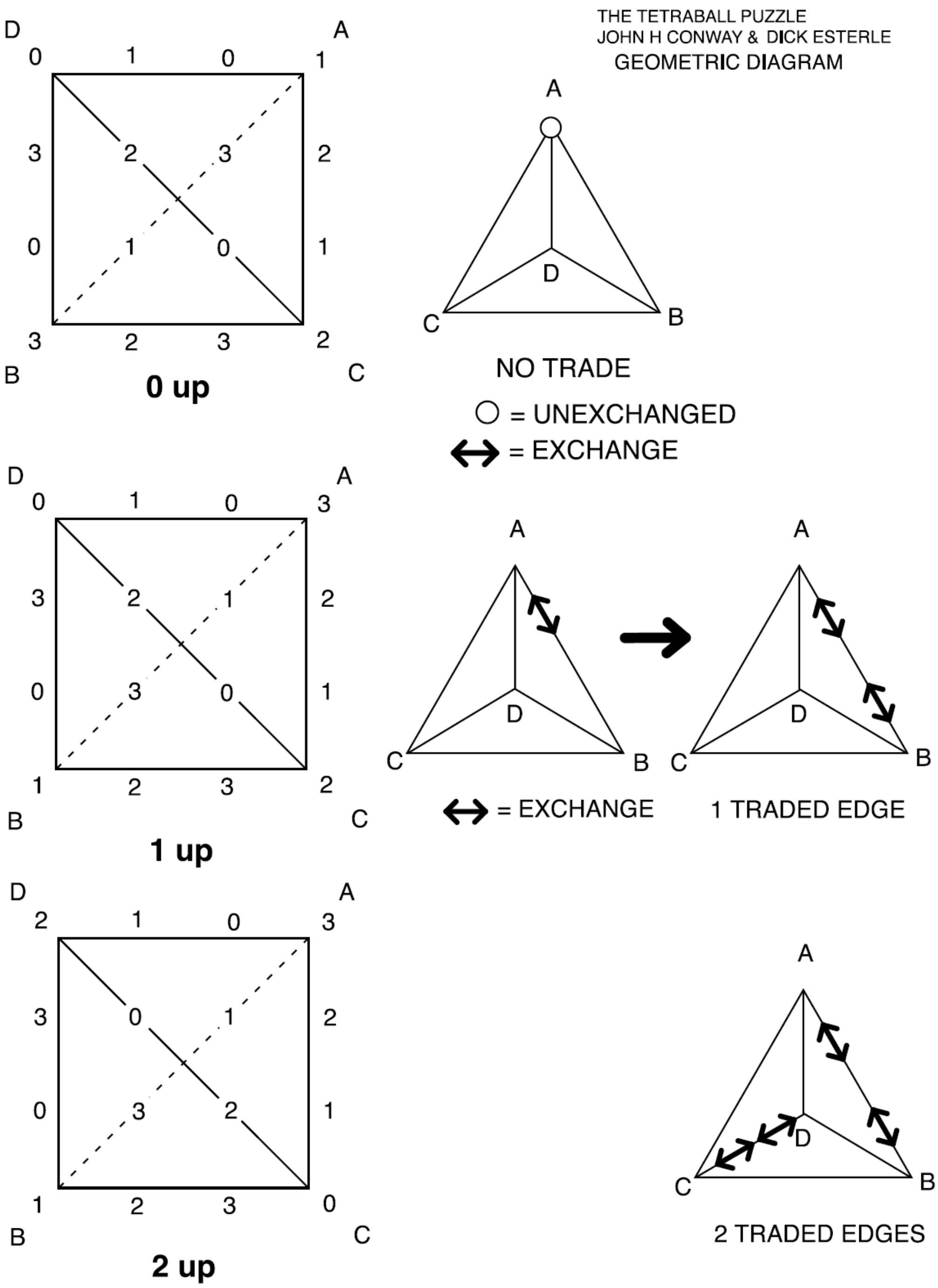

Figure 8: Geometric diagram.

References

[1] Conway, J., Burgiel, H., Goodman-Strauss, C. *The Symmetries of Things*, A K Peters, 2008.