# Recreational Mathematics Magazine

## Mathematics of Soccer

*Alda Carvalho, Carlos Pereira dos Santos, Jorge Nuno Silva*
ISEL & Cemapre-ISEG, ISEC & Cima, FCUL
acarvalho@adm.isel.pt, carlos.santos@isec.universitas.pt, jnsilva@cal.berkeley.edu

Number 1
March, 2014

Ludus CIUHCT

# *Articles*

## Mathematics of Soccer

*Alda Carvalho, Carlos Pereira dos Santos, Jorge Nuno Silva*
ISEL & Cemapre-ISEG, ISEC & Cima, FCUL
acarvalho@adm.isel.pt, carlos.santos@isec.universitas.pt, jnsilva@cal.berkeley.edu

**Abstract:** *Soccer, as almost everything, presents mathematizable situations (although soccer stars are able to play wonderfully without realizing it). This paper presents some, exemplifying each with situations of official matches.*

**Key-words:** Magnus effect, shot angles, soccer balls.

## 1 Introduction

There is some work done about mathematics and soccer. We highlight the book *The Science of Soccer* [10] which forms a extensive set of non-artificial examples of soccer situations where scientific methods may be useful. Also, the articles [8] and [5] are very interesting: the first one studies optimal shot angles and the second one discusses the spatial geometry of the soccer ball and its relation to chemistry subjects. In this paper, we expose some details of these works complementing it with *some situations from practice*. We strongly recommend the reader to accompany the reading of this paper with the visualization of the video [9]. Also, the section about goal line safes presents a new approach just using elementary geometry.

## 2 Shot Angles

Consider a situation in which a soccer player runs straight, with the ball, towards the bottom line of the field. Intuitively, it is clear that there is an optimal point maximizing the shot angle, providing the best place to kick in order to improve the chances to score a goal. If the player chooses the bottom line, the angle is zero and his chances are just horrible; if the player is kicking far way, the angle is also too small.

The geometry of the situation was studied in [8]. Briefly, we will expose the details of this, adding some well-chosen examples from real practice. In fig. 1, the point $P$ is the player's position. The straight line containing $P$ is the player's path. Considering the circle passing through $A$, $B$ and $P$, where $A$ and $B$ are the goalposts, it is a trivial geometric fact that the shot angle is half of the central angle determined by $A$ and $B$. Therefore, maximizing the shot angle is the same as maximizing the correspondent central angle.

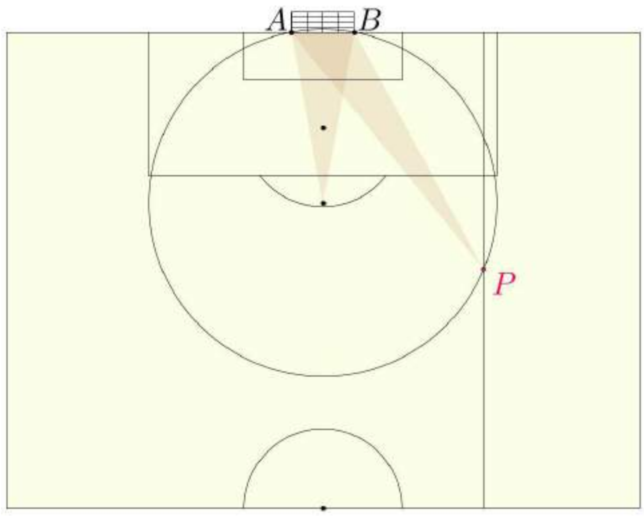


Figure 1: Related central angle.

Also, it is known from the Euclidian Geometry that the perpendicular bisector of $[AP]$ is a tangent line to the parabola whose focus is $A$ and the directrix is the player's path (fig. 2). Because the center of the circle belongs to the perpendicular bisector and the tangent line is *"below"* the parabola, the optimal point is the intersection between the parabola and the central axis of the field.

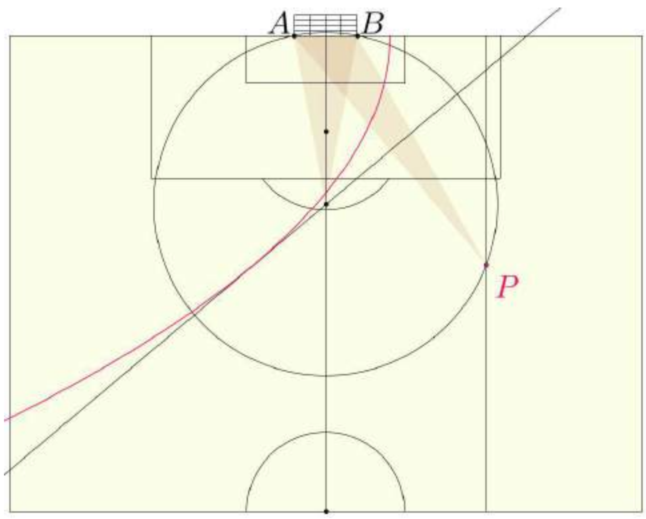


Figure 2: Fundamental parabola.

So, the distance between the optimal point and the point $A$ is exactly equal to the distance between the central axis and the player's path (it lies on parabola). Therefore, it is possible to construct the optimal point with a simple euclidian construction (fig. 3).

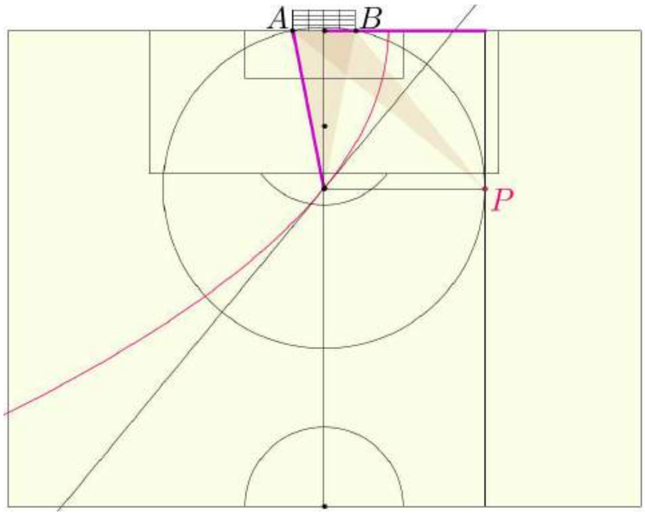


Figure 3: Optimal point.

Figure 4 plots the locus of the optimal points.

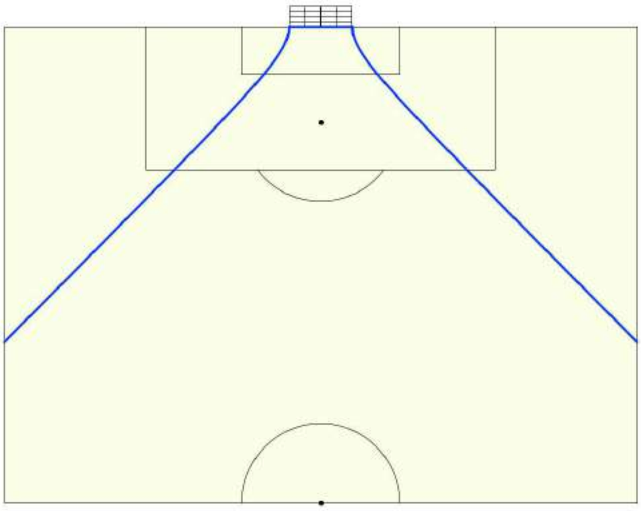

Figure 4: Locus of the optimal points.

Following, three peculiar episodes related to this geometric situation.

1. The second goal of the game Napoli 2 - Cesena 0, Serie A 1987/88, was scored by Diego Maradona [9]. It is an amazing example of the

"Maradona's feeling" about the optimal place to kick.


Figure 5: Maradona's kick.

2. The Van Basten's goal in the final of the Euro 88 is a classic [9]. It is a real example showing that mathematics *is not* useful for all situations. The forward kicked with a shot angle of roughly 6 degrees. Very far from the optimal!

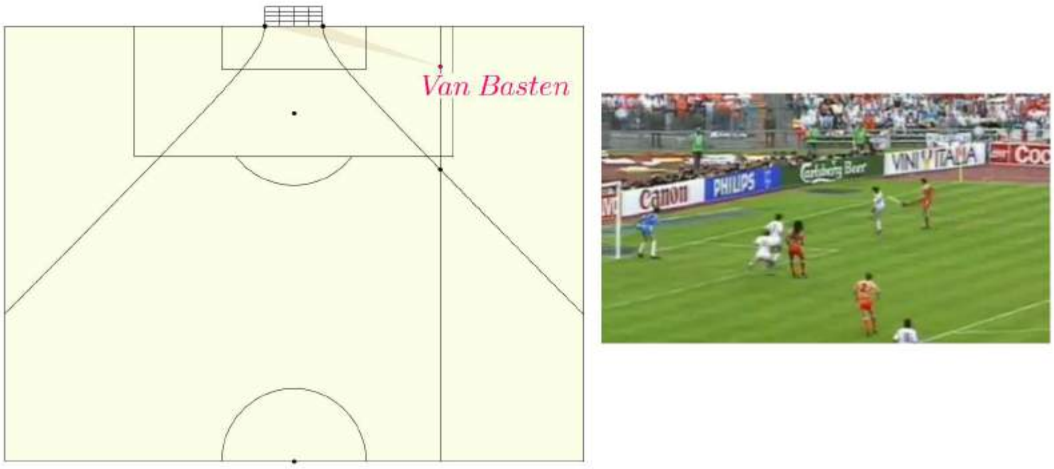


Figure 6: Van Basten's kick.

3. Some people say that Khalfan Fahad's miss was the worst ever [2]. It occurred in Qatar - Uzbekistan 0-1, Asian Games 2010 [9]. The funny kick was performed with an huge shot angle of roughly 90 degrees!

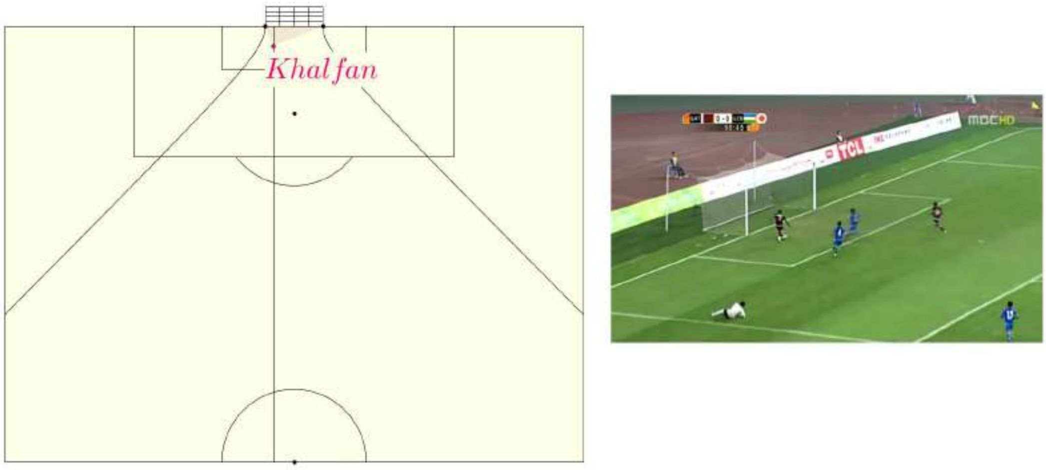


Figure 7: Khalfan Fahad's kick.

## 3 Goal Line Safes

In this section, we analyze the situation where a striker is kicking the ball into an empty goal and there is a defender who tries to intercept the ball in time. Consider $r = \frac{ball's\ speed}{defender's\ speed}$. In fig. 8, the point $S$ is the position where the striker is kicking the ball and the point $D$ is the position where the defender is when the kick is done.

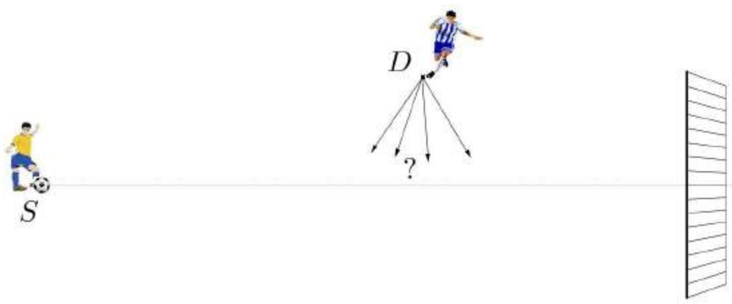


Figure 8: Goal Line Safes.

To construct geometrically the zone where the defender can intercept the ball it is possible to use similarity of triangles. First, consider arbitrarily a point $U$ and let $\overline{SU}$ be the unit. After, construct the point $R$ in such way that $\frac{\overline{SR}}{\overline{SU}}$ is equal to the ratio of speeds $r$. Constructing the unit circle with center in $R$, and considering $A$ and $B$, the intersections between the circle and the straight line $SD$, we have $\frac{\overline{SR}}{\overline{RA}} = \frac{\overline{SR}}{\overline{RB}} = r$. Therefore, if we consider the points $L$ and $Q$ in the ball's path in such way that $DL \parallel BR$ and $DQ \parallel AR$, the conclusion is that $\frac{\overline{SL}}{\overline{DL}} = \frac{\overline{SQ}}{\overline{DQ}} = r$. The segment $[LQ]$ is the zone where the ball can be intersected by the defender (fig. 9).

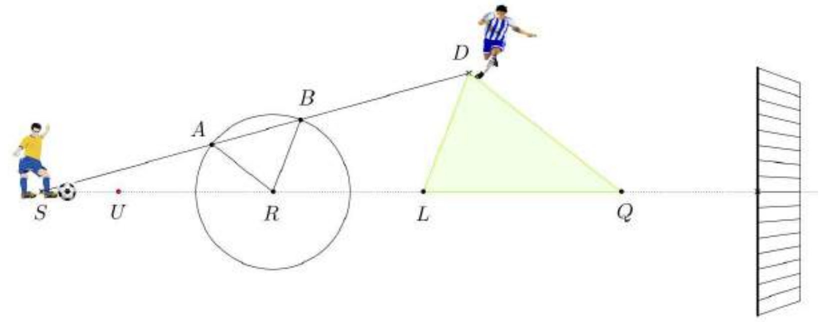


Figure 9: Intersecting the ball.

The indicated construction shows that the defender just has the possibility of success *when A and B exists*. If the defender is behind the tangent to the circle passing trough $S$, the intersection is just impossible (fig. 10).

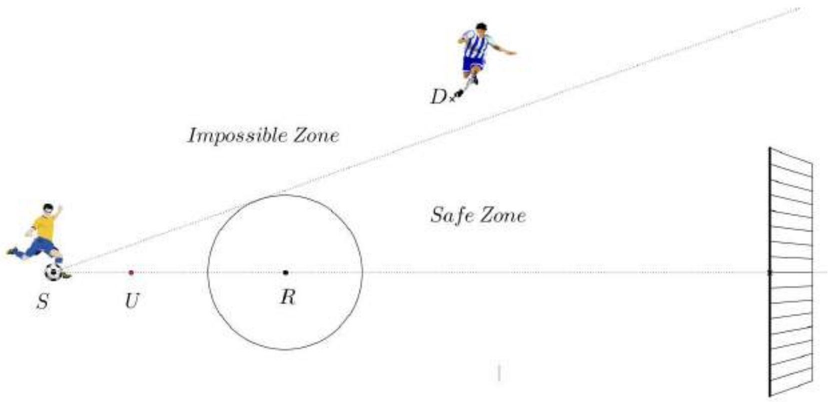


Figure 10: Safe zone.

In practice, there are several examples of nice goal line safe situations. One of the most impressive happened in a match West Ham - Aston Villa [9]. It is not an ideal situation: the ball goes through the air and we have no accurate way to assess the ratio of speeds. But it is a move that is worth seeing.


Figure 11: Great goal line safe.

## 4 Magnus Effect

The Magnus effect is observed when a spinning ball curves away from its flight path. It is a really important effect for the striker that charges a free kick.

For a ball spinning about an axis perpendicular to its direction of travel, the speed of the ball, relative to the air, is different on opposite sides. In fig 12 the lower side of the ball has a larger speed relative to the air than the upper side. This results in non-symmetric sideways forces on the ball. In order to have momentum conservation, we observe a sideways reaction acting downwards. German physicist Heinrich Magnus described the effect in 1852 [6].

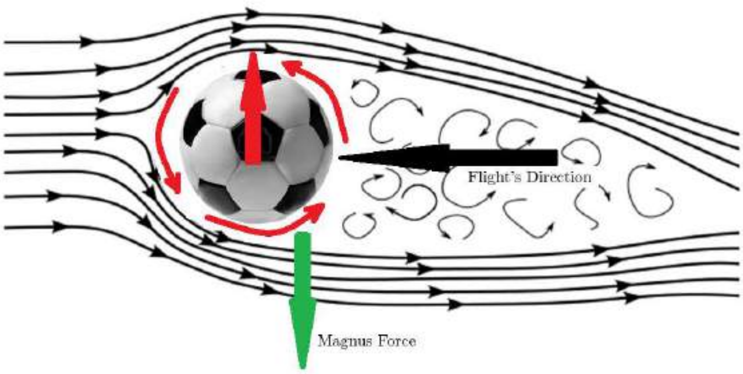


Figure 12: Magnus effect.

Suppose that a striker wants to kick as showed in fig 13. He chooses the length $q$, directly related to the angle relative to the straight direction, and he chooses the initial speed of the ball $S$. In [10] we can see a formula that nicely describes the flight path of the ball under the effect of the Magnus force. The formula uses the following physical parameters:

- $a$, radius $(0,11$ m$)$;
- $A$, cross-sectional area $(0,039\ m^2)$;
- $m$, mass of the ball $(0,43\ kg)$;
- $\rho$, air density $(1,2\ kgm^{-3})$.

The curve is described by

$$x = \frac{d\omega}{KS} y \left(1 - \frac{y}{d}\right)$$

where $K = \frac{8m}{a\rho A}$.

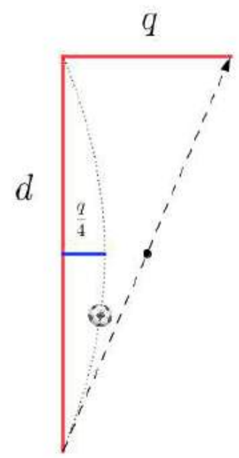


Figure 13: Free kick.

To obtain the path plotted in fig. 13, the striker must kick with the angular speed

$$\omega = \frac{KSq}{d^2}$$

in order to obtain $x = \frac{q}{d} y \left(1 - \frac{y}{d}\right)$. The number of revolutions during the complete flight is given by $\frac{d\omega}{2\pi S}$.

So, an expert must calibrate the direction (related to $q$), the initial speed and the angular speed (related to the number of revolutions during the flight). Of course an expert doesn't calculate anything but scores goals, a mathematician enjoys these concepts but doesn't score any goal!

One amazing example is the famous Roberto Carlos's free kick against France in 1997 [9]. This free kick was shot from a distance of 35 $m$. Roberto Carlos strongly hits the ball with an initial speed of 136 $km/h$ with an angle of about 12 degrees relative to the direction of the goal [3]. With a dynamic geometry software [4] it is easy to construct a simulator of the exposed situation. Figure 14 shows an interpretation of the famous free kick. Roberto Carlos needed an angular speed about 88 $rad/s$. Very difficult, but plausible.

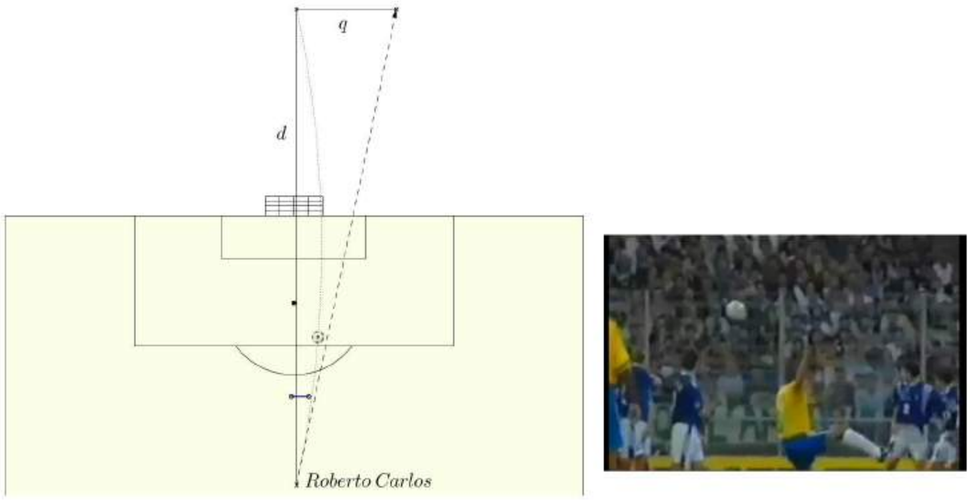


Figure 14: Roberto Carlos' free kick.

## 5 The Soccer Ball

The history of the geometric shape of the soccer ball has distinct phases [7]. Usually, we think in the truncated icosahedron as "the soccer ball", the design proposed by the architect Richard Buckminster Fuller. However, the first "official world cup" occurrence of this spatial shape (Adidas Telstar) was just in the Mexico, 1970. Before, the soccer balls had different shapes. Also, nowadays, the official balls have different shapes.

Figure 15: Charles Goodyear (1855), FA Cup final (1893), World Cup (1930), panel balls (1950), World Cup (1966).

Following, in the rest of this section, we will analyze some mathematics related to the Buckminster's proposal.

Figure 16: Eusébio, Portuguese player, World Cup, England, 1966 (before the Buckminster design).

An Archimedean solid is a very symmetric convex polyhedron composed of two or more types of regular polygons meeting in identical vertices. They are distinct from the Platonic solids, which are composed of only one type of polygon meeting in identical vertices. The solid is obtained truncating an icosahedron (fig. 17). The truncating process creates 12 new pentagon faces, and leaves the original 20 triangle faces of the icosahedron as regular hexagons. Thus the length of the edges is one third of that of the original icosahedron's edges.

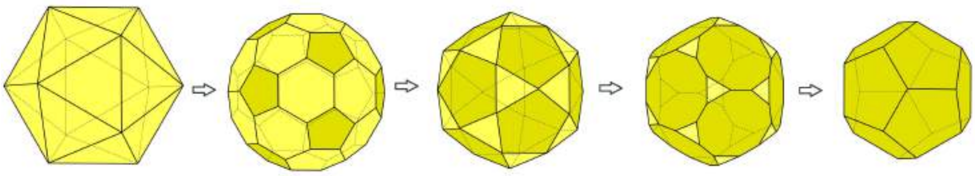

Figure 17: The "travel" from the icosahedron to dodecahedron: truncated icosahedron (1st), icosidodecahedron (2nd), truncated dodecahedron (3rd).

Figure 18: World Cup Final, Mexico 1970, Tostão with the ball, Pelé watching, (Buckminster's ball).

The Buckminster's proposal was well accepted. Just in the modernity, the soccer ball was improved (fig. 19).

Figure 19: Ball of the FIFA World Cup 2012 (South Africa), conduced by the Portuguese player Cristiano Ronaldo.

In [5], related to polyhedrons and soccer balls, because of the sphericity and aesthetical reasons, Dieter Kotschick proposed that the standard soccer ball should have three important properties:

1. The standard soccer ball should have only pentagons and hexagons (guaranteing a good sphericity) ;
2. The sides of each pentagon should meet only hexagons (isolating, for aesthetical reasons, the pentagons);
3. The sides of each hexagon should alternately meet pentagons and hexagons (for aesthetical reasons, in order to avoid groups of 3 joint hexagons).

Dieter Kotschick wrote that he first encountered the above definition in 1983, in a problem posed in *Bundeswettbewerb Mathematik* (a German mathematics competition). The definition captures the iconic image of the soccer ball.

In [5], we can see the explanation of a curious relation to a chemistry subject. In the 1980's, the 60-atom carbon molecule, the "buckyball" $C_{60}$ was discovered. The spatial shape of $C_{60}$ is identical to the standard soccer ball. This discover, honored by the 1996 Nobel Prize in chemistry (Kroto, Curl, and Smalley), created interest about a class of carbon molecules called fullerenes. By chemical properties, the stable fullerenes present the following properties:

1. The stable fullerenes have only pentagons and hexagons;
2. The sides of each pentagon meet only hexagons;
3. Precisely three edges meet at every vertex.

Only the third item is different in the two definitions. Some natural questions arise:

1. Are there stable fullerenes other than buckyball?
2. Are there soccer balls other than the standard one?
3. How many soccer balls are also stable fullerenes?

It is possible to use the famous and classical Euler's formula, $V - E + F = 2$, to answer to these questions. About the first one, consider $P$ and $H$, the number of pentagons and hexagons. Of course, $F = P + H$. Also, because each pentagon has 5 edges and each hexagon has 6 edges, and because we don't want to count each edge twice, $E = \frac{5P+6H}{2}$. By a similar argument, because precisely three edges meet at every vertex, $V = \frac{5P+6H}{3}$. Using the Euler's formula, replacing and canceling, we obtain the fundamental equality for stable fullerenes:

$$P = 12.$$

The number of pentagons must be 12, but there is an unlimited number of possibilities for the number of hexagons compatible with Euler's formula. After the $C_{60}$, other fullerenes were discovered and object of research. For instance, $C_{70}$ is a fullerene molecule consisting of 70 carbon atoms. It is a structure which resembles a rugby ball, made of 25 hexagons and 12 pentagons (fig. 20).

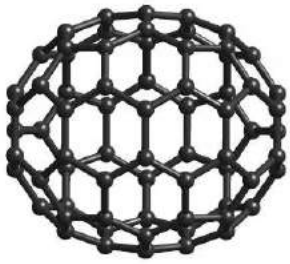

Figure 20: $C_{70}$

The first question is solved.

About soccer balls, we also have $F = P + H$ and $E = \frac{5P+6H}{2}$. However, the third item of the definition is different and we can have more than three edges meeting at a vertex. So, $V \leqslant \frac{5P+6H}{3}$ and $P \geqslant 12$. It is possible to use the third item of the definition of the soccer ball: exactly half of the edges of hexagons are also edges of pentagons so, $\frac{6H}{2} = 5P \Leftrightarrow 3H = 5P$. The fundamental conditions for Kotschick's soccer balls are

$$P \geqslant 12 \quad \textit{and} \quad 3H = 5P.$$

Again, there is an unlimited number of possibilities for Kotschick's soccer balls. In fig 17, we observe a well known process to have archimedean solids from platonic solids. In [1], it is proposed a topological way, *branched covering*, to have a new soccer ball from a previous one. Topology is the branch of mathematics that studies properties of objects that are unchanged by continuous deformation, so, a"elastic ball" is considered. In fig. 21 the process is exemplified.

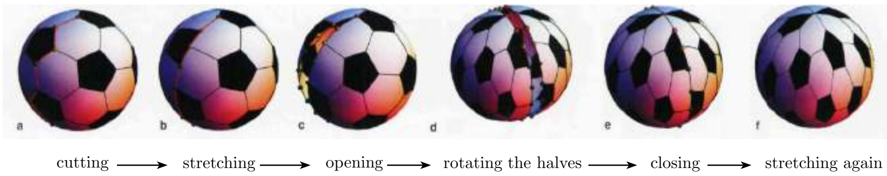


Figure 21: New soccer ball compatible with the definition and Euler's formula obtained by the branching covering process (pictures from [5]).

Therefore, the second question is solved. About the third one, for fullerenes, we have $P = 12$; for soccer balls we have $3H = 5P$. So, an object that is simultaneously a fullerene and a soccer ball must have the Buckminster's design ($P = 12$ and $H = 20$). There is something essential in the truncated icosahedron!

## 6 (Illogical) Rules

This final part is about the rules of competitions. Strangely, the choice of rules is not as simple as we might think. Sometimes, poorly chosen tiebreakers can generate totally bizarre situations. An impressive situation occurred in a match Barbados vs Grenada (Shell Caribbean Cup 1994). The situation after 2 games is described in the fig. 22.

Barbados needed to win by two goals the last match against Grenada to progress to the finals. But a trouble arises...

1. The organizers had stated that all games must have a winner. Also, all games drawn over 90 minutes would go to sudden death extra time;
2. There was an unusual rule which stated that in the event of a game going to sudden death extra time, *the goal would count double.*

Barbados was leading 2-0 until the 83rd minute, when Grenada scored, making it 2-1.

Approaching the final of the match, the Barbadians realized they had little chance of scoring past Grenada's mass defense in the time available, so they deliberately scored an own goal to tie the game at 2-2.

Figure 21: Shell Caribbean Cup 1994 after 2 games.

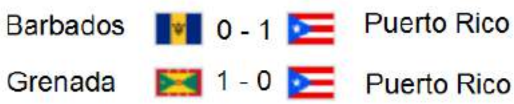


Figure 22: Shell Caribbean Cup 1994 after 2 games.

The Grenadians realized what was happening and attempted to score an own goal as well. However, the Barbados players started defending their opposition's goal to prevent this.

During the game's last five minutes, a crazy situation happened: Grenada trying to score in either goal while Barbados defended both ends of the soccer field.

After 4 minutes of extra time, Barbados scored the golden goal and qualified for the finals. The thing was unbelieving. It is possible to see a video showing Grenada's second goal [9]. If the reader has a video with the final five minutes of the regular time, please send to us!

Figure 23: The strangest goal ever.

# References

[1] Braungardt, V.; Kotschick, D. "The Classification of Soccer Ball Patterns". Math. Semesterber. 54, No. 1, pp. 53-68, 2007.

[2] Daily Record, http://www.dailyrecord.co.uk/sport/football/worst-miss-ever-costs-qatar-1075944

[3] G. Dupeux, C. Cohen, A. Goff, D. Quéré, C. Clanet. "Football Curves". Journal of Fluids and Structures, 27, pp. 659-667, 2011.

[4] Geogebra, Dynamic mathematics & science for learning and teaching,http://www.geogebra.org

[5] D. Kotschick. "The Topology and Combinatorics of Soccer Balls". The American Scientist, 2006.

[6] G. Magnus. "Über die Abweichung der Geschosse". Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin, pp. 1–23, 1852.

[7] Soccer Ball World, http://www.soccerballworld.com

[8] E. Veloso. "Ronaldo e o SketchPad". Educação Matemática, 46, pp. 39–42, 1998.

[9] Soccer Episodes, http://jnsilva.ludicum.org/LUDUS/Opuscula/RMM/Soccer.m4v

[10] J. Wesson. *The Science of Soccer*. Taylor & Francis, 2002.