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# Games and puzzles

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## EXPLORING MOD 2 $n$ -QUEENS GAMES

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**Abstract:** *We introduce a two player game on an  $n \times n$  chessboard where queens are placed by alternating turns on a chessboard square whose availability is determined by the parity of the number of queens already on the board which can attack that square. The game is explored as well as its variations and complexity.*

**Keywords:** Chess, independence,  $n$ -queens.

### 1 Introduction

The  $n$ -queens problem is the challenge of placing  $n$  queens on an  $n$  by  $n$  chessboard such that none are attacking each other. This puzzle originated as the 8-queens problem, played on a standard  $8 \times 8$  chessboard, and was proposed by Bezzel [2] in 1848. The 8-queens problem and the more general  $n$ -queens problem attracted the interest of notable mathematicians of that time. In 1850, Nauck [3] was the first to publish all 92 solutions for the standard  $8 \times 8$  chessboard and in 1874 Pauls [5, 6] gives the first set of solutions for the general  $n$ -queens problem published in two articles.

The problem of finding more solutions in various dimensions has continued to hold the interest of mathematicians and computer scientists. Solutions sets have been found using graph theory, magic squares, Latin squares, and group theory, among other techniques. (See a survey paper by Bell and Stevens [1] for more details.) In a brute force approach, solution sets for a given  $n$  can be found use the backtracking algorithm; a standard technique taught in computer science classes. In more modern times, a player can even find mobile apps to test his or her ability to find solutions on  $8 \times 8$  or other dimensional boards.

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In this paper, we look at a game variant of the  $n$ -queens problem that can be played on chessboards of varying dimensions, although here, we limit our focus to square chessboards. The basic version of the  $n$ -queens game is described by Noon [4], who, observing that not all placements of queens will lead to a full solutions with  $n$  queens, suggests a two player game where each player successively places queens in non-attacking positions. The first player who cannot place a queen loses the game. In the next section, we introduce a modification of this  $n$ -queens game. Sections 3 and 4 explore two states a game board may take. Then Section 5 discusses alternate versions of the game, while Section 6 looks at the complexity of the game. Open questions are given throughout.

## 2 The mod 2 $n$ -queens game

First, we describe a variation of the  $n$ -queens game. For any game, a player should consider the motivation of his or her opponent. In particular, in a game involving placing a queen on a chessboard, when is that queen safe from attack; that is, would it be a good move for an enemy queen to attack this piece? If you place a queen on a square that cannot be attacked by any queen, clearly it is safe. However, if you place your queen on a square that may be attacked by exactly one queen, the enemy queen can attack your piece while remaining safe on the square after your piece is taken. Continuing along these lines, what if you place your queen on a square that may be attacked by exactly two queens? Neither of the enemy queens would wish to attack first because then she would be vulnerable to the other. Hence, you could safely hold this square. Next, consider placing your queen on a square that may be attacked by exactly three queens. Any one of these enemy queens has incentive to attack, because after she claims the square, the remaining two queens are once again at a stalemate. Therefore, it would not be wise to place your queen on the square in that case. We see that for any number of queens on the board, the safety of a square is determined by the parity of the number of queens which may attack that square, so we propose the following additional rule to the two-player  $n$ -queens game.

Rule: Let every non-occupied square take on a value given by the number of queens who are directly attacking that square. If the value is congruent to zero modulo two, then the square is open and a queen may be placed on it. If the value is congruent to one modulo two the square is closed and we may not place a queen on that square.

Of course this rule implies squares with an even number of attacking queens are open and with an odd number of attacking queens are closed. Play continues as before until a losing player cannot place another queen on the board. We will refer to this game as the **mod 2  $n$ -queens game**.

This rule opens up the possibility for more play compared to Noon's original  $n$ -queens game. Figure 1 (left) shows an example of a set of queens played on an  $8 \times 8$  chessboard. This figure uses a gradient to indicate the number of queens attacking each square on the chessboard, with white indicating an open square and darker shades indicating an increasing number of queens attacking that square.

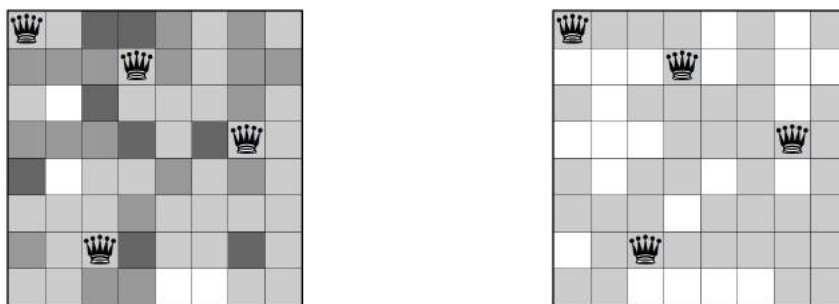


Figure 1: Left: A chessboard with a gradient indicating attacking queens. Right: A chessboard with a mod two gradient indicating attacking queens.

With the new rule, we are able to simplify this illustration as well as open up more squares on the board. Figure 1 (right) illustrates the same configuration of queen with a now much simpler and more open gradient given by the mod 2  $n$ -queens game. We note, in traditional play, placing another queen will only cause squares to become progressively darker in the gradient, but once a square is closed it remains closed. The mod 2  $n$ -queens game is much more dynamic. Squares may easily change from open to closed and vice versa as the game progresses.

There are other differences between this version of the game and the original  $n$ -queens game. For one, the order of the placement of the queens now matters as the parity of a square may change throughout the game. Next, the maximum number of moves in the game increases. A maximum of  $n^2$  queens can be placed to fill the board with the new rule, whereas the original game could have at most  $n$  queens.

Before exploring the game play, we define three game states for any arrangement of queens upon a square chessboard created by a mod 2  $n$ -queens game. We call such chessboards complete, unlocked, and locked as follows:

**Definition.** Given an arrangement of queens upon a chessboard, the chessboard is **complete** if all  $n^2$  square are covered by queens. The chessboard is **unlocked** if it has fewer than  $n^2$  queens placed on it, and there exist empty open squares in which a queen may be placed. Finally, the chessboard is **locked** if it has fewer than  $n^2$  queens, but no legal moves remain.

Finally, before we look at some complete chessboards, let us review some chessboard terminology. Each square on a  $n \times n$  chessboard will be indexed by an **ordered pair**  $(i, j)$  where  $1 \leq i \leq n$  indicates the row numbered from top to bottom and  $1 \leq j \leq n$  indicates the column numbered from left to right. The  **$k$ -sum diagonal** is the diagonal running from left to right and bottom to top in which the sum of the indices of each square is  $k$  for some integer  $1 \leq k \leq 2n$ . The  **$k$ -difference diagonal** is the diagonal running from left to right and top to bottom in which the difference of the indices of each square is  $k$  for some integer  $-(n-1) \leq k \leq n-1$ . The  $(n+1)$ -sum diagonal is known as the **main sum diagonal**, and similarly the 0-difference diagonal is called the **main difference diagonal**.

### 3 Complete Chessboards

We begin our exploration of the mod 2  $n$ -queens games by considering complete chessboards. By definition, we know that a complete chessboard contains  $n^2$  queens, so we ask, given an  $n \times n$  chessboard is it always possible to achieve a complete chessboard state through legal play?

In the case where the size of the board is odd, the answer is yes.

**Proposition 1.** If  $n$  is an odd positive integer, the  $n \times n$  chessboard has a complete solution of  $n^2$  queens.

*Proof.* Thinking inductively, a  $1 \times 1$  chessboard may be filled with one queen. In the next case, if we want to fill a  $3 \times 3$  board we need to fill the top two rows and left two columns with queens as well as the  $1 \times 1$  board in the lower right corner. More generally, to fill a  $n \times n$  board we need to fill the top two rows, the left two columns, and a  $(n - 2) \times (n - 2)$  board in the lower right corner. Looking at the game position where queens have filled the first two rows and the first two columns of the chessboard as illustrated in Figure 2 in the case  $n = 11$ , we observe that every uncovered square on this chessboard can be attacked by an even number of queens.

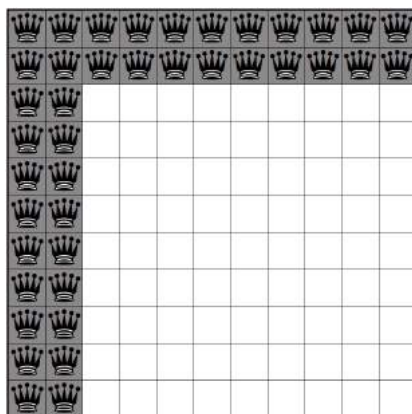


Figure 2: An unlocked  $11 \times 11$  chessboard.

Specifically, each square can be attacked vertically, horizontally, and along the difference diagonal by exactly two queens on each line. Squares to left or on the main diagonal are attacked by four more queens on the sum diagonal, squares on the  $n + 2$  sum diagonal are attacked by exactly two more queens, and all other squares are attacked by zero more queens. Therefore game play from the chessboard with queens in this starting configuration is equivalent to the game play on a  $(n - 2) \times (n - 2)$  board with no queens. Because we can proceed inductively to a  $1 \times 1$  board, now we only need to show that this starting board can be obtained from a sequence of legal game moves.

Begin by filling the upper left corner of a  $n \times n$  chessboard with the eight queens in the sequence given by Figure 3 (left). In the following steps, we will

successively fill in the next four squares from the first two rows and column. We proceed as listed in the steps below and illustrated in Figures 3 (center) and 3 (right) for an increasing integer  $k$  where  $1 \leq k \leq \frac{n-2}{2}$ .

Step 1: Place a queen in square  $(2, 2k + 2)$  and then in square  $(1, 2k + 2)$ .

Step 2: Place a queen in square  $(2k + 2, 1)$  and then in square  $(2k + 2, 2)$ .

Step 3: Place a queen in square  $(1, 2k + 3)$  and then in square  $(2, 2k + 3)$ .

Step 4: Place a queen in square  $(2k + 3, 2)$  and then in square  $(2k + 3, 1)$ .

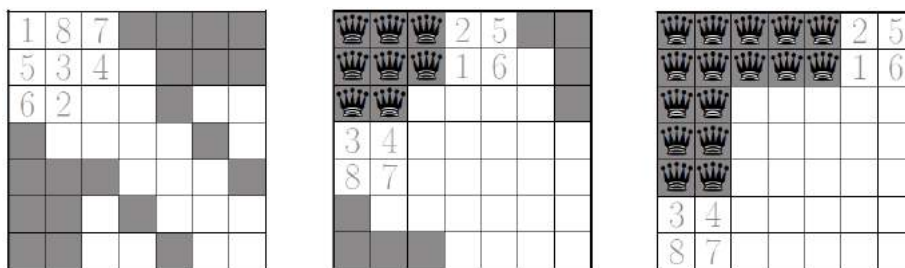


Figure 3: Left: Place the first eight queens. Center: Place the next eight queens. Right: Steps in the construction of a complete board for  $n$  odd.

Once this process is finished, we repeat on the  $(n - 2) \times (n - 2)$  chessboard until we have reduced to the one empty square in the lower right corner which can then be filled with a queen.  $\square$

**Proposition 2.** If  $n$  is an even positive integer, the  $n \times n$  chessboard does not have a complete solution of  $n^2$  queens.

*Proof.* Suppose to the contrary, that there is an open square in which to place  $(n^2)^{\text{th}}$  queen on a  $n \times n$  chessboard, and further suppose that square is on the outer edge of the chessboard. Each square on the edge has  $n - 1$  queens attacking along the horizontal line and  $n - 1$  queens attacking along the vertical line. If we chose a corner square, it only has one diagonal containing  $n - 1$  attacking queens. As you move from a corner square along the edge, the larger diagonal decreases by one and the lesser diagonal increases by one which always keeps the sum of the diagonals of a square on the edge equal to  $n - 1$ . Thus, the open square cannot be one of the squares on the edge of the chessboard because the number of attacking queens is  $(n - 1) + (n - 1) + (n - 1) \equiv_2 1$ . Thus we can assume the outer ring is filled with queens, so we remove this ring without changing the parity of the chessboard, as each square in the middle is attacked by exactly eight queens from the outer ring. We are left with  $(n - 2) \times (n - 2)$  chessboard. As  $n - 2$  is still even, we repeat, removing the outer rings, until we are left with the contradiction, a  $2 \times 2$  chessboard which cannot be filled with four queens.  $\square$

Although we cannot have a complete solution in the case of an even length chessboard, we can identify a locked solution that comes close.

**Proposition 3.** If  $n$  is an even positive integer,  $n^2 - 2$  queens can be placed legally on a  $n \times n$  chessboard.

One such construction could follow similarly to the odd case by applying induction and filling the topmost two rows and leftmost two columns, so we leave it to the reader. This case is interesting, however, and we pose the following question.

**Question 4.** Is the solution of  $n^2 - 2$  queens on an  $n \times n$  chessboard for  $n$  an even positive integer a maximal locked solution?

Empirical data and a computer simulation for  $n = 4$  suggest that the answer is yes. However, with our modest computational power the program could run the simulation for  $n = 4$  in eleven minutes but it would take approximately 22 years to run for  $n = 6$ , so verifying computationally was not an option.

In the next section, we consider locked chessboards.

## 4 Locked Chessboards

Another interesting game state is a locked chessboard. For strategy reasons, a player would be interested to know of locked solutions in order to lead an opponent into such a solution or avoid them himself. For small  $n = 1, 2$  or  $3$ , there is a simple first-player win strategy, that is, placing only one queen as seen in Figures 4 (left) and 4 (right).



Figure 4: Left: A locked  $2 \times 2$  chessboard. Right: A locked  $3 \times 3$  chessboard.

Of course, these small examples cannot be generalized and locked boards with only one queen do not exist for  $n > 3$ . We now look at a more complex locked chessboard. Again, as we did for complete boards, we separate the chessboards into two classes by their even or odd lengths.

Consider the chessboards where queens fill the top row and leftmost column for  $n$  odd or almost fill the top row and leftmost column for  $n$  even. These game positions are illustrated in Figure 5.

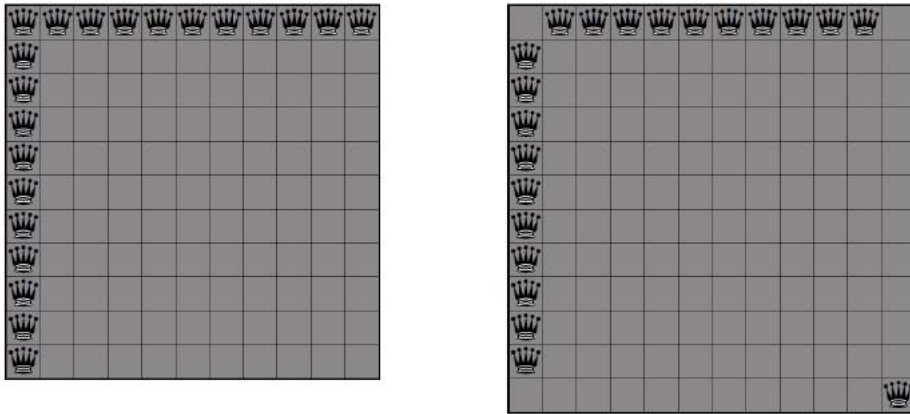


Figure 5: Left: A locked  $11 \times 11$  chessboard. Right: A locked  $12 \times 12$  chessboard.

**Proposition 5.** For  $n$  an odd positive integer, the chessboard with queens exactly on squares in the set  $\{(1, i), (i, 1) | 1 \leq i \leq n\}$  is a legal, locked game position.

*Proof.* We need to check that this board is locked as well as check that the queens can be placed in this configuration by a legal set of game moves. First we confirm that an odd number of queens attack every uncovered square. Observe that every unoccupied square has exactly one queen which can attack vertically and exactly one queen that can attack horizontally. On the positive diagonal, if the uncovered square is on or to the left of the main sum diagonal exactly two queens can attack along the positive diagonal. However, if the uncovered square is the right of the main diagonal, no queens can attack along the positive diagonal. Further every empty square can be attacked by exactly one queen on the difference diagonal, so all uncovered square are attacked by either exactly five or exactly three queens and hence the board is locked.

Next, we need to show that this game board can arise from a sequence of legal placements of queens onto the board. First consider the case where  $n = 3$ . Figure 6 illustrates a sequence of five moves to lock a  $3 \times 3$  chessboard by playing queens on the first row and column.

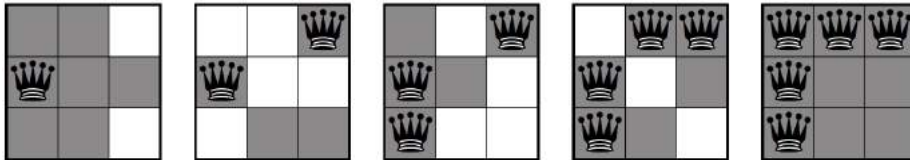


Figure 6: A sequence of game positions to create a locked  $3 \times 3$  chessboard.

We want to extend the game play in Figure 6 to occur in a chessboard of any size. Our strategy will be to repeat the first four moves of Figure 6 for the next two vacant squares in the first row and the corresponding two vacant squares in the first column. Suppose we wish to fill the four squares from the set



$\{(1, i), (1, i+1), (i, 1), (i+1, 1)\}$  with queens. In each stage will fill two squares in column one and two squares in row one, so the total number of queens attacking either horizontally or vertically must be even and further there are no queens which can attack from either of the two diagonals. Squares  $(i, 1)$  and  $(1, i+1)$  are not on a line vertically, horizontally, or diagonally, so a queen can be placed on each of these squares. The remaining squares  $(i+1, 1)$  and  $(1, i)$  can be attacked by both queens that were just played and also are not on a line with each other vertically, horizontally, or diagonally. We can fill these squares with queens. After repeating this process  $\frac{n-1}{2}$  times, each square in the first row and first column, except the square in the upper left corner, contains a queen. As the square in the upper left can be attacked by all  $2(n-1)$  queens on the board, we can place the final queen in this position and hence lock the board.  $\square$

We can find a locked board in the case where  $n$  is even as well.

**Proposition 6.** For  $n$  an even positive integer and  $n > 2$ , the chessboard with queens on squares in the set  $\{(1, i), (i, 1) | 2 \leq i \leq n-1\} \cup \{(n, n)\}$  is a legal, locked game position.

The proof is similar to that of Proposition 5, so we leave to the reader to check that this chessboard is locked and arises from a sequence of legal game moves.

We can now make the following observation:

**Corollary 1.** When  $n$  is an odd positive integer, at most  $2n-1$  queens are needed to lock the board, and when  $n$  is an even positive integer, at most  $2n-3$  queens are needed to lock the board.

This corollary gives an upper bound on the number of queens need to lock an even or odd chessboard, but is this bound strict? We know for very small values of  $n$  the chessboard can be locked with fewer queens, but what about larger values? We pose the next question.

**Question 7.** What is the minimum number of queens needed to lock a  $n \times n$  chessboard?

Next, we will look at different versions of the game with alternate rules and the connections between them.

## 5 In the Alternate Universe

Thus far, we have seen that there is a difference in outcomes for chessboards with even and odd length. We wish to modify the rules of the game to look at an alternative version of the game where queens can be placed on squares that are attacked by an *odd* number of queens and cannot be placed on squares attacked by an even number of queens. We will call such a game an **alternate universe mod 2  $n$ -queens game**.

The first observation is that generally this cannot happen as an empty board has no squares attacked by an odd number of queens. So we will modify the rules again to start with one queen already on the board before we commence play.

A second way to play the game would be to start with  $n^2$  queens on the board and try to remove the queens one by one following the rule that a queen may be removed as long as she can be attacked by an even number of queens. We call such a game an **complementary mod 2  $n$ -queens game**. The relationship between alternate universe, complementary games, and standard mod 2  $n$ -queens games is dependent on the size of the chessboard.

**Proposition 8.** When  $n$  is an even positive integer, the alternate universe mod 2  $n$ -queens games are in bijection with complementary mod 2  $n$ -queens games.

*Proof.* Earlier we saw that if an even length chessboard is fully covered with queens, each square is attacked by an odd number of queens. The initial step for both games is the same. We place one queen in the alternate universe and remove one queen from the same square in the full complementary game. While playing the complementary game, queens can only be removed if the number of queens attacking is even and hence because of the parity of the board, it means the number of empty squares “attacking” must be odd. By taking the complement of the board, removing queens to make empty squares, and placing queens on otherwise empty squares, we can go back and forth from the complementary game to the alternate universe game.  $\square$

We note that the alternate universe game on the odd length chessboards does not have the same relationship. In fact, we have the following:

**Proposition 9.** When  $n$  is an odd positive integer, the complementary mod 2  $n$ -queens games are in bijection with standard mod 2  $n$ -queens games.

*Proof.* In this case, the beginning parity for every square on a complementary game board is even. We obtain the bijection by placing a queen in the same square of the standard game from which we removed it in the complementary game.  $\square$

Thus for each case, even or odd length chessboard, there is really only two different games to be played, standard and alternate universe, as the complementary game can be completely reconstructed from one of the other two games.

We conclude this section with an open-ended question.

**Question 10.** What other  $n$ -queens games can be played on a  $n \times n$  chessboard? In particular, how could we adapt play for a **mod  $k$   $n$ -queens game**?

In the next section we will discuss the complexity of the game and give examples of a game tree and a graph created from this tree.

## 6 Game Trees, Graphs, and Complexity

Chess and other games played with chess pieces are complex games. We will discuss several standard measures of complexity of a game for the mod 2  $n$ -queens game. A **game tree** is the rooted tree whose root is the empty board and whose leaves are locked or complete boards such that the directed paths from the root to a leaf display all possible games. Figure 7 shows the first three levels of the game tree in the case where  $n = 3$ .

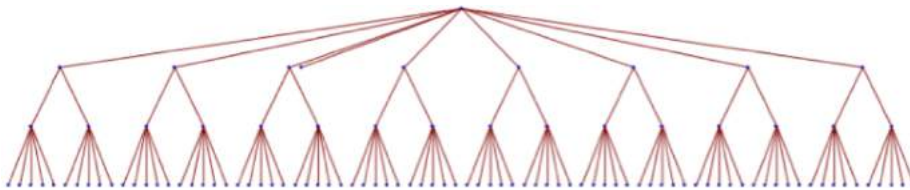


Figure 7: Partial game tree for  $n = 3$ .

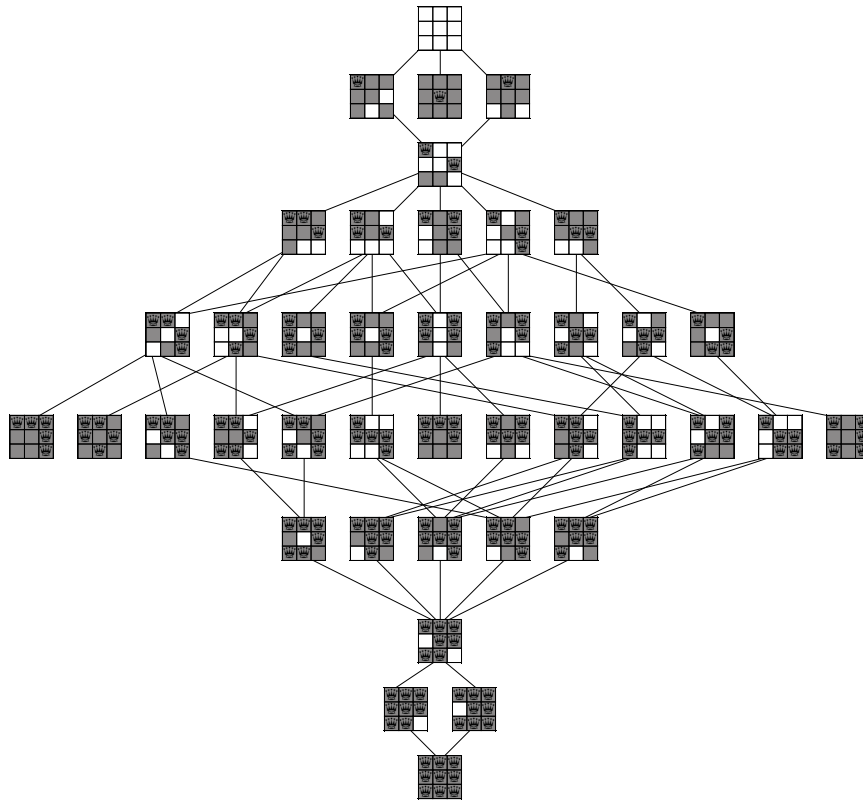
Because there are  $n^2$  squares on the board, we can get an upper bound for the number of leaf nodes or **game tree size** at  $n^2!$ .

Many of the game boards represented by these nodes are identical or symmetric by rotation or reflection. In fact, any given game board has up to eight symmetric boards. To simplify the structure of this tree, we propose to combine nodes describing boards which are equivalent by symmetry. In this case, we have fewer nodes, but the tree has become a graph. Figure 8 shows the full graph for  $n = 3$ . This graph is still quite large even for smallest non-trivial value for  $n$ .

We are also interested in the complexity of the game. Because, for some  $n$ , all possible squares may be filled, we have a finite set of  $n^2$  squares; any subset of which could be filled with queens. This set of all subsets of  $n^2$  elements is the power set which has size of  $2^{n^2}$ . Since defining rules for the board limits the number of game boards,  $2^{n^2}$  is the upper bound for the **state-space complexity** or number of legal game positions. Therefore we have a complexity  $O(2^{n^2})$  and this game is in EXPTIME.

We conclude with one final question.

**Question 11.** Clearly, the upper bounds for the game tree size and the state-space complexity are not firm for all  $n$ . Can these be improved?

Figure 8: Partial game tree for  $n = 3$ .

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