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Games and puzzles

The 5-Way Scale

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Abstract: *In this paper, we discuss coin-weighing problems that use a 5-way scale which has five different possible outcomes: MUCH LESS, LESS, EQUAL, MORE, and MUCH MORE. The 5-way scale provides more information than the regular 3-way scale. We study the problem of finding two fake coins from a pile of identically looking coins in a minimal number of weighings using a 5-way scale. We discuss similarities and differences between the 5-way and 3-way scale. We introduce a strategy for a 5-way scale that can find both counterfeit coins among 2^k coins in $k+1$ weighings, which is better than any strategy for a 3-way scale.*

Keywords: coin weighings

1 Introduction

Coin problems have been a fascination for mathematicians for a long time [1]. In most of them there are many coins that look identical and a balance scale with two pans. Usually all real coins weigh the same and all counterfeit coins weigh the same, though lighter than real coins. The clasical balance scale used in past models has three different outcomes: LESS, EQUAL, and MORE. If the left pan is lighter, we denote the outcome as LESS. If the right pan is lighter, we denote the outcome as MORE. If the two pans are of equal weights, the outcome is EQUAL.

In this paper we assume that the scale is more refined; that is, it has five different outcomes: MUCH LESS, LESS, EQUAL, MORE, and MUCH MORE. The LESS and MORE outcomes correspond to the weight difference between one fake coin and one real coin, while MUCH LESS and MUCH MORE to a larger difference.

In the most classical problem [1] it is known that one coin is fake and lighter than the rest and the goal is to find it in the smallest number of weighings. For the new scale this problem is not interesting, as MUCH MORE or MUCH LESS outcomes are never possible with the same number of coins on both pans. In other words, the 5-way scale doesn't give any advantage; it can only be used as a 3-way scale.

In this paper we study the problem of finding two counterfeit coins among many identical-looking coins. We introduce the problem in Section 2.

In Section 3 we find a natural information-theoretic bound. In Section 4, we study the first weighing and consider the outcome that gives us the least information.

In Section 5 we study a special case when the two counterfeit coins are separated into two different piles. In Section 6 we give a strategy using $k+1$ weighings for the case of separated coins when we have two piles of equal sizes 2^k. This allows to create a strategy for our general case in Section 7 which is better than any strategy for a 3-way scale: it uses $k+1$ weighings for 2^k coins.

2 Problem Formulation

We assume that the coins are identical and there are two types of coins: real and counterfeit. Real coins weigh the same. Counterfeit coins also weigh the same, but lighter than real ones.

We consider a new type of scale with five possible outcomes for each weighing: MUCH LESS, LESS, EQUAL, MORE, and MUCH MORE. We put the same number of coins on the pans. LESS means that the number of counterfeit coins on the left pan is one more than the number of counterfeit coins on the right pan. MUCH LESS means that the left pan has at least two more counterfeit coins than the right pan. MORE and MUCH MORE are similarly defined. EQUAL means that there are an equal number of counterfeit coins on both pans.

As this scale is more precise than the usual 3-way scale in the standard coin problems, the number of coins that we can process in a given number of weighings is at least the same. Indeed, if we treat MUCH LESS as LESS and MUCH MORE as MORE, then the 5-way scale becomes a 3-way scale.

The problem of finding one counterfeit coin is not interesting, as we never can get MUCH LESS or MUCH MORE. Therefore, we assume that we have at least 2 counterfeit coins, and at least 2 real coins. We note that the case of n counterfeit and k real coins is equivalent to the case of k counterfeit and n real coins.

In this paper we study the simplest case of two counterfeit coins:

> There are $N > 3$ identical-looking coins, two of them are counterfeit. What is the smallest number of weighings so that we are guaranteed to find both counterfeit coins on a 5-way scale?

3 The Information Theoretic Bound

The standard way to find a lower bound for the number of coins that can be processed in a given number of weighings is to count outcomes. Here is the argument for a 3-way scale.

Each weighing has 3 outcomes. That means, there are a total of 3^w possible outcomes after w weighings. We need to find our two counterfeit coins after these weighings. What is the maximum number of coins we can process?

Suppose we found our two coins. That means one of 3^w possible outcomes led us to these pair of bad counterfeit coins pretending to be real. From this observation follows an information theoretical bound of $\binom{N}{2} \leq 3^w$, as each outcome must point to a different pair of coins. That means $N(N+1) \leq 2 \cdot 3^w$, or

$$N \leq \frac{\sqrt{8 \cdot 3^w + 1} - 1}{2} \approx \sqrt{2 \cdot 3^w}. \quad (1)$$

Computational results show that the bound is very close to the actual number of how many coins we can process for a 3-way scale [2, 3, 4]. Given that we can ignore the distinction between MUCH LESS and LESS, any strategy that works for a 3-way scale, also works for a 5-way scale.

But our scale is spiffier. Does that mean we can move the bound up? A similar argument replacing 3 with 5 shows that we can process not more than

$$N \leq \frac{\sqrt{8 \cdot 5^w + 1} - 1}{2} \approx \sqrt{2 \cdot 5^w}$$

coins.

This is a larger number than the bound for the 3-way scale. But is our new bound tight? For the rest of the paper our goal is to find a strategy for a 5-way scale that is better than any strategy for a 3-way scale.

Let us consider an example. Suppose we have five coins with two counterfeits among them. How many weighings do we need? There are 10 possibilities for pairs of counterfeit coins. Our bound proves that one weighing is not enough. Can we do it in two weighings?

First compare two coins against two coins. If if the outcome is MUCH LESS or MUCH MORE, then, hooray, we found the two counterfeit coins. This is the beauty of the 5-way scale. There is no way to find two counterfeit coins out of five in one weighing using a regular 3-way scale.

Can our luck continue? If the first weighing is LESS or MORE, then we know that one counterfeit coin is outside, and the other is on the lighter pan. Indeed, comparing the two coins on the lighter pan would lead us to our second counterfeit coin.

If the first outcome is EQUAL, then we know that one counterfeit coin is on each pan, and one real coin must be outside. Denote the coins on the left side

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as 1 and 2, the right side as 3 and 4, and the coin outside as 5. In the second weighing we compare 1 and 3 with 2 and 5: if it is MUCH LESS, then our counterfeit coins are 1 and 3; if it is LESS, they are 1 and 4; if it is EQUAL, they are 2 and 3; if it is MORE, they are 2 and 4. Therefore, our strategy allows us to find our two counterfeit coins from five using only two weighings!

The total number of possibilities for two counterfeit coins among 5 coins is 10. In contrast with our 5-way scale, we can't find the two counterfeit coins in two weighings using a 3-way scale.

Now we want to find a strategy that is better than the bound in Eq. (1) for any number of coins. We start by analyzing the first weighing.

4 The First Weighing

Consider the first weighing where we compare k coins against k coins. We calculate the information produced by each outcome, that is, the number of possible pairs of fake coins. By symmetry, we only need to consider three cases.

1. If it is MUCH LESS, then both fake coins are on the left pan, corresponding to $\binom{k}{2}$ different possibilities.
2. If it is LESS, then one fake coin is on the left pan and the other one is not on the scale, corresponding to $k(N-2k)$ possibilities.
3. If it is EQUAL, then either each pan has one fake coin or both fake coins are not on the scale, corresponding to $k^2 + \binom{N-2k}{2}$ possibilities.

The maximum out of the three expressions above is provided by the last one as shown in the next lemma.

Lemma 1. $\max(\binom{k}{2}, k(N-2k), k^2 + \binom{N-2k}{2}) = k^2 + \binom{N-2k}{2}$.

Proof. The first value is not more than the third as $\binom{k}{2} \leq k^2 + \binom{N-2k}{2}$. Now we want to show that the second value is not more than the third.

Both N and k are integers, and $\frac{1}{4} \leq \left(x - \frac{1}{2}\right)^2$ for any integer x, so

$$\frac{1}{2} \leq \left(N - 3k - \frac{1}{2}\right)^2 + \left(k - \frac{1}{2}\right)^2 \implies$$
$$0 \leq N^2 + 10k^2 - 6Nk - N + 2k \implies$$
$$2Nk - 6k^2 \leq N^2 - 4Nk + 4k^2 - N + 2k \implies$$
$$Nk - 2k^2 \leq k^2 + \frac{(N-2k)(N-2k-1)}{2},$$

as desired. □

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The EQUAL outcome is the one that gives us the least information. But how much info?

Now we want to find the minimum possible value for Case (3). We have that after the first weighing, if it is EQUAL, then the remaining possibilities are

$$k^2 + \frac{(N-2k)(N-2k-1)}{2}, \tag{2}$$

where k is the number of coins on each side of the scale. The next step is to make our life more complicated by assuming that k is not an integer, but a real number. As a function of k, Eq. (2) is a parabola and it reaches its minimum where the derivative, $6k - 2N + 1$, is zero. That is the minimum for a real k is at $k = \frac{N}{3} - \frac{1}{6}$.

The real k told us that we need to divide all the coins into approximately three equal parts. If $k \approx N/3$, then EQUAL leaves us with approximately $\frac{N^2 - N}{6} = \frac{\binom{N}{2}}{3}$ bits of information. Though we have five different outcomes, with the first weighing we cannot divide the number of possibilities into five equal parts. In the best division, the largest pile of information is one third: We can't reduce the number of possibilities by more than a third.

This is unfortunate. In the worst case the 5-way scale behaves similar to a 3-way scale. If every weighing behaved like this then the final answer would be close to the bound in Eq. (1), and the 5-way scale wouldn't be much better than a 3-way scale.

Let's not despair yet and see how information is distributed if the first weighing has $N/3$ coins on each pan:

- MUCH LESS or MUCH MORE: one pile of size $N/3$ with two counterfeit coins, about $N^2/18$ possibilities.
- LESS or MORE: two piles of size $N/3$ with one counterfeit coin each, $N^2/9$ possibilities.
- EQUAL: either one pile of size $N/3$ contains both coins, which corresponds to about $N^2/18$ possibilities or two other piles of size $N/3$ contain one fake coin each, corresponding to $N^2/9$ possibilities. Summing up we get $N^2/6$ possibilities

We just discovered a situation that requires our attention:

- One fake coin in one pile and one fake coin in another pile.

We will call this case the case of *separated coins* and sstudy it in the next section.

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5 Separated coins

Suppose one fake coin is in pile P_1 and the other in P_2, where P_1 and P_2 are disjoint. The two piles P_1 and P_2 are not assumed to have equal size. We denote the size of a pile P as $|P|$.

We divide P_1 into three piles denoted A_1, A_2, and A_3, where $|A_1| = |A_2| = x|P_1|$, and P_2 into three piles B_1, B_2, and B_3, where $|B_1| = |B_2| = y|P_2|$. We are interested in how much we can reduce the number of possibilities $|P_1| \cdot |P_2|$ for the case two separated coins using one weighing.

In our weighing we compare A_1B_1 versus A_2B_2. From now on we write two or more piles together to indicate the union of those piles.

MUCH LESS means one fake coin is in pile A_1 and one in B_1, thus reducing the total number of possibilities by xy.

LESS means that one coin is in A_3 and another in B_1, or one coin in A_1 and another in B_3. We have two disjoint cases, with possibilities reduced by $(1-2x)y$ and $x(1-2y)$ correspondingly.

EQUAL yields either one coin in A_1 and the other in B_2, or one coin in A_2 and the other in B_1, or one coin in A_3 and the other in B_3. We got three disjoint groups with reductions xy, xy and $(1-2x)(1-2y)$.

Our x and y range between 0 and $\frac{1}{2}$. We want to find the minimum of

$$\max\{xy, (1-2x)y + x(1-2y), 2xy + (1-2x)(1-2y)\}.$$

We used a program to get us the answer we unfortunately expected: the minimum of $\frac{1}{3}$ is achieved for $x = y = \frac{1}{3}$. In the best case we can divide the number of possibilities by 3.

Again, the possibilities are not divided evenly and EQUAL is the worst case.

If we had a 3-way scale, the strategy for the first weighing were the same. And in the worst case we would have divided our information by 3.

Are we doomed? Maybe we can't do better than a 3-way scale. Though we can divide the information into five groups, we can't divide it evenly. It seems that the worst case is the same as the worst case for the 3-way scale.

6 Separated Coins Strategy

In this section we suggest a strategy for separated coins when two piles of coins are of the same size $N = 2^k$, which is a power of 2.

In this section we will encounter another case, which we call the *doubly separated coins.* In this case we have four piles A, B, C, and D, all of size 2^k.

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It is known that there is one coin in each of A and B or there is a coin in each of C and D.

Let $f(k)$ be the maximum number of weighings in the strategy below to find both counterfeit coins in the separated case, where each pile has 2^k coins. Let $g(k)$ be the maximum number of weighings needed to find both counterfeit coins in the strategy below in the doubly separated case, where each pile has 2^k coins.

Let us calculate f and g for small k.

First, we observe that $f(0) = 0$: we do not need any weighings if the fake coins are separated into two piles of size 1 each.

Next, let's calculate $f(1)$. We have four coins labeled 1, 2, 3, and 4, so that one fake coin is in the pile 12 and the other in the pile 34. We can find the fake coins in two weighings: first comparing 1 versus 2, then 3 versus 4. An exhaustive search shows that we can't find both coins in one weighings. That means $f(1) = 2$.

Next we want to calculate $g(0)$. In this case we have four coins total, and the two fake coins are either 1 and 2 or 3 and 4. In one weighing, comparing 12 versus 34, we can find the fake coins. That is, $g(0) = 1$.

Now we are ready to show you a cool strategy for separated coins, where each of the piles is size 2^k, where $k > 1$. We split the first pile containing one fake coin into four equal groups labeled as 1, 2, 3, and 4. We do the same with the second pile and labels 5, 6, 7, 8.

First weigh 12 vs 56. The result cannot be MUCH LESS or MUCH MORE.

Case 1. Suppose the first weighing shows LESS. This must mean that we are reduced to a problem where one fake coin is in 12, and the other is in 78. This is the case of separated coins with twice fewer coins. Thus, in this case we need $f(k-1)+1$ weighings. The outcome MORE has similar results.

Case 2. If the scale shows EQUAL in the first weighing, then either one coin is in 12 and the other is in 56, or that one coin is in 34 and the other is in 78. This corresponds to the case of doubly separated coins with 2^{k-1} coins in each pile: we need $g(k-1)+1$ weighings.

This means
$$f(k) = \max\{f(k-1)+1, g(k-1)+1\}.$$

Now consider the case of doubly separated coins, where each pile is of size 2^k, and $k > 0$. We split each pile into two so that A is 1 and 2, B is 5 and 6, C is 3 and 4, and D is 7 and 8. Now weigh 15 vs 37.

Case $1'$. If the weighing is MUCH LESS, then one fake coin is in 1 and the other is in 5. We have a case of separated coins where each pile has 2^{k-1} coins:

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we need total of $f(k-1)+1$ weighings. Similarly, if the weighing is MUCH MORE, then one fake coin is in 3 and the other in 7.

Case $2'$**.** If the weighing is LESS, one coin is in 1 and one coin is in 6, or one coin is in 2 and the other is in 5. We have a case of doubly separated coins with piles half as large. Thus, we need $g(k-1)+1$ weighings.

Case $3'$**.** If the weighing is EQUAL, then 15 can't contain a fake coin. Therefore, either one coin is in 2 and the other is in 6, or one coin is in 4 and the other is in 8. Again, we have a case of doubly separated coins with piles half as large and need $g(k-1)+1$ weighings in total. It follows that

$$g(k) = \max\{f(k-1)+1, g(k-1)+1\}.$$

By continuing this process, we either get to separated coins with each pile of size 2, or to doubly separated coins with each pile of size 1. Looking at the starting conditions of $f(1) = 2$ and $g(0) = 1$ and using induction, we get

$$f(k) = k+1 \quad \text{and} \quad g(k) = k+1.$$

We produced a strategy that solves the separated coins case for the total of 2^w coins in w weighings. This strategy is better than any strategy on a 3-way scale.

Notice that the first weighing for separated coins is not very effective. Depending on the outputs LESS, MORE, or EQUAL, it divides information into 1/4, 1/4, and 1/2 respectively. But the worst case of EQUAL corresponds to the case of doubly separated coins.

Let us see how the information is divided in the doubly separated case. We start with four piles A, B, C, and D of size 2^k. There is one fake coin in either A and B, giving 2^{2k} possibilities, or C and D, giving another 2^{2k} possibilities, the total being 2^{2k+1}.

If the first weighing is MUCH LESS or MUCH MORE we get to the case of separated coins with pile sizes 2^{k-1}, for a total of 2^{2k-2} possibilities each. Other weighing outcomes direct us back to doubly separated coins with sizes reduced by a factor of 2. That means they divide the information by 4. That is, five different outcomes divide the information into portions 1/8, 1/8, 1/4, 1/4, and 1/4. This is way better than a 3-way scale.

7 The Strategy

Now that we have a strategy that works fast for separated coins, we want to extend it to the general case of 2 fake coins in a group of 2^k coins. Let $h(k)$ denote the maximum number of weighings necessary to find both coins in one pile of size 2^k. We assume that $k > 1$.

Let us split the pile into four equal piles, labeling them 1, 2, 3, and 4. First weigh 1 versus 2.

Case 1. If the weighing is LESS or MORE, we weigh 3 versus 4 to enter the scenario in Section 6 with four piles of size 2^{k-2} each. They can be processed in $k-1$ weighings. That means all given coins can be processed in $k+1$ weighings.

Case 2. If the weighing is MUCH LESS or MUCH MORE, our strategy allows us to find the coins in $h(k-2)+1$ weighings.

Case 3. If the first weighing is EQUAL, then either there is exactly one counterfeit coin in each of 1 and 2, or both coins are in 3 and 4. Next, we weigh 2 versus 3 to reduce the problem to the following cases

- LESS—one counterfeit coin in each of 1 and 2. We need $f(k-2)+2$ weighings in total.
- EQUAL—both counterfeit coins are in 4. We need $h(k-2)+2$ weighings in total.
- MORE—one counterfeit coin in each of 3 and 4. We need $f(k-2)+2$ weighings in total.
- MUCH MORE—both counterfeit coins are in 3. We need $h(k-2)+2$ weighings in total.

That means after two weighings in Case 3, we either reduce the pile by a factor of 4, or split the half of the pile into two equal piles each containing one fake coin, which we know how to process by Section 6. In particular, in the worst case we have

$$h(k) = \max(f(k-2)+2, h(k-2)+2),$$

where f is defined in Section 6.

We can use an exhaustive search to see that $h(1)=0$, and $h(2)=2$. It follows that $h(3) = \max(f(1)+2, h(1)+2) = 4$. Similarly, $h(4) = \max(f(2)+2, h(2)+2) = 5$. Continuing by induction, we get that

$$h(k) = k+1,$$

for $k > 2$.

Thus we can always solve a problem of finding two fake coins among $N = 2^w$ coins in $w+1$ weighings using a 5-way scale.

Let us see how the information is divided in the first weighing of our strategy. At the beginning we had $\binom{N}{2}$ or about $N^2/2$ possibilities. If the first weighing is MUCH LESS or MUCH MORE the number of possibilities is divided by 16. If it is LESS or MORE, then one coin is in the pile of size $N/4$, and the other in the pile of size $N/2$. So the information is divided by about 4. So EQUAL is the worst case with 3/8 of the initial information.

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In the first weighing, we are not doing better than a 3-way scale. But later we get to the case of doubly separated coins, which allows us to become much more effective.

8 Conclusion

Working on this project was an emotional rollercoaster. We started with a small example of 5 coins that gave us hope that a 5-way scale was noticeably better than a 3-way scale. Then we started analyzing the first weighing in different situations and were extremely discouraged. It looked like a 5-way scale couldn't do better than a 3-way scale. We were so discouraged that for some time we were not looking into what the second weighing might do. That was a mistake, as later weighings make the strategy we found faster and better. Despite the fact that the 5-way scale can't divide the possibilities evenly, there is a strategy for a 5-way scale that is much better than any strategy with a 3-way scale.

We gave our example of a fast strategy only for a specific number of coins $N = 2^k$. We leave it to the reader and future researchers to see if this strategy can be extended to any number of coins.

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Games and puzzles

Exploring mod 2 n-queens games

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Abstract: *We introduce a two player game on an $n \times n$ chessboard where queens are placed by alternating turns on a chessboard square whose availability is determined by the parity of the number of queens already on the board which can attack that square. The game is explored as well as its variations and complexity.*

Keywords: Chess, independence, n-queens.

1 Introduction

The n-queens problem is the challenge of placing n queens on an n by n chessboard such that none are attacking each other. This puzzle originated as the 8-queens problem, played on a standard 8×8 chessboard, and was proposed by Bezzel [2] in 1848. The 8-queens problem and the more general n-queens problem attracted the interest of notable mathematicians of that time. In 1850, Nauck [3] was the first to publish all 92 solutions for the standard 8×8 chessboard and in 1874 Pauls [5, 6] gives the first set of solutions for the general n-queens problem published in two articles.

The problem of finding more solutions in various dimensions has continued to hold the interest of mathematicians and computer scientists. Solutions sets have been found using graph theory, magic squares, Latin squares, and group theory, among other techniques. (See a survey paper by Bell and and Stevens [1] for more details.) In a brute force approach, solution sets for a given n can be found use the backtracking algorithm; a standard technique taught in computer science classes. In more modern times, a player can even find mobile apps to test his or her ability to find solutions on 8×8 or other dimensional boards.

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In this paper, we look at a game variant of the n-queens problem that can be played on chessboards of varying dimensions, although here, we limit our focus to square chessboards. The basic version of the n-queens game is described by Noon [4], who, observing that not all placements of queens will lead to a full solutions with n queens, suggests a two player game where each player successively places queens in non-attacking positions. The first player who cannot place a queen loses the game. In the next section, we introduce a modification of this n-queens game. Sections 3 and 4 explore two states a game board may take. Then Section 5 discusses alternate versions of the game, while Section 6 looks at the complexity of the game. Open questions are given throughout.

2 The mod 2 n-queens game

First, we describe a variation of the n-queens game. For any game, a player should consider the motivation of his or her opponent. In particular, in a game involving placing a queen on a chessboard, when is that queen safe from attack; that is, would it be a good move for an enemy queen to attack this piece? If you place a queen on a square that cannot be attacked by any queen, clearly it is safe. However, if you place your queen on a square that may be attacked by exactly one queen, the enemy queen can attack your piece while remaining safe on the square after your piece is taken. Continuing along these lines, what if you place your queen on a square that may be attacked by exactly two queens? Neither of the enemy queens would wish to attack first because then she would be vulnerable to the other. Hence, you could safely hold this square. Next, consider placing your queen on a square that may be attacked by exactly three queens. Any one of these enemy queens has incentive to attack, because after she claims the square, the remaining two queens are once again at a stalemate. Therefore, it would not be wise to place your queen on the square in that case. We see that for any number of queens on the board, the safety of a square is determined by the parity of the number of queens which may attack that square, so we propose the following additional rule to the two-player n-queens game.

> Rule: Let every non-occupied square take on a value given by the number of queens who are directly attacking that square. If the value is congruent to zero modulo two, then the square is open and a queen may be placed on it. If the value is congruent to one modulo two the square is closed and we may not place a queen on that square.

Of course this rule implies squares with an even number of attacking queens are open and with an odd number of attacking queens are closed. Play continues as before until a losing player cannot place another queen on the board. We will refer to this game as the **mod 2 n-queens game**.

This rule opens up the possibility for more play compared to Noon's original n-queens game. Figure 1 (left) shows an example of a set of queens played an 8×8 chessboard. This figure uses a gradient to indicate the number of queens attacking each square on the chessboard, with white indicating an open square and darker shades indicating an increasing number of queens attacking that square.

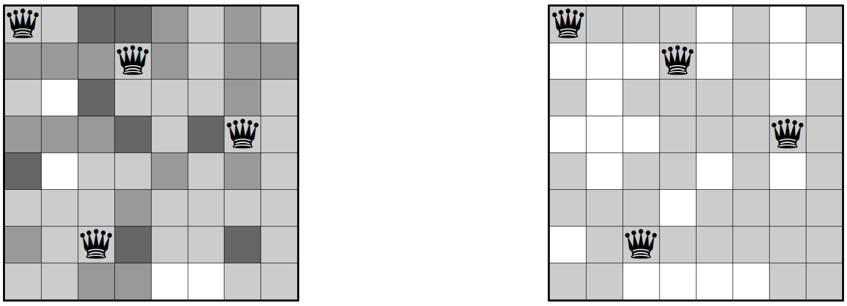

Figure 1: Left: A chessboard with a gradient indicating attacking queens. Right: A chessboard with a mod two gradient indicating attacking queens.

With the new rule, we are able to simplify this illustration as well as open up more squares on the board. Figure 1 (right) illustrates the same configuration of queen with a now much simpler and more open gradient given by the mod 2 n-queens game. We note, in traditional play, placing another queen will only cause squares to become progressively darker in the gradient, but once a square is closed it remains closed. The mod 2 n-queens game is much more dynamic. Squares may easily change from open to closed and vice versa as the game progresses.

There are other differences between this version of the game and the original n-queens game. For one, the order of the placement of the queens now matters as the parity of a square may change throughout the game. Next, the maximum number of moves in the game increases. A maximum of n^2 queens can be placed to fill the board with the new rule, whereas the original game could have at most n queens.

Before exploring the game play, we define three game states for any arrangement of queens upon a square chessboard created by a mod 2 n-queens game. We call such chessboards complete, unlocked, and locked as follows:

Definition. Given an arrangement of queens upon a chessboard, the chessboard is **complete** if all n^2 square are covered by queens. The chessboard is **unlocked** if it has fewer than n^2 queens placed on it, and there exist empty open squares in which a queen may be placed. Finally, the chessboard is **locked** if it has fewer than n^2 queens, but no legal moves remain.

Finally, before we look at some complete chessboards, let us review some chessboard terminology. Each square on a $n \times n$ chessboard will be indexed by an **ordered pair** (i, j) where $1 \leq i \leq n$ indicates the row numbered from top to bottom and $1 \leq j \leq n$ indicates the column numbered from left to right. The **k-sum diagonal** is the diagonal running from left to right and bottom to top in which the sum of the indices of each square is k for some integer $1 \leq k \leq 2n$. The **k-difference diagonal** is the diagonal running from left to right and top to bottom in which the difference of the indices of each square is k for some integer $-(n-1) \leq k \leq n-1$. The $(n+1)$-sum diagonal is known as the **main sum diagonal**, and similarly the 0-difference diagonal is called the **main difference diagonal**.

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3 Complete Chessboards

We begin our exploration of the mod 2 n-queens games by considering complete chessboards. By definition, we know that a complete chessboard contains n^2 queens, so we ask, given an $n \times n$ chessboard is it always possible to achieve a complete chessboard state through legal play?

In the case where the size of the board is odd, the answer is yes.

Proposition 1. If n is an odd positive integer, the $n \times n$ chessboard has a complete solution of n^2 queens.

Proof. Thinking inductively, a 1×1 chessboard may be filled with one queen. In the next case, if we want to fill a 3×3 board we need to fill the top two rows and left two columns with queens as well as the 1×1 board in the lower right corner. More generally, to fill a $n \times n$ board we need to fill the top two rows, the left two columns, and a $(n-2) \times (n-2)$ board in the lower right corner. Looking at the game position where queens have filled the first two rows and the first two columns of the chessboard as illustrated in Figure 2 in the case $n = 11$, we observe that every uncovered square on this chessboard can be attacked by an even number of queens.

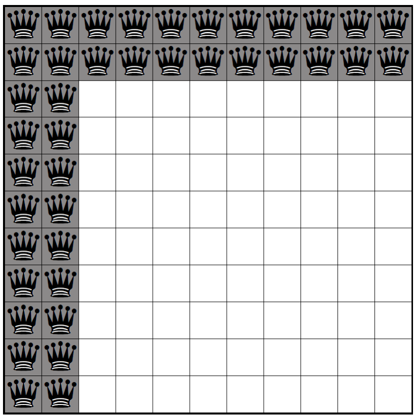

Figure 2: An unlocked 11×11 chessboard.

Specifically, each square can be attacked vertically, horizontally, and along the difference diagonal by exactly two queens on each line. Squares to left or on the main diagonal are attacked by four more queens on the sum diagonal, squares on the $n+2$ sum diagonal are attacked by exactly two more queens, and all other squares are attacked by zero more queens. Therefore game play from the chessboard with queens in this starting configuration is equivalent to the game play on a $(n-2) \times (n-2)$ board with no queens. Because we can proceed inductively to a 1×1 board, now we only need to show that this starting board can be obtained from a sequence of legal game moves.

Begin by filling the upper left corner of a $n \times n$ chessboard with the eight queens in the sequence given by Figure 3 (left). In the following steps, we will

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successively fill in the next four squares from the first two rows and column. We proceed as listed in the steps below and illustrated in Figures 3 (center) and 3 (right) for an increasing integer k where $1 \leq k \leq \frac{n-2}{2}$.

Step 1: Place a queen in square $(2, 2k+2)$ and then in square $(1, 2k+2)$.

Step 2: Place a queen in square $(2k+2, 1)$ and then in square $(2k+2, 2)$.

Step 3: Place a queen in square $(1, 2k+3)$ and then in square $(2, 2k+3)$.

Step 4: Place a queen in square $(2k+3, 2)$ and then in square $(2k+3, 1)$.

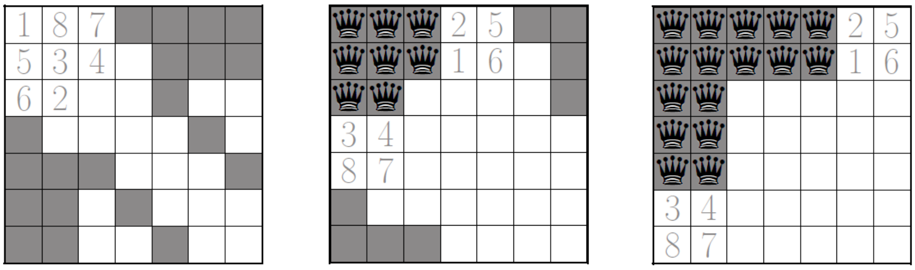

Figure 3: Left: Place the first eight queens. Center: Place the next eight queens. Right: Steps in the construction of a complete board for n odd.

Once this process is finished, we repeat on the $(n-2) \times (n-2)$ chessboard until we have reduced to the one empty square in the lower right corner which can then be filled with a queen.

□

Proposition 2. If n is an even positive integer, the $n \times n$ chessboard does not have a complete solution of n^2 queens.

Proof. Suppose to the contrary, that there is an open square in which to place $(n^2)^{\text{th}}$ queen on a $n \times n$ chessboard, and further suppose that square is on the outer edge of the chessboard. Each square on the edge has $n-1$ queens attacking along the horizontal line and $n-1$ queens attacking along the vertical line. If we chose a corner square, it only has one diagonal containing $n-1$ attacking queens. As you move from a corner square along the edge, the larger diagonal decreases by one and the lesser diagonal increases by one which always keeps the sum of the diagonals of a square on the edge equal to $n-1$. Thus, the open square cannot be one of the squares on the edge of the chessboard because the number of attacking queens is $(n-1)+(n-1)+(n-1) \equiv_2 1$. Thus we can assume the outer ring is filled with queens, so we remove this ring without changing the parity of the chessboard, as each square in the middle is attacked by exactly eight queens from the outer ring. We are left with $(n-2) \times (n-2)$ chessboard. As $n-2$ is still even, we repeat, removing the outer rings, until we are left with the contradiction, a 2×2 chessboard which cannot be filled with four queens. □

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Although we cannot have a complete solution in the case of an even length chessboard, we can identify a locked solution that comes close.

Proposition 3. If n is an even positive integer, $n^2 - 2$ queens can be placed legally on a $n \times n$ chessboard.

One such construction could follow similarly to the odd case by applying induction and filling the topmost two rows and leftmost two columns, so we leave it to the reader. This case is interesting, however, and we pose the following question.

Question 4. Is the solution of $n^2 - 2$ queens on an $n \times n$ chessboard for n an even positive integer a maximal locked solution?

Empirical data and a computer simulation for $n = 4$ suggest that the answer is yes. However, with our modest computational power the program could run the simulation for $n = 4$ in eleven minutes but it would take approximately 22 years to run for $n = 6$, so verifying computationally was not an option.

In the next section, we consider locked chessboards.

4 Locked Chessboards

Another interesting game state is a locked chessboard. For strategy reasons, a player would be interested to know of locked solutions in order to lead an opponent into such a solution or avoid them himself. For small $n = 1, 2$ or 3, there is a simple first-player win strategy, that is, placing only one queen as seen in Figures 4 (left) and 4 (right).

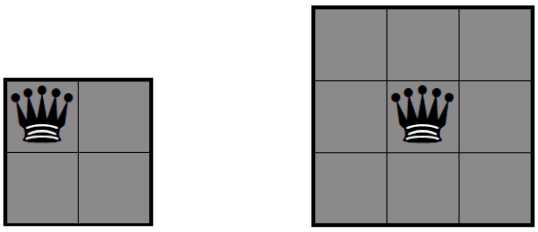

Figure 4: Left: A locked 2×2 chessboard. Right: A locked 3×3 chessboard.

Of course, these small examples cannot be generalized and locked boards with only one queen do not exist for $n > 3$. We now look at a more complex locked chessboard. Again, as we did for complete boards, we separate the chessboards into two classes by their even or odd lengths.

Consider the chessboards where queens fill the top row and leftmost column for n odd or almost fill the top row and leftmost column for n even. These game positions are illustrated in Figure 5.

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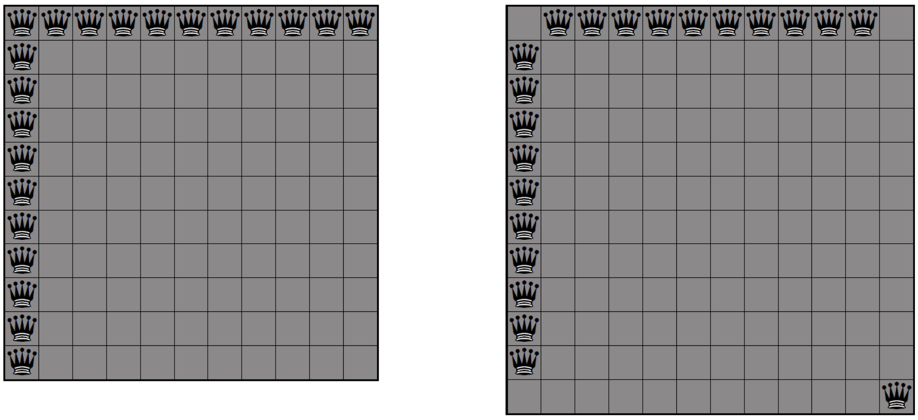

Figure 5: Left: A locked 11×11 chessboard. Right: A locked 12×12 chessboard.

Proposition 5. For n an odd positive integer, the chessboard with queens exactly on squares in the set $\{(1,i),(i,1) | 1 \leq i \leq n\}$ is a legal, locked game position.

Proof. We need to check that this board is locked as well as check that the queens can be placed in this configuration by a legal set of game moves. First we confirm that an odd number of queens attack every uncovered square. Observe that every unoccupied square has exactly one queen which can attack vertically and exactly one queen that can attack horizontally. On the positive diagonal, if the uncovered square is on or to the left of the main sum diagonal exactly two queens can attack along the positive diagonal. However, if the uncovered square is the right of the main diagonal, no queens can attack along the positive diagonal. Further every empty square can be attacked by exactly one queen on the difference diagonal, so all uncovered square are attacked by either exactly five or exactly three queens and hence the board is locked.

Next, we need to show that this game board can arise from a sequence of legal placements of queens onto the board. First consider the case where $n = 3$. Figure 6 illustrates a sequence of five moves to lock a 3×3 chessboard by playing queens on the first row and column.

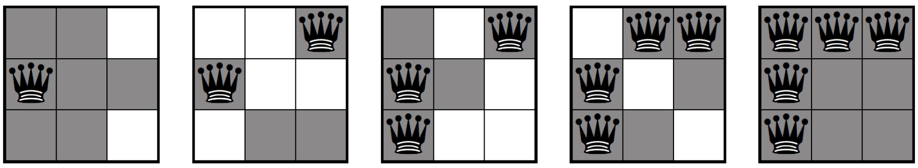

Figure 6: A sequence of game positions to create a locked 3×3 chessboard.

We want to extend the game play in Figure 6 to occur in a chessboard of any size. Our strategy will be to repeat the first four moves of Figure 6 for the next two vacant squares in the first row and the corresponding two vacant squares in the first column. Suppose we wish to fill the four squares from the set

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$\{(1,i),(1,i+1),(i,1),(i+1,1)\}$ with queens. In each stage will fill two squares in column one and two squares in row one, so the total number of queens attacking either horizontally or vertically must be even and further there are no queens which can attack from either of the two diagonals. Squares $(i,1)$ and $(1,i+1)$ are not on a line vertically, horizontally, or diagonally, so a queen can be placed on each of these squares. The remaining squares $(i+1,1)$ and $(1,i)$ can be attacked by both queens that were just played and also are not on a line with each other vertically, horizontally, or diagonally. We can fill these squares with queens. After repeating this process $\frac{n-1}{2}$ times, each square in the first row and first column, except the square in the upper left corner, contains a queen. As the square in the upper left can be attacked by all $2(n-1)$ queens on the board, we can place the final queen in this position and hence lock the board. $\square$

We can find a locked board in the case where n is even as well.

Proposition 6. For n an even positive integer and $n > 2$, the chessboard with queens on squares in the set $\{(1,i),(i,1)|2 \leq i \leq n-1\} \cup \{(n,n)\}$ is a legal, locked game position.

The proof is similar to that of Proposition 5, so we leave to the reader to check that this chessboard is locked and arises from a sequence of legal game moves.

We can now make the following observation:

Corollary 1. When n is an odd positive integer, at most $2n-1$ queens are needed to lock the board, and when n is an even positive integer, at most $2n-3$ queens are needed to lock the board.

This corollary gives an upper bound on the number of queens need to lock an even or odd chessboard, but is this bound strict? We know for very small values of n the chessboard can be locked with fewer queens, but what about larger values? We pose the next question.

Question 7. What is the minimum number of queens needed to lock a $n \times n$ chessboard?

Next, we will look at different versions of the game with alternate rules and the connections between them.

5 In the Alternate Universe

Thus far, we have seen that there is a difference in outcomes for chessboards with even and odd length. We wish to modify the rules of the game to look at an alternative version of the game where queens can be placed on squares that are attacked by an *odd* number of queens and cannot be placed on squares attacked by an even number of queens. We will call such a game an **alternate universe mod 2 n-queens game**.

The first observation is that generally this cannot happen as an empty board has no squares attacked by an odd number of queens. So we will modify the rules again to start with one queen already on the board before we commence play.

A second way to play the game would be to start with n^2 queens on the board and try to remove the queens one by one following the rule that a queen may be removed as long as she can be attacked by an even number of queens. We call such a game an **complementary mod 2 n-queens game**. The relationship between alternate universe, complementary games, and standard mod 2 n-queens games is dependent on the size of the chessboard.

Proposition 8. When n is an even positive integer, the alternate universe mod 2 n-queens games are in bijection with complementary mod 2 n-queens games.

Proof. Earlier we saw that if an even length chessboard is fully covered with queens, each square is attacked by an odd number of queens. The initial step for both games in the same. We place one queen in the alternate universe and remove one queen from the same square in the full complementary game. While playing the complementary game, queens can only be removed if the number of queens attacking is even and hence because of the parity of the board, it means the number of empty squares "attacking" must be odd. By taking the complement of the board, removing queens to make empty squares, and placing queens on otherwise empty squares, we can go back and forth from the complementary game to the alternate universe game. □

We note that the alternate universe game on the odd length chessboards does not have the same relationship. In fact, we have the following:

Proposition 9. When n is an odd positive integer, the complementary mod 2 n-queens games are in bijection with standard mod 2 n-queens games.

Proof. In this case, the beginning parity for every square on a complementary game board is even. We obtain the bijection by placing a queen in the same square of the standard game from which we removed it in the complementary game. □

Thus for each case, even or odd length chessboard, there is really only two different games to be played, standard and alternate universe, as the complementary game can be completely reconstructed from one of the other two games.

We conclude this section with an open-ended question.

Question 10. What other n-queens games can be played on a $n \times n$ chessboard? In particular, how could we adapt play for a **mod k n-queens game**?

In the next section we will discuss the complexity of the game and give examples of a game tree and a graph created from this tree.

6 Game Trees, Graphs, and Complexity

Chess and other games played with chess pieces are complex games. We will discuss several standard measures of complexity of a game for the mod 2 n-queens game. A **game tree** is the rooted tree whose root is the empty board and whose leaves are locked or complete boards such that the directed paths from the root to a leaf display all possible games. Figure 7 shows the first three levels of the game tree in the case where $n = 3$.

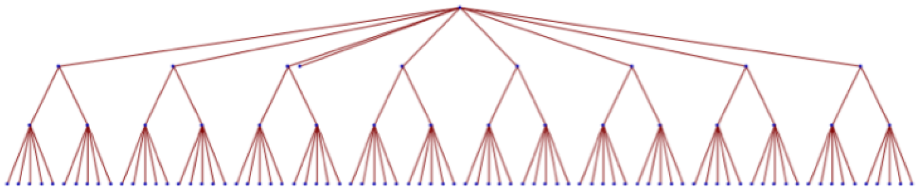

Figure 7: Partial game tree for $n = 3$.

Because there are n^2 squares on the board, we can get an upper bound for the number of leaf nodes or **game tree size** at $n^2!$.

Many of the game boards represented by these nodes are identical or symmetric by rotation or reflection. In fact, any given game board has up to eight symmetric boards. To simplify the structure of this tree, we propose to combine nodes describing boards which are equivalent by symmetry. In this case, we have fewer nodes, but the tree has become a graph. Figure 8 shows the full graph for $n = 3$. This graph is still quite large even for smallest non-trivial value for n.

We are also interested in the complexity of the game. Because, for some n, all possible squares may be filled, we have a finite set of n^2 squares; any subset of which could be filled with queens. This set of all subsets of n^2 elements is the power set which has size of 2^{n^2}. Since defining rules for the board limits the number of game boards, 2^{n^2} is the upper bound for the **state-space complexity** or number of legal game positions. Therefore we have a complexity $O(2^{n^2})$ and this game is in EXPTIME.

We conclude with one final question.

Question 11. Clearly, the upper bounds for the game tree size and the state-space complexity are not firm for all n. Can these be improved?

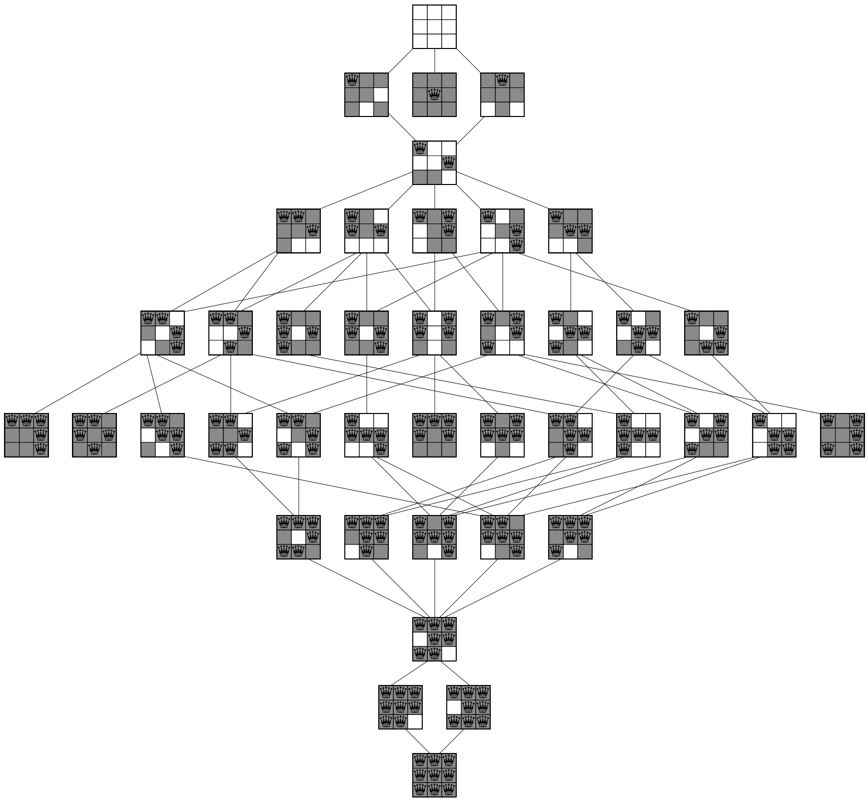

Figure 8: Partial game tree for $n = 3$.

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Articles

New Year Mathematical Card or V Points Mathematical Constant

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Abstract: *The article describes an attempt to define a new mathematical constant - the probability of obtaining a hyperbola or an ellipse when throwing five random points on a plane.*

Keywords: second-order plane curve, hyperbola, ellipse, Mathcad, pseudo-random number.

The author has such a long pre-New Year's habit. He and his students develop and post on the site of the mathematical package [1] greeting animated cards with some entertaining mathematical meaning.

At the end of 2017, such New Year's postcard was published: in a square on a plane, five points are randomly scattered 1000 times, through which a second-order plane curve was drawn. All who recognize all the seven possible curves on this postcard (seven is a beautiful number!), could expect happiness and luck in the New Year [2, 3][1].

The prehistory of this New Year's postcard is as follows. In one lesson with his students, the author showed a sagging chain (see Figure 1) and asked a question, the graph of what function this all resembles. The students answered in chorus that it was a parabola [4]. Then the coordinates of five points (left, right, bottom and two points in the middle of the "branches" of the chain) were taken from the photograph of the chain in the environment of the graphic editor Paint, through which a plane curve of the second order was drawn. These five points were taken from the photographs of the sagging chain in different positions and with varying degrees of tension. After processing the data, it turned out that the curve of the second order passing through the five "chain" points was, of course, not a parabola and not even one of the branches of the

[1] On the site `www.demonstrations.wolfram.com/FivePointsDetermineAConicSection` we can see only two curves.

hyperbola, but. . . an arc of an ellipse. The lesson ended with the description of the function of the chain line (a catenary), which more or less accurately (taking into account measurement errors) passes through three, four, five and more points "taken" from the photograph of the sagging chain.

A question arose: which second-order curve passes through five randomly chosen points on a plane? Having at hand a computer with the program *Mathcad* [5], it can be tried to determine that by statistical tests (Monte Carlo) [6].

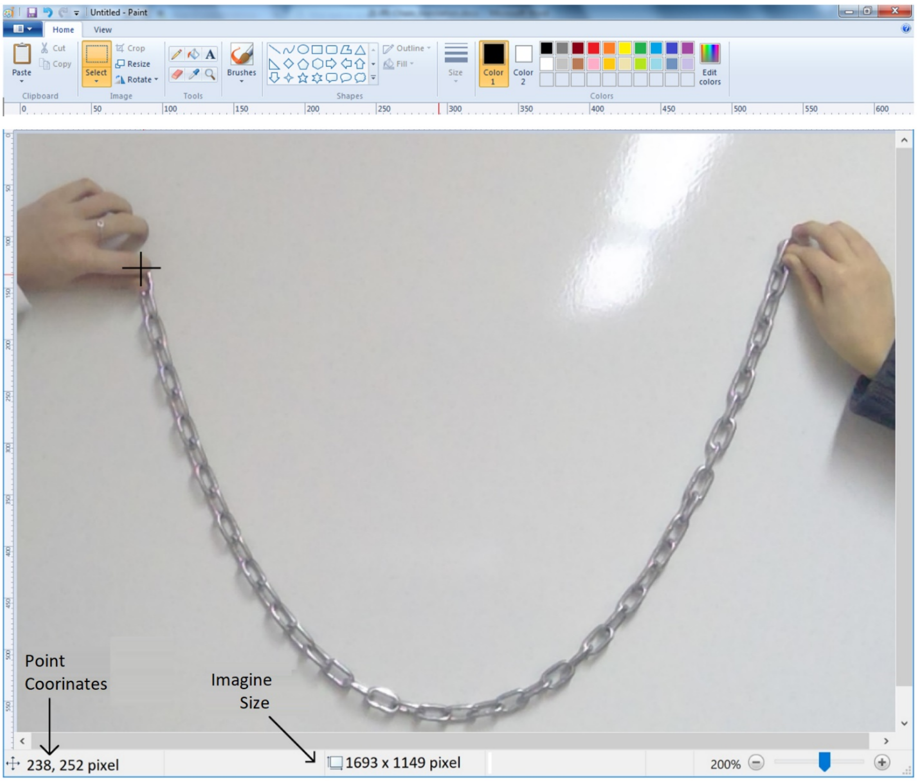

Figure 1: Experiment with a sagging chain.

It is known that through five points on the plane one can draw the following curves [2]:

1. Two branches of a hyperbola;
2. An ellipse;
3. A parabola (transitional case from hyperbola to ellipse);
4. A circle (special case of an ellipse);
5. Two intersecting lines (degenerate case of two branches of a hyperbola);
6. Two parallel straight lines;
7. One straight line (a special case of cases 5 and 6).

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In practice (not in theory), five randomly points can hold only two curves: a hyperbola with two branches and an ellipse (Figure 2). The author and his students "drew" manually the other five curves on the New Year's card, asking the "exact" coordinates of five points.

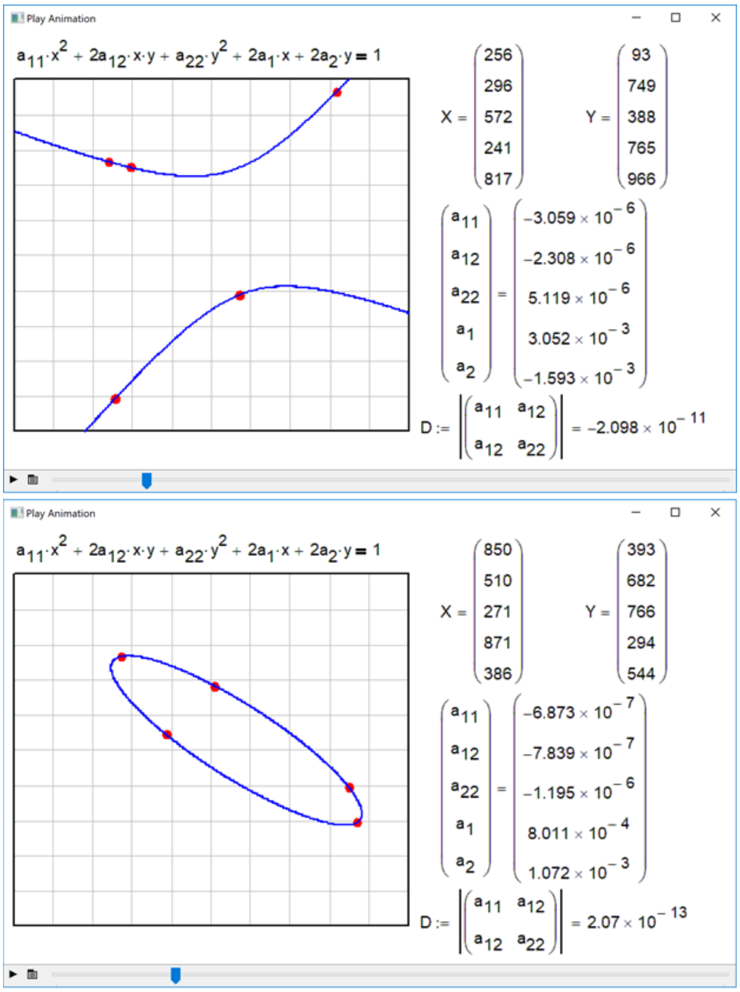

Figure 2: A hyperbola (above) and an ellipse drawn through five random points.

Figure 2 shows two frames of the New Year's animation card with a hyperbola and an ellipse. The coordinates of the five random points (the values of the vectors X and Y) were also shown on the card, the values of the five coefficients of the second-order curve equation (a_{11}, a_{12}, a_{22}, a_1, and a_2): see the equation on the top of Figure 2), as well as the values of the invariant D over which the curve was determined: $D < 0$ is a hyperbola and $D > 0$ is an ellipse [2].

For $D = 0$ (and under other additional conditions), a parabola (the transient case from the ellipse to the hyperbola) must be obtained. But that case, we emphasize, took place only with an artificial, and not random assignment of the values of the vectors X and Y.

Those who received the New Year card were asked to calculate how many times they saw a hyperbola, and how many times they saw an ellipse (for "full happiness" in the coming 2018).

It turned out that the hyperbola appeared in approximately 71.84% of the cases, and the ellipse in the remaining 28.16%. This was calculated, of course, not through viewing animation frames, but through a statistical computer experiment (Figure 3).

$$
(H\ E\ P) := \left|\begin{array}{l}
\text{for } i \in 1..1000000 \\
\quad \left|\begin{array}{l}
X \leftarrow \mathrm{runif}(5,-1,1) \\
Y \leftarrow \mathrm{runif}(5,-1,1) \\
\begin{pmatrix} a_{11} \\ a_{12} \\ a_{22} \\ a_1 \\ a_2 \end{pmatrix} \leftarrow \mathrm{lsolve}\left[\begin{bmatrix}
(X_1)^2 & 2\cdot X_1\cdot Y_1 & (Y_1)^2 & 2\cdot X_1 & 2\cdot Y_1 \\
(X_2)^2 & 2\cdot X_2\cdot Y_2 & (Y_2)^2 & 2\cdot X_2 & 2\cdot Y_2 \\
(X_3)^2 & 2\cdot X_3\cdot Y_3 & (Y_3)^2 & 2\cdot X_3 & 2\cdot Y_3 \\
(X_4)^2 & 2\cdot X_4\cdot Y_4 & (Y_4)^2 & 2\cdot X_4 & 2\cdot Y_4 \\
(X_5)^2 & 2\cdot X_5\cdot Y_5 & (Y_5)^2 & 2\cdot X_5 & 2\cdot Y_5
\end{bmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right] \\
D \leftarrow \left|\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}\right| \\
\mathrm{if}(D < 0, H \leftarrow H + 1, \mathrm{if}(D > 0, E \leftarrow E + 1, P \leftarrow P + 1))
\end{array}\right. \\
(H\ E\ P)
\end{array}\right.
$$

$$H = 719484 \qquad E = 280516 \qquad P = 0 \qquad H + E + P = 1000000 \qquad \frac{E}{H} = 0.389885$$

Figure 3: Ocurrences of hyperbolas and ellipses – Mathcad layout.

Figure 3 shows the number of dropped hyperbolas (H) and ellipses (E) when throwing five random points ten million times in a square of 2×2. At the same time (just in case!), the number of dropped parabolas (P) was counted.

Regarding Mathcad layout (Figure 3), it suffices to explain the essence of the following operators and functions:

1. The FOR command makes a loop parameter i (throwing points into a square);
2. The function RUNIF generates 5 numbers (the first argument of the function) with a random (pseudo-random) distribution in the interval from -1 to 1 (the second and third arguments of the RUNIF function);
3. The LSOLVE function returns the solution of the system of linear algebraic equations (SLAE), the coefficient matrix for the unknowns is the first argument of the LSOLVE function, and the vector of the free terms is the second argument. The SLAE solutions is the vector of the coefficients of the required second-order equation (a_{11}, a_{12}, a_{22}, a_1 and a_2);
4. The IF function counts the "dropped" hyperbolas, ellipses and parabolas.

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At the bottom of Figure 3, it is shown that, with ten million tossing, the five points of the hyperbola (H) fell 719 484 times, the ellipse (E) 280 516 times, and parabola (P), as expected, never. The same approximate figures are obtained with other amounts of casts, and with squares of different sizes, where five points were thrown. We'll discuss the problematic of the shape of the region later.

Many people, after receiving a greeting card, read the message, admire the picture and. . . throw a postcard into a desk drawer or even throw it away. The publication on the Mathcad user forum of the "New Year" postcard with hyperbolas and ellipses had other consequences:

1. It was suggested that the new mathematical constant of 0.2806 should have a name: V Points [2]. One colleague of the author from the department of higher mathematics asserts that he has found an analytical way to calculate this constant, but he did not have time to write that down [3].

2. One visitor of the forum, Frank Purcell from Chicago, suggested that this constant can be determined (estimated) in another way. When solving the problem of four points (Sylvester's Four-Point Problem [4]), through which two intersecting parabolas are held, breaking a square area into certain zones [7], the fifth random point can fall into one of these zones, determining if the curve through the five points is a hyperbola or a parabola. The sum of the areas of these zones gives our constant. Sylvester's Four-Point Problem is described on [8].

3. At the forum of Mathcad, a race started – after throwing some points into the plane the idea was to draw through them a curve of higher order (Figure 4).

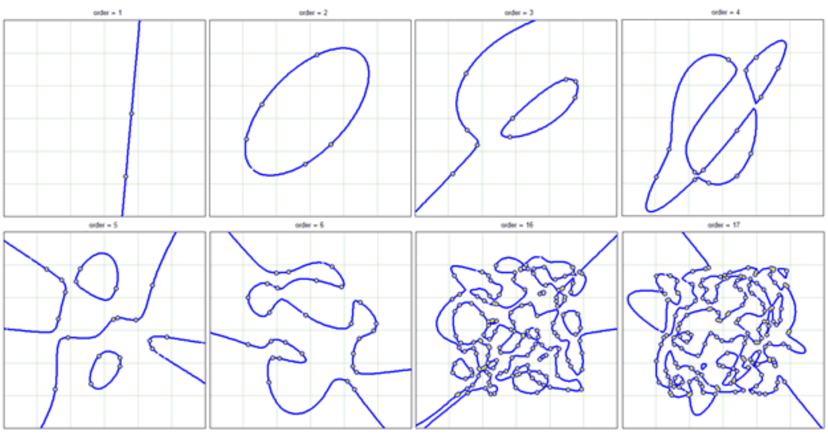

Figure 4: Curves of different orders.

[2]V Points – Google translator gives the Russian author's name В Очков.

[3]This man wants to be like Pierre Fermat, who wrote down his great theorem, but did not give a proof. Only a few centuries later a proof was found.

[4]Four points are randomly dashed to the plane and it is determined whether they form a convex quadrangle or one of the points is inside the triangle formed by the other three points.

4. If the shape of the region is not a square, but a circle, then a slightly different value for the V Points constant is obtained (Figure 5). What is its value?

```
N := 10^6        r := 3
(H)
(E) :=  | for  i ∈ 1..N
        |   | while  1
        |   |   | X ← runif(5, −r, r)
        |   |   | Y ← runif(5, −r, r)
        |   |   | F ← 0
        |   |   | for  i ∈ 1..5
        |   |   |   F ← F + 1  if  (X_i)^2 + (Y_i)^2 > r^2
        |   |   | break  if  F = 0
        |   | (a11)            [ (X1)^2  2·X1·Y1  (Y1)^2  2·X1  2·Y1 ]   (1)
        |   | (a12)            [ (X2)^2  2·X2·Y2  (Y2)^2  2·X2  2·Y2 ]   (1)
        |   | (a22) ← lsolve ( [ (X3)^2  2·X3·Y3  (Y3)^2  2·X3  2·Y3 ] , (1) )
        |   | (a1 )            [ (X4)^2  2·X4·Y4  (Y4)^2  2·X4  2·Y4 ]   (1)
        |   | (a2 )            [ (X5)^2  2·X5·Y5  (Y5)^2  2·X5  2·Y5 ]   (1)
        |   | D ← | (a11  a12) |
        |   |     | (a12  a22) |
        |   | if(D < 0, H ← H + 1, E ← E + 1)
        | (H)
        | (E)
```

$H = 701742 \qquad E = 298258 \qquad \frac{H}{N} = 0.70174 \qquad \frac{E}{N} = 0.29826$

Figure 5: Hyperbolas and ellipses in a circular region.

By the way, the students were also asked about an impression: is a plane an infinite extension of a square or a circle? Again, without thinking aabout the contours of such a question, they answered in chorus that it was an infinite extension of a square. That is understandable - in school, they are used to the Cartesian coordinate system, but have little contact with polar, cylindrical or spherical coordinates. The lesson mentioned at the beginning of this article ended with a story about how the function describing the chain line (a catenary) was independently and almost simultaneously discovered by Bernoulli, Huygens and Leibniz [9].

Final Remarks

This mathematical study is included in a book that is being prepared for publication by CRC/Tailor & Francis with the title "2^5 Problems for STEM Education" [5]. The book introduces a new and emerging course for undergraduate STEM programs called "Physical-Mathematical Informatics", following a new direction in education called STE(A)M (Science, Technology, Engineering, [Art] and Mathematics). The book is for undergraduate students (and high school students), teachers of mathematics, physics, chemistry and other disciplines (humanities), and readers who have a basic understanding of mathematics and math software[5].

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Games and puzzles

Xor-Magic Graphs

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Abstract: *A connected graph on 2^n vertices is defined to be xor-magic if the vertices can be labeled with distinct n-bit binary numbers in such a way that the label at each vertex is equal to the bitwise xor of the labels on the adjacent vertices. We show that there is at least one 3-regular xor-magic graph on 2^n vertices for every $n \geqslant 2$. We classify the 3-regular xor-magic graphs on 8 and 16 vertices, and give multiple examples of 3-regular xor-magic graphs on 32 vertices, including the well-known Dyck graph.*

Keywords: cubic graph, regular graph, graph theory, combinatorics, linear algebra, binary, xor.

A special labeling of the cube

The vertices of the cube can be labeled using all eight 3-bit binary numbers in such a way that the bitwise xor of each vertex with its three neighbors is zero. Figure shows such a labeling. The xor condition is satisfied at the back lower right vertex because

$$111 \oplus 011 \oplus 010 \oplus 110 = 000,$$

and you can see that the condition is satisfied at the remaining vertices as well. Other such labelings can be obtained by symmetries of the cube. If you regard a 3-bit binary number as a vector in $\mathbb{F}_2^3$, then the xor operation is just vector addition, and so you can obtain further labelings of the cube by applying any element of $GL_3(\mathbb{F}_2)$ to all of the labels. But how special is the graph of the cube, that it admits any such labeling at all?

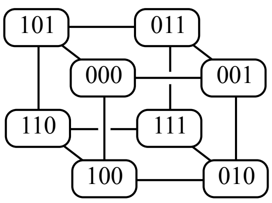

Figure 1: Xor-magic cube.

Let us define a graph G to be *xor-magic* (of order n) if G is connected, has 2^n vertices, and the vertices of G can be labeled with the distinct n-bit binary numbers in such a way that the bitwise xor of every vertex with its three neighbors is zero. Equivalently, if we view an n-bit number as a vector in $\mathbb{F}_2^n$, the magic condition is that all 2^n vectors are used, and the sum of every vertex with its three neighbors is the zero vector.

Although the xor-magic definition makes sense for any graph on 2^n vertices, this note will focus on 3-regular graphs. This is partly because the xor-magic property of the cube was discovered in the analysis of a tile-sliding game which can only be played on 3-regular graphs [4]. It is also partly just because regularity, with a low vertex degree, makes the graphs more appealing as puzzles, when they are to be worked out by hand.

We will show that 3-regular, xor-magic graphs of every order exist (Theorem 1). We also give necessary conditions for a graph to be xor-magic (Theorem 2), and, with computer assistance, show that 3-regular, xor-magic graphs of small order are rare. Finally, we offer an order 5 xor-magic graph without giving a labeling, for the reader who would like to work it out as a puzzle.

An infinite family

The complete graph on four vertices is a 3-regular, xor-magic graph of order 2, and the cube seen above is order 3. These can be seen as the first two in an infinite family, constructed inductively in a manner reminiscent of the reflected binary Gray code.

Theorem 1. *There is a 3-regular, xor-magic graph of order n for every $n \geqslant 2$.*

Proof. We begin with a different drawing of the cube graph, shown in Figure 2. The edges extending off the right end are meant to connect to the corresponding edges on the left end, as if the graph were wrapped around a cylinder.

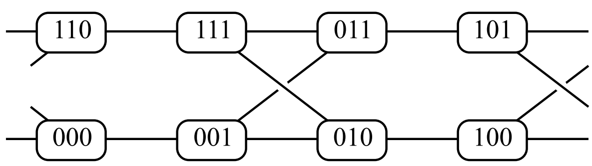

Figure 2: Cube as “crossed prism”.

To make an order 4 graph,

1. Make two copies of the order 3 graph. In one copy, append a 0 at the left of every label, and in the second copy, append a 1 at the left. This generates all possible four-bit labels.

2. Glue the two copies together, side by side (Figure 3).

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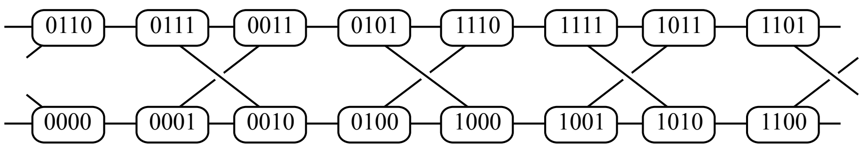


Figure 3: Two copies of the cube, glued together.

The resulting graph is clearly three-regular. At every vertex, the three rightmost bits satisfy the xor condition since they are copied directly from the cube. Any vertex not involved in the gluing has the same leftmost bit as all its neighbors, so the xor condition is satisfied for that bit as well. Any vertex which *is* on the gluing boundary has one neighbor with the same leftmost bit, and two neighbors with the opposite leftmost bit, so once again, the xor condition is satisfied. We have constructed an order 4 xor-magic graph.

Now, two copies of the order 4 graph can be extended to 5 bits and placed side by side to form an order 5 xor-magic graph, and so on. □

Necessary conditions

The next theorem provides simple calculations that can be used to show that many graphs are *not* xor-magic. Let G be a graph with vertices numbered 1 through 2^n. Let M be the $2^n \times 2^n$ matrix with

$$m_{ij} = \begin{cases} 1, & \text{if } i = j \text{ or if vertices } i \text{ and } j \text{ are adjacent in } G \\ 0, & \text{otherwise.} \end{cases}$$

In other words, M is the adjacency matrix of G, together with 1's on the main diagonal. An element in the nullspace of M over $\mathbb{F}_2$ is an assignment of single bits to the vertices of G in such a way that the xor condition is satisfied at every vertex. Since vectors are added componentwise, an n-bit labeling of G is equivalent to a choice of n vectors in the nullspace of M. The first vector gives the first bit in each label; the second vector gives the second bit; and so on.

Let $y_1, y_2, \ldots, y_k$ be any basis for the nullspace of M over $\mathbb{F}_2$ (so, k denotes the nullity of M). Finally, let Y be the $2^n \times k$ matrix having the y_i's as columns.

Theorem 2. *If G is xor-magic, then*
(Test 1) *$k \geqslant n$, and*
(Test 2) *The rows of Y are all distinct.*

Test 2 is particularly powerful because it applies to *any* basis for the nullspace.

Proof. Suppose that G is xor-magic. Let S be the $2^n \times n$ binary matrix in which the i-th row contains the bits in the label of vertex i.

For Test 1: Certainly S contains n linearly independent rows (for example, the rows containing a single 1). Thus, the rank of S is n, and the n column vectors of S are linearly independent. But each of those column vectors is in the nullspace of M, so the nullity k is at least n.

For Test 2: Let Y be a matrix whose columns are a basis for the nullspace of M, as described above. Since each column of S is in the nullspace of M, it can be expressed as a linear combination of the columns of Y. Hence, there is a $k \times n$ matrix J with $YJ = S$. Since the rows of S are all distinct, the rows of Y must all be distinct as well. □

Note: If a graph passes Test 2, then the rows of the Y matrix provide a set of k-bit labels which satisfy the xor condition and are all distinct. If $k = n$, then the graph is xor-magic. However, if $k > n$, there may or may not be a set of distinct n-bit labels which satisfy the xor condition.

3-regular graphs on 8 vertices

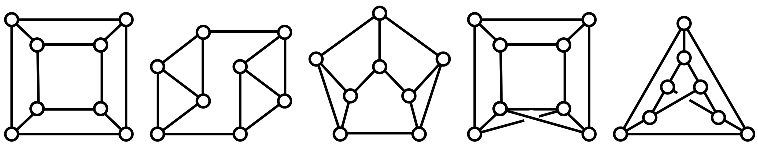
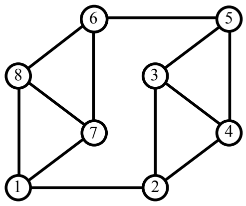


Figure 4: The five connected, 3-regular graphs on 8 vertices.

There are only five connected, 3-regular graphs on 8 vertices (sequence A002851 in the OEIS [5]), and they are shown in Figure 4.

The nullities of the M matrices for the five graphs are, respectively: 4, 4, 2, 2, and 1. Thus, all but the first two graphs fail Test 1, and cannot be xor-magic.

The second graph in Figure 4 has sufficient nullity to pass Test 1. This graph's M matrix (with respect to the vertex numbering shown at left) is



$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

from which a basis for the nullspace of M can be computed. One such basis is given by the columns of the Y matrix here:

$$Y = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

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Since this Y matrix has duplicate rows, the graph cannot be xor-magic. Thus, the cube is the only 3-regular, xor-magic graph on 8 vertices.

3-regular graphs on 16 vertices

There are 4,060 connected, 3-regular graphs on 16 vertices (see [5], or [2] for details of the calculation), and there are multiple sites on the internet where one can download a data file containing all of them. One such site is House of Graphs (`https://hog.grinvin.org/Cubic`), which is introduced in [1].

With the hard work of enumeration already done, it is easy to have a computer algebra system do the necessary nullspace calculations to establish the following:

Theorem 3. *There are only two connected 3-regular graphs on 16 vertices which pass both tests from Theorem 2.*

For example, in *Mathematica*, this can be done as follows, where the `cub16.g6` file comes from `https://hog.grinvin.org/Cubic`:

```
yMatrix[g_] := NullSpace[
    IdentityMatrix[Length[AdjacencyMatrix[g]]] + AdjacencyMatrix[g],
    Modulus -> 2]//Transpose;

test[g_] := UnsameQ @@ yMatrix[g]
allGraphs = Import["cub16.g6"];
passingGraphs = Select[allGraphs, test];
```

One of them, of course, is the crossed prism constructed in Theorem 1. The other, which we shall denote G_1, is shown in Figure 5. The nullity of the M

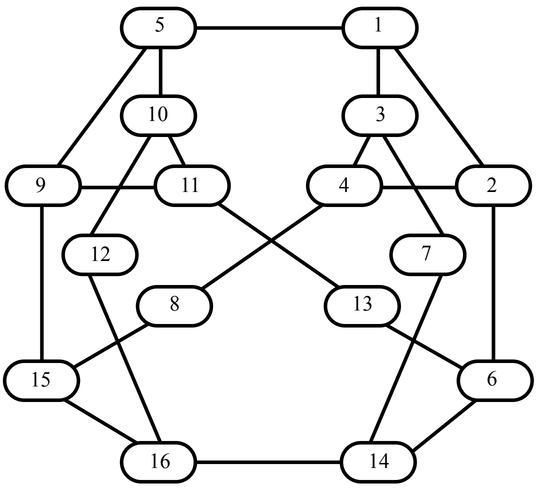

Figure 5: G_1 passes both tests in Theorem 2.

matrix for G_1 is 6, and computing a basis for the nullspace gives a Y matrix with distinct rows. However, this only goes to demonstrate that the necessary conditions of Theorem 2 are not sufficient:

Theorem 4. *G_1 is not xor-magic.*

The crossed prism is thus the only 3-regular, xor-magic graph of order 4. Before we begin the proof, let us note two generally useful transformations. Suppose that a graph G has a set of n-bit labels which satisfy the xor-magic condition.

1. Applying any invertible linear transformation in $GL_n(\mathbb{F}_2)$ to all the labels preserves the xor-magic property.

2. If every vertex in G has odd degree, then adding a constant n-bit vector to every label of G preserves the xor-magic property.

Proof of Theorem 4. Throughout the proof, we will refer to the numbering of the vertices shown in Figure 5. Suppose that G_1 has an xor-magic labeling, and let v_i denote the 4-bit label on vertex number i. By adding v_1 to every label in the graph, we can assume that $v_1 = 0000$.

If v_1, v_2, v_3, and v_4 are known, then v_5 through v_8 are determined by the xor condition – in fact, they are linear combinations of v_1 through v_4. Since all eight of these labels must be different, and we have normalized v_1 to 0000, v_2, v_3 and v_4 must be linearly independent. By applying a linear transformation, we can change those to any three linearly independent labels we like. After this transformation, we may assume that the first eight vertices are labeled as follows:

$$\begin{array}{llll} v_1 = 0000 & v_2 = 0001 & v_3 = 0010 & v_4 = 0100 \\ v_5 = 0011 & v_6 = 0101 & v_7 = 0110 & v_8 = 0111 \end{array}$$

Now, whatever v_9 is, it begins with a 1, so it is linearly independent from the labels at vertices 2, 3, and 4, and we can apply a linear transformation to assume that $v_9 = 1000$ (while preserving the labels on the first eight vertices). It then follows that $v_{10} = 1011$.

At this point, the xor condition requires $v_{11} \oplus v_{12} = v_5 \oplus v_{10} = 1000$. But this is impossible since v_{11} and v_{12} must both have a 1 in the leftmost position. □
As noted before Theorem 4, the M matrix for G_1 has nullity 6, and since it passes Test 2, its Y matrix provides a set of distinct 6-bit labels which satisfy the xor condition. In fact, one can improve this to give G_1 a set of distinct 5-bit labels which satisfy the xor condition (this is not too difficult, and is left as an exercise for the interested reader). G_1 is not magic, but it is very close. In general, for a graph which passes both tests from Theorem 2, with $k > n$, the author does not know a way to determine the minimum number of bits which suffice to give the vertices distinct labels which satisfy the xor condition.

More than just crossed prisms

At this point, one might guess that the crossed prisms of each order are the only 3-regular xor-magic graphs, but that is not so. An exhaustive search of 32-vertex graphs is not feasible. However, searching among the most symmetric 3-regular graphs is surprisingly fruitful. There are just ten connected, vertex-transitive cubic graphs on 32 vertices (A032355 in [6]). *Vertex-transitive* means that, for any two vertices u and v in the graph, there is a graph automorphism which carries u to v – an extra-strong regularity property. Among the ten are four xor-magic graphs: the crossed prism, the Dyck graph, and two others. Figures 6 and 7 show the show magic labelings of the Dyck graph and one of the others. Figure 8 is the remaining graph, given without a labeling for readers who might enjoy working out a solution for themselves, whether computer-assisted or entirely by hand.

Final Remarks

The xor-magic condition introduced in this note may seem unrelated to the old recreational mathematics standby of magic squares. However, Conway, Norton, and Ryba [3] prove that when the numbers 0–15 are placed into a traditional 4×4 magic square, the numbers in each row and each column xor to zero; they call this the *Nimm0 property.* The present article is inspired by their exposition. With regard to 3-regular, xor-magic graphs, it would be interesting to know if other infinite families with simple constructions like the crossed prisms might exist – perhaps some or all of the 32-vertex graphs in Figures 6–8 represent the beginning of some such family. We have already acknowledged that the focus on 3-regular graphs in this article is somewhat arbitrary, and many other xor-magic constructions may be waiting to be found among other classes of graphs.

Most interesting of all would be better theory to determine the minimum number of bits required to give a graph a set of distinct labels which satisfies the xor condition. An upper bound is given by Theorem 2, but the 5-bit labeling of G_1 shows that it is not sharp. Finding good theorems or algorithms to lower this bound should be challenging work for the future.

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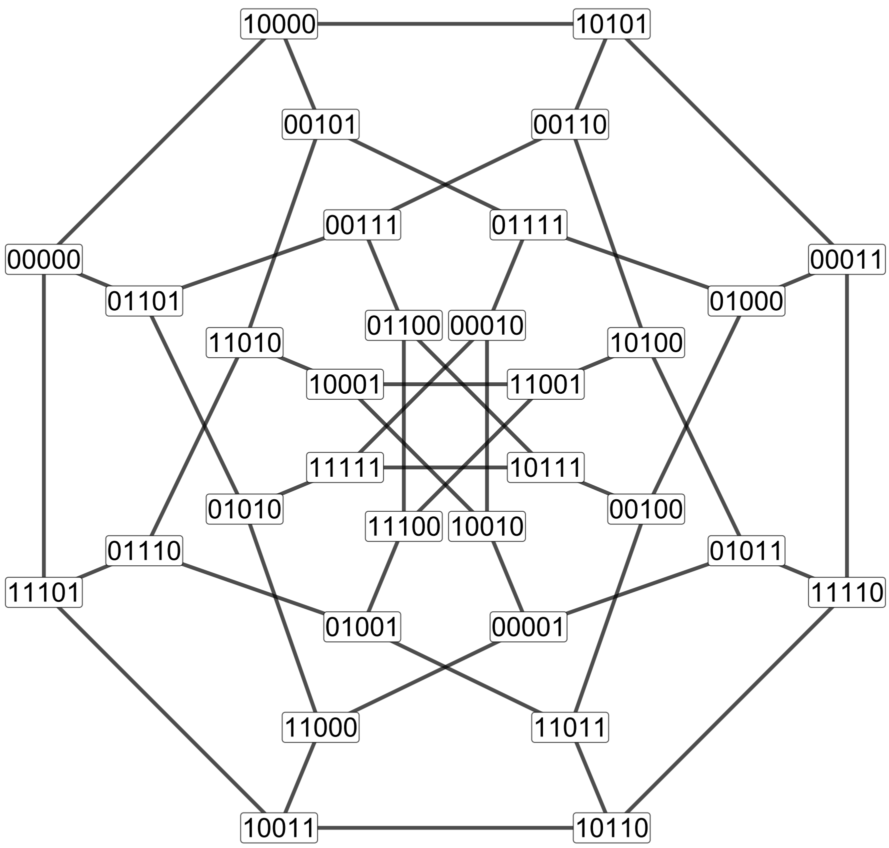

Figure 6: The Dyck graph is xor-magic.

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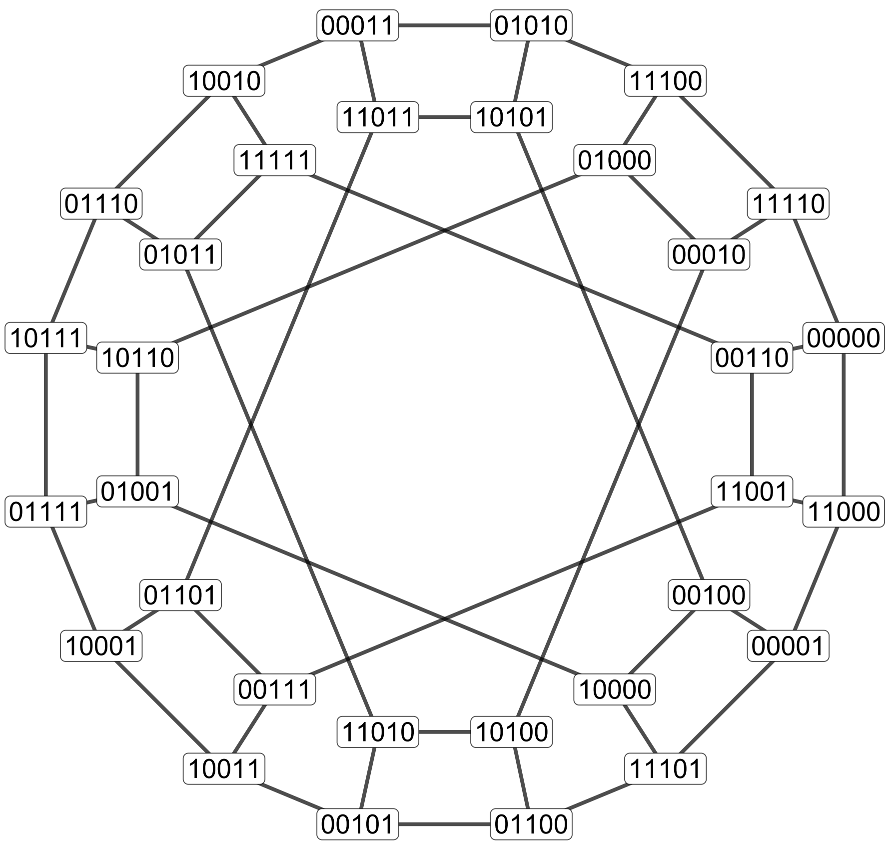

Figure 7: Another vertex-transitive xor-magic graph.

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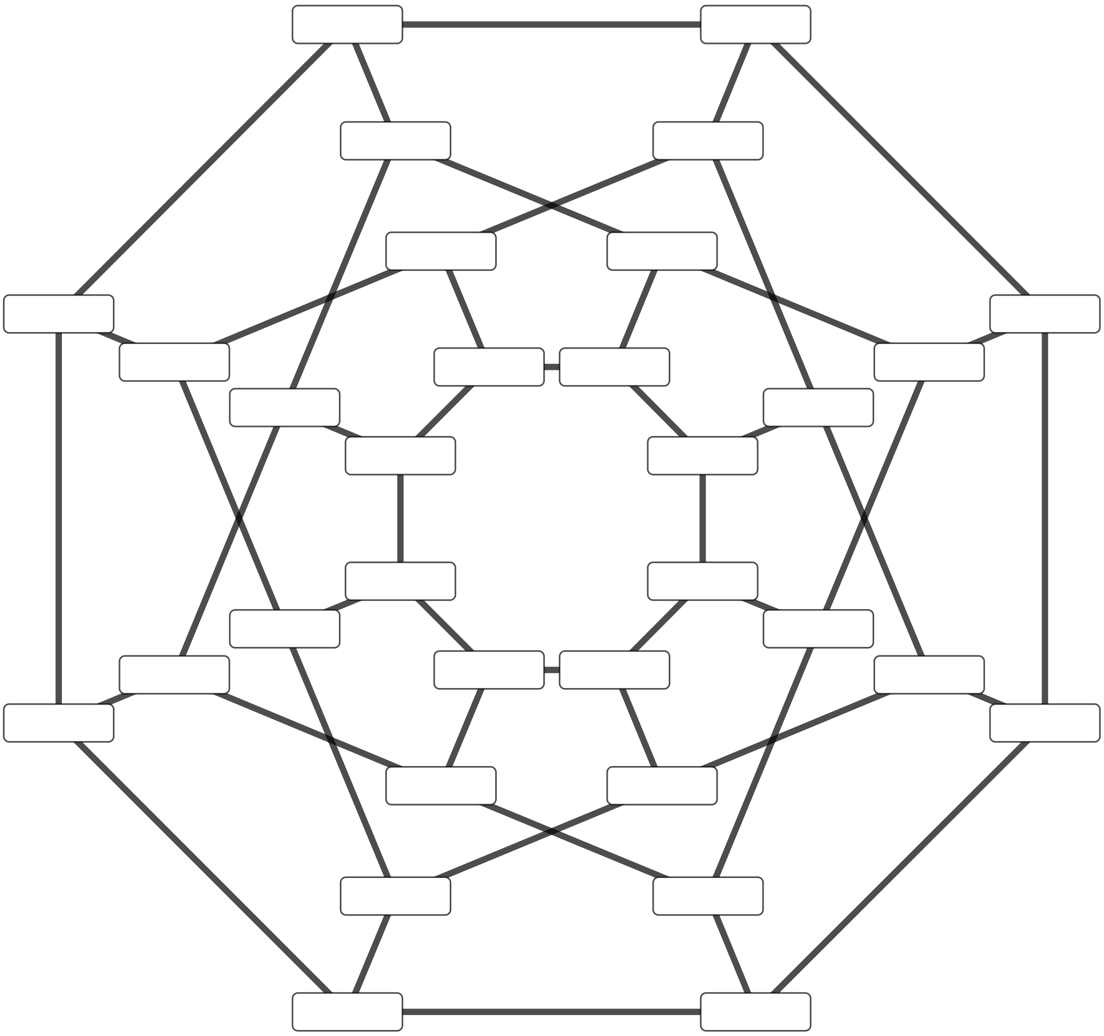

Figure 8: Order 5 xor-magic graph, without its labels.